UNIT 4 TRIGONOMETRIC FUNCTIONS

Structure

4.1 INTRODUCTION

In this unit, we shall be studying trigonometry. It is convenient to use trigonometry to measure distances between two landmarks or width or depth of rivers or heights of mountain etc. It was first started in India. Elements of the subject can be found even in Rigveda. All the ancient Indian Mathematicians like Aryabhata, Bhaskara I and II and Brahamagupta got important results.

Trigonometry means the science of measuring triangles. Given some of the sides and angles of a triangle, trigonometry helps us to calculate the remaining sides and angles.

In this unit, we will deal with the various systems of measuring angles, then we will define various circular functions and develop their properties. In the end, we will talk about periodicity, graph of circular functions and inverse trigonometric functions.

Objectives

After studying of this unit, you should be able to

- define an angle,
- define trigonometric ratios,
- derive the value of trigonometric ratios of some standard angles, allied and multiple angles,
- learn the basic properties of trigonometric ratios,
- define the inverse of trigonometric functions, and
- have an idea of the graphs of trigonometric functions and inverse trigonometric functions.

4.2 ANGLES

Definition 1

Let a revolving line starting from OX, revolve about O in a plane in the direction of the arrow and occupy the position OP, then it is said to trace out an angle XOP. OP is called the final (or terminal) position and OX, the initial position. The point O is called the vertex.

To be more rigorous an angle may be defined as a measure of the rotation of a half-ray about its origin.

Figure 4.1

An angle XOP is called *positive* if it is traced out by ray revolving in the anti-clockwise direction and *negative* if it is traced out by a ray revolving in the clockwise direction. *It may be noted that an angle can have any magnitude*.

There are two systems of measurement of an angle which are of importance in mathematics.

Sexagesimal System

In this system, an angle is measured in *degrees*, *minutes* and *seconds*. One degree (written as 1^{o}) is $\frac{1}{360}$th of a complete rotation, i.e. one complete rotation = 360^{o}.

Since a right angle is $\frac{1}{4}$th of a revolution, therefore, **1 right angle = 90^{o}**. A degree is further subdivided as follows :

1 degree = 60 minutes, written as $60'$ and

1 minute = 60 seconds, written as $60''$.

Circular System

In this system, an angle is measured in radians.

A radian is an angle subtended at the centre of a circle by an arc whose length is equal to the radius.

Let AB be an arc of a circle of radius r such that length of arc $AB = r$, then $\angle AOB$ = 1 radian (written as 1^{c}).

Since the whole circle subtends an angle of 360^{o} (= 4 right angles) at the centre and the angles at the centre of a circle are in the ratio of subtending arcs, therefore,

$$\frac{\angle AOB}{4 \text{ right angle}} = \frac{\text{arc } AB}{\text{circumference}}$$

$$\Rightarrow \quad \angle \text{AOB} = \frac{r}{2\pi r} \times 4 \text{ right angles } (\because \quad \text{circumferemce} = 2\pi r)$$

$$\Rightarrow \quad 1 \text{ radian} = \frac{2}{\pi} \text{ right angles}.$$

This means that a radian is a constant angle, independent of the radius of the circle. Also, we find that

π radians = 2 right angles = 180°

Figure 4.2

From here, we get

$$1 \text{ radian} = \frac{180}{\pi} \text{ degree} = \frac{180 \times 7}{22} = \frac{630^{\circ}}{11}$$

$$= \left(57 + \frac{3}{11}\right)^{\circ} = 57^{\circ} + \frac{3}{11} \times 60'$$

$$= 57^{\circ} + \left(16 + \frac{4}{11}\right)'$$

$$= 57^{\circ} + 16' + \frac{4}{11} \times 60''$$

$$= 57^{\circ}\ 16'\ 22'' \text{ nearly.}$$

Remark

The symbol 'π' stands for the ratio of circumference of a circle to its diameter. It is an irrational number. However, for all practical purposes, unless otherwise mentioned, the value of π is taken as $\frac{22}{7}$.

A better approximation for π is $\frac{355}{113}$. When we take this value of π, we find that

$$1 \text{ radian} = 57^{\circ}\ 17'\ 45'' \text{ nearly.}$$

Definition 2 : Circular Measure of an Angle

The circular measure of an angle is the number of radians it contains.
Thus the circular measure of a radian is 1.

Circular measure of some standard angles is given in the following Table 4.1.

Angle in Degrees	0^{o}	30^{o}	45^{o}	60^{o}	90^{o}	120^{o}	135^{o}	150^{o}	180^{o}	270^{o}	360^{o}
Circular Measures	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

Remember

When the unit in terms of which an angle is measured, is not mentioned, radian is understood.

Theorem 1

Prove that the number of radians in an angle subtended by an arc of a circle at the centre $= \dfrac{\textbf{arc}}{\textbf{radius}}$ **.**

Proof

Let $\angle AOP = \theta$ radians be the angle subtended by an arc AP (= l) of a circle at the centre O. Cut off arc AB = radius (= r) and join OB, then

$$\angle AOB = 1 \text{ radian (by definition).}$$

Now $$\frac{\angle AOP}{\angle AOB} = \frac{\text{arc } AP}{\text{arc } AB}$$

($\because$ *angles at the centre of a circle are proportional to the arcs on which they stand*.)

$$\Rightarrow \quad \frac{\theta \text{ radian}}{1 \text{ radian}} = \frac{l}{r}$$

$$\Rightarrow \quad \theta = \frac{l}{r}.$$

Figure 4.3

Hence number of radian in

$$\angle AOP = \frac{\text{arc } AP}{\text{radius}}$$

4.3 CIRCULAR FUNCTIONS OR TRIGONOMETRIC RATIOS

Definition 3 : Definition of Circular Functions

Let a revolving line, starting from OX, trace out an angle XOP = θ in any of the four quadrants. Let M be the foot of perpendicular from P upon X′OX.

Regarding OM and MP as directed lengths (OP always +ve), the ratios of OM, MP and OP with one another are called *circular functions* or *trigonometrical ratios* (abbreviated as *t*-ratios) of the angle θ.

Figure 4.4

Let OM = x, MP = y and OP = $r > 0$, then we define the various circular functions as follow :

(i) $\frac{\text{MP}}{\text{OP}}$ is called **sine of** θ and is written as sin θ, i.e. $\mathbf{sin}\ \theta = \frac{\boldsymbol{y}}{\boldsymbol{r}}$.

(ii) $\frac{\text{OM}}{\text{OP}}$ is called **cosine of** θ and is written as cos θ, i.e. $\mathbf{cos}\ \theta = \frac{\boldsymbol{x}}{\boldsymbol{r}}$.

(iii) $\frac{\text{MP}}{\text{OM}}$ is called **tangent of** θ and is written as tan θ, i.e.

$\mathbf{tan}\ \theta = \frac{\boldsymbol{y}}{\boldsymbol{x}}, \boldsymbol{x} \neq \mathbf{0}$.

(iv) $\frac{\text{OM}}{\text{MP}}$ is called **cotangent of** θ and is written as cot θ, i.e.

$\mathbf{cot}\ \theta = \frac{\boldsymbol{x}}{\boldsymbol{y}}, \boldsymbol{y} \neq \mathbf{0}$.

(v) $\frac{\text{OP}}{\text{OM}}$ is called **secant of** θ and is written as sec θ, i.e.

$\mathbf{sec}\ \theta = \frac{\boldsymbol{r}}{\boldsymbol{x}}, \boldsymbol{x} \neq \mathbf{0}$.

(vi) $\frac{\text{OP}}{\text{MP}}$ is called **cosecant of** θ and is written as cosec θ, i.e.

$\mathbf{cosec}\ \theta = \frac{\boldsymbol{r}}{\boldsymbol{y}}, \boldsymbol{y} \neq \mathbf{0}$.

Remarks

(1) From the above definitions, it is clear that

(i) $\operatorname{cosec} \theta = \frac{1}{\sin \theta}$

(ii) $\sec\theta = \dfrac{1}{\cos\theta}$

(iii) $\cot\theta = \dfrac{1}{\tan\theta}$

(iv) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

(v) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

(2) $\sin\theta$ is one symbol, i.e. $\sin\theta \neq \sin\times\theta$. Similar is the case for other *t*-ratios.

(3) $(\sin\theta)^n$ is usually written as $\sin^n\theta, n \in N$ and $(\sin\theta)^{-1} = \dfrac{1}{\sin\theta}$.

(4) We observe that the above functions depend only on the value of the angle θ and not on the point P chosen on the terminal side of θ. For example, if we take another point $P'(x', y')$ on the terminal line with $OP' = r'$, then considering similar triangles we have

$$\frac{y}{r} = \frac{y'}{r'}, \frac{x}{r} = \frac{x'}{r'}, \frac{y}{x} = \frac{y'}{x'}$$

(5) If the terminal side coincides with one of the axes say if it coincides with *x*-axis, then cosec and cot are not defined while if it coincides with *y*-axis, then sec and tan are not defined.

(6) The signs of trigonometric ratios depend on the quadrant in which the terminal line of the angle lies. They depend upon the sign of *x* and *y* as *r* is always +ve.

Table 4.2 describes the signs of various *t*-ratios in different quadrants (refer to Figure 4.4).

Quadrant	**I**	**II**	**III**	**IV**
MP = y	+ ve	+ ve	− ve	− ve
OM = x	+ ve	− ve	− ve	+ ve
$\sin\theta = \dfrac{y}{r}$	$\dfrac{+\text{ve}}{+\text{ve}} = +\text{ve}$	$\dfrac{+\text{ve}}{+\text{ve}} = +\text{ve}$	$\dfrac{-\text{ve}}{+\text{ve}} = -\text{ve}$	$\dfrac{-\text{ve}}{+\text{ve}} = -\text{ve}$
$\cos\theta = \dfrac{x}{r}$	$\dfrac{+\text{ve}}{+\text{ve}} = +\text{ve}$	$\dfrac{-\text{ve}}{+\text{ve}} = -\text{ve}$	$\dfrac{-\text{ve}}{+\text{ve}} = -\text{ve}$	$\dfrac{+\text{ve}}{+\text{ve}} = +\text{ve}$
$\tan\theta = \dfrac{y}{x}, x \neq 0$	$\dfrac{+\text{ve}}{+\text{ve}} = +\text{ve}$	$\dfrac{+\text{ve}}{-\text{ve}} = -\text{ve}$	$\dfrac{-\text{ve}}{-\text{ve}} = +\text{ve}$	$\dfrac{-\text{ve}}{+\text{ve}} = -\text{ve}$

The signs of other *t*-ratios can be found by using reciprocal relations, i.e.

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \sec\theta = \frac{1}{\cos\theta} \text{ and } \cot\theta = \frac{1}{\tan\theta}.$$

So, we have

Quadrant : →	I	II	III	IV
t-ratios which arc +ve	All	sin θ cosec θ	tan θ cot θ	cos θ sec θ

In the first quadrant as the angle increases from 0^o to 90^o, sin θ increases from 0 to 1. In the second quadrant as θ increases from 90^o to 180^o, sin θ decreases from 1 to 0. In the third quadrant as θ increases from 180^o to 270^o, sin θ decreases from 0 to – 1 and finally in the fourth quadrant sin θ increases from – 1 to 0 as θ increases from 270^o to 360^o. In fact we have the following table.

I quadrant		**II quadrant**	
Sine	increases from 0 to 1	Sine	decreases from 1 to 0
Cosine	decreases from 1 to 0	Cosine	decreases from 0 to – 1
Tangent	increases from 0 to ∞	Tangent	increases from – ∞ to 0
Cotangent	decreases from ∞ to 0	Cotangen t	decreases from 0 to – ∞
Secant	increases from 1 to ∞	Secant	increases from – ∞ to –1
Cosecant	decreases from ∞ to 1	Cosecant	decreases from 1 to ∞
III quadrant		**IV quadrant**	
Sine	decreases from 0 to – 1	Sine	increases from – 1 to 0
Cosine	increases from – 1 to 0	Cosine	increases from 0 to 1
Tangent	increases from 0 to ∞	Tangent	increases from – ∞ to 0
Cotangent	decreases from ∞ to 0	Cotangen t	increases from 0 to – ∞
Secant	decreases from – 1 to – ∞	Secant	increases from ∞ to 1
Cosecant	increases from – ∞ to –1	Cosecant	decreases from – 1 to – ∞

Remark

In the above table we see the symbol ∞. Observe that ∞ is not a real number and is just a symbol. Statement like tan θ increases from 0 to ∞ for $\theta \in \left(0, \frac{\pi}{2}\right)$ simply means that tan θ increases as θ increases in the interval $\left(0, \frac{\pi}{2}\right)$ and assumes arbitrarily large positive values as θ increases to $\frac{\pi}{2}$. Similarly, to say that cosec decreases from – 1 to – ∞ in the fourth quadrant means that cosec θ is a *decreasing function* for $\theta \in \left(\frac{3\pi}{2}, 2\pi\right)$ and assumes arbitrarily large negative values as θ approaches 2π.

Theorem 2 : Fundamental Identities

Prove that

(i) $\sin^2\theta + \cos^2\theta = 1$

(ii) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

(iii) $1 + \tan^2\theta = \sec^2\theta$

Figure 4.5

Proof

(i) $\sin^2\theta + \cos^2\theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$

(ii) $1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$ (Dividing by $\sin^2\theta$)

i.e. $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

(iii) $\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}$ (Dividing by $\cos^2\theta$)

i.e. $1 + \tan^2\theta = \sec^2\theta$.

4.3.1 Trigonometric Ratios of Standard Angles

Value of *t*-ratio for 45° or $\frac{\pi}{4}$

Let a revolving line, starting from OX, trace out $\angle$ XOP = 45°.

From *P*, draw PM $\perp$ OX.

Figure 4.6

Then $\angle \text{OPM} = 90^\circ - \angle \text{MOP}$

$= 90^\circ - 45^\circ = 45^\circ$

$= \angle \text{MOP}$,

$\therefore$ OM = MP = a (say), $a > 0$

$\therefore \text{OP}^2 = \text{OM}^2 + \text{MP}^2$

$= a^2 + a^2 = 2a^2$

$$\text{OP} = a\sqrt{2}$$

(*Taking +ve sign with the square root,* $\because$ OP *is always +ve*).

$$\therefore\ \sin 45^{o} = \frac{\text{MP}}{\text{OP}} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}};$$

$$\cos 45^{o} = \frac{\text{OM}}{\text{OP}} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}};$$

$$\tan 45^{o} = \frac{\text{MP}}{\text{OM}} = \frac{a}{a} = 1;$$

$$\cot 45^{o} = \frac{\text{OM}}{\text{MP}} = \frac{a}{a} = 1;$$

$$\sec 45^{o} = \frac{\text{OP}}{\text{OM}} = \frac{a\sqrt{2}}{a} = \sqrt{2};$$

$$\operatorname{cosec} 45^{o} = \frac{\text{OP}}{\text{MP}} = \frac{a\sqrt{2}}{a} = \sqrt{2}.$$

Value of *t*-ratio for 30^{o} or $\frac{\pi}{6}$

Let a revolving line, starting from OX, trace out $\angle$ XOP = 30^{o}.

From *P*, draw PM $\perp$ OX.

Then $\quad \angle \text{OPM} = 90^{o} - \angle \text{MOP}$

$$= 90^{o} - 30^{o} = 60^{o}.$$

AT *O*, make $\angle$ P′OM = 30^{o} and produce PM to meet OP′ in P′.

Then Δ*s*, OMP, OMP′ are congruent

$$(30^{o} = 30^{o},\ rt\ \angle = rt\ \angle \text{ and } \text{OM} = \text{OM})$$

$$\therefore\ \text{MP} = \text{P'M} = a \qquad \text{(say)}$$

Figure 4.7

and $\quad \angle \text{MP'O} = \angle \text{OPM} = 60^{o}$ (Proved above)

$\therefore\ \Delta$ OPP′ is equilateral,

$\therefore\ \text{OP} = \text{P'P} = 2\text{MP} = 2a,\ a > 0$

$$\therefore\ \text{OM}^2 = \text{OP}^2 - \text{MP}^2 = 4a^2 - a^2 = 3a^2$$

$$\Rightarrow\ \text{OM} = a\sqrt{3},$$

(*Taking +ve sign, with the square root, because* OM *being drawn to the right of O, is +ve*).

$$\therefore \ \sin 30^{\circ} = \frac{MP}{OM} = \frac{a}{2a} = \frac{1}{2};$$

$$\cos 30^{\circ} = \frac{OM}{OP} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30^{\circ} = \frac{MP}{OM} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}};$$

$$\cot 30^{\circ} = \frac{OM}{MP} = \frac{a\sqrt{3}}{a} = \sqrt{3};$$

$$\sec 30^{\circ} = \frac{OP}{OM} = \frac{2a}{a\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$\operatorname{cosec} 30^{\circ} = \frac{OP}{MP} = \frac{2a}{a} = 2.$$

Value of *t*-ratio for 60^{o} or $\frac{\pi}{3}$

Let a revolving line, starting from OX, trace out $\angle$ XOP = 60^{o}.

From P, draw PM $\perp$ OX.

Then $\angle \text{OPM} = 90^{\circ} - \angle \text{MOP}$

$$= 90^{\circ} - 60^{\circ} = 30^{\circ}.$$

At P make $\angle \text{MPP}' = 30^{\circ}$ and let PP′ meet OX in P′.

Figure 4.8

Then Δs OMP, MP′P are congruent

$(\because\ 30^{\circ} = 30^{\circ},\ rt\ \angle = rt\ \angle$ and $\text{MP} = \text{MP})$

$\therefore$ OM = MP′ = a (say) and

$\angle \text{PP}'\text{M} = \angle \text{MOP} = 60^{\circ}$

$\therefore$ Δ OPP′ is equilateral,

$\Rightarrow$ OP = OP′ = 2OM = $2a$, $a > 0$

$$\therefore\ \text{MP}^2 = \text{OP}^2 - \text{OM}^2 = 4a^2 - a^2 = 3a^2$$

$$\Rightarrow\ \text{MP} = a\sqrt{3}$$

(*Taking +ve sign, with the square root,* $\because$ MP *being drawn above OX is +ve*).

$$\therefore \ \sin 60^{o} = \frac{MP}{OP} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2};$$

$$\cos 60^{o} = \frac{OM}{OP} = \frac{a}{2a} = \frac{1}{2}$$

$$\tan 60^{o} = \frac{MP}{OM} = \frac{a\sqrt{3}}{a} = \sqrt{3};$$

$$\cot 60^{o} = \frac{OM}{MP} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}};$$

$$\sec 60^{o} = \frac{OP}{OM} = \frac{2a}{a} = 2$$

$$\operatorname{cosec} 60^{o} = \frac{OP}{MP} = \frac{2a}{a\sqrt{3}} = \frac{2}{\sqrt{3}}.$$

Value of *t*-ratio for 0°

Let a revolving line, starting from OX, trace out ∠ XOP = 0° so that *P* lies on OX.

From *P*, draw PM perpendicular on OX, so that M coincides with *P*.

Then OM = OP = a (say), MP = 0

OP = a, $a > 0$

Figure 4.9

$$\therefore \ \sin 0^{o} = \frac{MP}{OM} = \frac{0}{a} = 0;$$

$$\cos 0^{o} = \frac{OM}{OP} = \frac{a}{a} = 1$$

$$\tan 0^{o} = \frac{MP}{OM} = \frac{0}{a} = 0;$$

$$\sec 0^{o} = \frac{OP}{OM} = \frac{a}{a} = 1$$

cot 0° and cosec 0° are not defined.

Value of *t*-ratio for 90° or $\frac{\pi}{2}$

Let a revolving line, starting from OX, trace out ∠ XOP = 90°.

From *P*, draw PM ⊥ on OX, so that M coincides with *O*.

Figure 4.10

$\therefore$ OM = 0, MP = OP = a (say), $a > 0$

$$\therefore \ \sin 90^o = \frac{MP}{OP} = \frac{a}{a} = 1;$$

$$\cos 90^o = \frac{OM}{OP} = \frac{0}{a} = 0$$

$$\cot 90^o = \frac{OM}{MP} = \frac{0}{a} = 0;$$

$$\operatorname{cosec} 90^o = \frac{OP}{MP} = \frac{a}{a} = 1.$$

tan 90^o and sec 90^o are not defined.

We summarise the values of sin θ and cos θ in Table 4.3 for ready reference.

θ	0^o	30^o	45^o	60^o	90^o
sin θ	$\sqrt{\frac{0}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{4}{4}}$
cos θ	$\sqrt{\frac{4}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{0}{4}}$

Example 4.1

In a right angled triangle, the difference between two acute angles is $\frac{\pi}{9}$ in circular measure. Find the angles in degrees.

Solution

Since the triangle is right angled, so the sum of the acute angles is 90^o.

Let the two acute angles be x and y, $x > y$.

Then $\qquad x + y = 90^o$

Also $\qquad x - y = \frac{\pi}{9}$ radian $= 20^o$

$$\therefore \ 2x = 90^o + 20^o = 110^o$$

i.e. $\qquad x = 55^o$

$$\therefore \ y = 90^o - 55^o = 35^o$$

Example 4.2

Given $\cot\theta = \frac{12}{5}$, θ in the IIIrd quadrant, find the value of the other trigonometric functions.

Solution

$$\tan\theta = \frac{5}{12} \qquad \ldots (4.1)$$

As $\qquad \sec^2\theta = 1 + \tan^2\theta = 1 + \frac{25}{144} = \frac{169}{144}$

In quadrant III, sin θ, cos θ, sec θ, cosec θ are all negative.

$$\therefore \quad \sec\theta = \frac{-13}{12} \text{ and } \cos\theta = \frac{-12}{13} \qquad \ldots (4.2)$$

$$\sin\theta = \tan\theta\cos\theta = \frac{5}{12}\left(\frac{-12}{13}\right) = \frac{-5}{13}$$

$$\therefore \quad \sin\theta = \frac{-5}{13} \text{ and } \operatorname{cosec}\theta = \frac{-13}{5} \qquad \ldots (4.3)$$

Example 4.3

Prove that

$$\frac{\sin\theta}{1-\cos\theta} + \frac{\tan\theta}{1+\cos\theta} = \sec\theta\operatorname{cosec}\theta + \cot\theta$$

$$\text{L. H. S.} \quad = \frac{\sin\theta}{1-\cos\theta} + \frac{\tan\theta}{1+\cos\theta}$$

$$= \frac{\sin\theta\,(1+\cos\theta) + \tan\theta\,(1-\cos\theta)}{1-\cos^2\theta}$$

$$= \frac{\sin\theta + \sin\theta\cos\theta + \tan\theta - \sin\theta}{\sin^2\theta}$$

$$= \frac{\sin\theta\cos\theta + \tan\theta}{\sin^2\theta}$$

$$= \frac{\sin\theta\cos\theta}{\sin^2\theta} + \frac{\tan\theta}{\sin^2\theta}$$

$$= \cot\theta + \frac{1}{\cos\theta\sin\theta} = \cot\theta + \sec\theta\operatorname{cosec}\theta$$

SAQ 1

(a) Find the radian measure correspondingly to the following degree measures

(i) 15°,

(ii) $-22^{\circ}.\,30'$.

(b) Find the degree measure correspondingly to the following radian measures

(i) $\frac{7\pi}{3}$, (ii) $\frac{1}{4}$.

(c) Find the value of the other five trigonometric functions in each of the following

(i) $\cos\theta = -\frac{1}{2}$, θ is in quadrant II

(ii) $\tan\theta = \frac{3}{4}$, θ is in quadrant III

(iii) $\sin\theta = \frac{3}{5}$, θ is in quadrant I.

(d) Prove the following trigonometric identities

(i) $\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \operatorname{cosec}\theta - \cot\theta$

(ii) $\frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \sec^2\theta - \operatorname{cosec}^2\theta$

(iii) $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$

(iv) $\frac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta} = 1 - 2\sec\theta\tan\theta + 2\tan^2\theta$

4.3.2 Trigonometric Ratios of Allied Angles

We shall now find t-ratios of $-\theta$, $90^o \pm \theta$, $180^o \pm \theta$ and $360^o \pm \theta$in terms of those of θ.

T-ratios of ($-\theta$) in terms of those of θ

Let a revolving line OP, starting from OX, trace out an $\angle$ XOP = θ in any of the four quadrants.

Let another revolving line OP′ = OP, starting from OX, revolve in the opposite direction through θ (as shown in Figure 4.11), so that $\angle \mathrm{XOP'} = -\theta$.

From P and P′ draw PM and P′M′ $\perp$ on XOX′

Then Δs OMP, OM′P′ are congruent.

($\because$ *in figure (i)*, $\angle$ POM *in magnitude* $= \theta$ $= \angle \mathrm{P'OM'}$, $rt = \angle\ rt\ \angle$ and $\mathrm{OP} = \mathrm{OP'}$)

OM′ = OM ($\because$ They *have the same sign*)

M′P′ = – MP ($\because$ They *have the opposite signs*)

and OP′ = OP

$$\therefore\ \sin(-\theta) = \frac{\mathrm{M'P'}}{\mathrm{OP'}} = \frac{-\mathrm{MP}}{\mathrm{OP}} = -\sin\theta\,;$$

$$\cos(-\theta) = \frac{\mathrm{OM'}}{\mathrm{OP'}} = \frac{\mathrm{OM}}{\mathrm{OP}} = \cos\theta\,;$$

$$\tan(-\theta) = \frac{\mathrm{M'P'}}{\mathrm{OM'}} = \frac{-\mathrm{MP}}{\mathrm{OM}} = -\tan\theta\,;$$

$$\cot(-\theta) = \frac{\mathrm{OM'}}{\mathrm{M'P'}} = \frac{\mathrm{OM}}{-\mathrm{MP}} = -\cot\theta\,;$$

$$\sec(-\theta) = \frac{OP'}{OM'} = \frac{OP}{OM} = \sec\theta\text{; and}$$

$$\operatorname{cosec}(-\theta) = \frac{OP'}{M'P'} = \frac{OP}{-MP} = -\operatorname{cosec}\theta$$

Figure 4.11

T-ratios of $(90^0 - \theta)$ in terms of those of θ

Let a revolving line OP, starting from OX, trace out an $\angle$ XOP = θ in any of the four quadrants.

Let another revolving line $OP' = OP$, starting from OX, trace out $\angle$ XOY = 90^o and then revolve back through θ so that $\angle XOP' = 90^o - \theta$.

From P, P′ draw PM, $P'M' \perp s$ on $X'OX$.

Then Δs OMP, $OM'P'$ are congruent.

($\because$ *in figure (i),* $\angle$ POM *in magnitude* = θ
$= \angle YOP' = \angle OP'M'$, (*alt.* $\angle s$), *rt* $\angle = rt$ $\angle$ and $OP = OP'$)

$\therefore$ $OM' = OM$ ($\because$ They *have the same sign*)

$M'P' = OM$ ($\because$ They *have the same sign*)

and $OP' = OP$.

$$\therefore\ \sin(90^o - \theta) = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos\theta;$$

$$\cos(90^o - \theta) = \frac{OM'}{OP'} = \frac{MP}{OP} = \sin\theta;$$

$$\tan(90^o - \theta) = \frac{M'P'}{OM'} = \frac{OM}{MP} = \cot\theta;$$

$$\cot(90^o - \theta) = \frac{OM'}{M'P'} = \frac{MP}{OM} = \tan\theta;$$

$$\sec(90^{\circ} - \theta) = \frac{OP'}{OM'} = \frac{OP}{MP} = \operatorname{cosec}\theta;\ \text{and}$$

$$\operatorname{cosec}(90^{\circ} - \theta) = \frac{OP'}{M'P'} = \frac{OP}{OM} = \sec\theta.$$

Figure 4.12

T-ratios of $(90^{\circ} + \theta)$ in terms of those of θ

Let a revolving line OP, starting from OX, trace out an $\angle XOP = \theta$ in any of the four quadrants.

Figure 4.13

Let another revolving line $OP' = OP$, starting from OX, trace out $\angle XOY = 90^{\circ}$ and then revolved further through θ so that $\angle XOP' = 90^{\circ} + \theta$

From P, P' draw $PM, P'M' \perp s$ on $X'OX$.

Then $\Delta\ OM'P'$ and $\Delta\ OMP$ are congruent.

[$\because$ $OP' = OP, \angle OM'P' = \angle OMP = 90^{\circ}$ *and* $\angle OP'M' = \angle MOP = \theta$ (*in figure(i)*)]

$\therefore \quad OM' = -MP \qquad (\because$ *They have opposite signs*)

$\quad M'P' = OM \qquad (\because$ *They have same sign*)

and $\quad OP' = OP.$

$$\therefore \ \sin(90^o + \theta) = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos\theta\,;$$

$$\cos(90^o + \theta) = \frac{OM'}{OP'} = \frac{-MP}{OP} = -\sin\theta\,;$$

$$\tan(90^o + \theta) = \frac{M'P'}{OM'} = \frac{OM}{-MP} = -\cot\theta\,;$$

$$\cot(90^o + \theta) = \frac{OM'}{M'P'} = \frac{-MP}{OM} = -\tan\theta\,;$$

$$\sec(90^o + \theta) = \frac{OP'}{OM'} = \frac{OP}{-MP} = -\operatorname{cosec}\theta\,; \text{ and}$$

$$\operatorname{cosec}(90^o + \theta) = \frac{OP'}{M'P'} = \frac{OP}{MP} = \sec\theta\,.$$

T-ratios of $(180^o - \theta)$ in terms of θ

Let a revolving line OP, starting from OX, trace out an $\angle\, XOP = \theta$, in any of the four quadrants.

Figure 4.14

Let another revolving line, $OP' = OP$, starting from OX, trace out $\angle\, XOX' = 180^o$ and then revolve back through θ so that $\angle\, XOP' = 180^o - \theta$.

From $P,\ P'$ draw $PM,\ P'M' \perp s$ on $X'OX$.

Then $\Delta s\ OMP', OM'P$ are congruent.

$(\because$ *in figure (i),* $\angle\, POM$ *in magnitude* $= \theta$
$= \angle\, P'OM'$, *rt* $\angle =$ *rt* $\angle$ *and* $OP = OP'$)

$\therefore$ OM′ = – OM (∵ *They have opposite signs*)

M′P′ = MP (∵ *They have same sign*)

and OP′ = OP.

$$\therefore \ \sin(180^o - \theta) = \frac{M'P'}{OP'} = \frac{MP}{OP} = \sin\theta\,;$$

$$\cos(180^o - \theta) = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos\theta\,;$$

$$\tan(180^o - \theta) = \frac{M'P'}{OM'} = \frac{MP}{-OM} = -\tan\theta\,;$$

$$\cot(180^o - \theta) = \frac{OM'}{M'P'} = \frac{-OM}{MP} = -\cot\theta\,;$$

$$\sec(180^o - \theta) = \frac{OP'}{OM'} = \frac{OP}{-OM} = -\sec\theta\,;\text{ and}$$

$$\operatorname{cosec}(180^o - \theta) = \frac{OP'}{M'P'} = \frac{OP}{MP} = \operatorname{cosec}\theta\,.$$

T-ratios of $(n\,.\,360^o + \theta), n \in I$

Since increasing or decreasing an angle by an integral multiple of 360^o amounts to only complete revolutions of the revolving line, therefore, *t*-ratios of $n\,.\,360^o + \theta, n \in I$ will remain the same as those of θ, i.e. for all $n \in I$, we have

$$\sin(n\,.\,360^o + \theta) = \sin\theta,$$

$$\cos(n\,.\,360^o + \theta) = \cos\theta,$$

$$\tan(n\,.\,360^o + \theta) = \tan\theta,$$

$$\sec(n\,.\,360^o + \theta) = \sec\theta, \text{ and}$$

$$\operatorname{cosec}(n\,.\,360^o + \theta) = \operatorname{cosec}\theta,$$

$$\cot(n\,.\,360^o + \theta) = \cot\theta,$$

T-ratios of $(360^o + \theta)$

Putting $n = 1$, we get

$$\sin(360^o + \theta) = \sin\theta,$$

$$\cos(360^o + \theta) = \cos\theta,$$

$$\tan(360^o + \theta) = \tan\theta,$$

$$\sec(360^o + \theta) = \sec\theta, \text{ and}$$

$$\operatorname{cosec}(360^o + \theta) = \operatorname{cosec}\theta,$$

$\cot(360^\circ + \theta) = \cot\theta,$

T-ratios of $(360^\circ - \theta)$

Replacing θ by − θ, we have

$\sin(360^\circ + (-\theta)) = \sin(-\theta) = -\sin\theta$

i.e. $\sin(360^\circ - \theta) = -\sin\theta,$

Similarly,

$\operatorname{cosec}(360^\circ - \theta) = -\operatorname{cosec}\theta,$

$\cos(360^\circ - \theta) = \cos\theta,$

$\sec(360^\circ - \theta) = \sec\theta,$

$\tan(360^\circ - \theta) = -\tan\theta,$

$\cot(360^\circ - \theta) = -\cot\theta.$

T-ratios of $(180^\circ + \theta)$

Now, $\sin(180^\circ + \theta) = \sin(180^\circ - (-\theta)) = \sin(-\theta) = -\sin\theta$

Similarly, we have

$\cos(180^\circ + \theta) = -\cos\theta,$

$\tan(180^\circ + \theta) = \tan\theta,$

$\cot(180^\circ + \theta) = \cot\theta,$

$\sec(180^\circ + \theta) = -\sec\theta,$ and

$\operatorname{cosec}(180^\circ + \theta) = -\operatorname{cosec}\theta.$

4.3.3 Trigonometric Ratios of Compound Angles

Theorem 3 : Addition and Subtraction Formula

Prove geometrically that :

(i) $\sin(A + B) = \sin A\cos B + \cos A\sin B$

(ii) $\cos(A + B) = \cos A\cos B - \sin A\sin B$

(iii) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

Proof

Let a revolving line, starting from its initial position OX, trace out $\angle XOY = A$. Let it further revolve through $\angle YOZ = B$, so that $\angle XOZ = A + B$.

From any point P on OZ, draw PM ⊥ OX, PN ⊥ OY.

From N, draw NQ ⊥ OX, NR ⊥ PM.

Then $\angle \text{RPN} = 90^{\circ} - \angle \text{PNR} = \angle \text{RNO} = \angle \text{NOQ} = \text{A}$ (*alt.* $\angle$ *s*)

Figure 4.15

From right $\angle d\ \Delta$ OMP ,

(i) $\sin(\text{A}+\text{B}) = \sin \angle \text{XOZ}$

$$= \frac{\text{MP}}{\text{OP}} = \frac{\text{MR}+\text{RP}}{\text{OP}} = \frac{\text{QN}+\text{RP}}{\text{OP}}$$

$$= \frac{\text{QN}}{\text{OP}} + \frac{\text{RP}}{\text{OP}} = \frac{\text{QN}}{\text{ON}} \times \frac{\text{ON}}{\text{OP}} + \frac{\text{RP}}{\text{PN}} \times \frac{\text{PN}}{\text{OP}}$$

$$= \sin \text{A} \cos \text{B} + \cos \text{A} \sin \text{B}\,.$$

(ii) $\cos(\text{A}+\text{B}) = \cos \angle \text{XOZ}$

$$= \frac{\text{OM}}{\text{OP}} = \frac{\text{OQ}-\text{MQ}}{\text{OP}} = \frac{\text{OQ}-\text{RN}}{\text{OP}}$$

$$= \frac{\text{OQ}}{\text{OP}} - \frac{\text{RN}}{\text{OP}} = \frac{\text{OQ}}{\text{ON}} \times \frac{\text{ON}}{\text{OP}} - \frac{\text{RN}}{\text{PN}} \times \frac{\text{PN}}{\text{OP}}$$

$$= \cos \text{A} \cos \text{B} - \sin \text{A} \sin \text{B}\,.$$

(iii) $\tan(\text{A}+\text{B}) = \tan \angle \text{XOZ}$

$$= \frac{\text{MP}}{\text{OM}} = \frac{\text{MR}+\text{RP}}{\text{OQ}-\text{MQ}} = \frac{\text{QN}+\text{RP}}{\text{OQ}-\text{RN}}$$

$$= \frac{\dfrac{\text{QN}}{\text{OQ}} + \dfrac{\text{RP}}{\text{OQ}}}{1 - \dfrac{\text{RN}}{\text{OQ}}} \text{ [Dividing num. and denom. By OQ]}$$

$$= \frac{\tan \text{A} + \dfrac{\text{RP}}{\text{OQ}}}{1 - \dfrac{\text{RN}}{\text{RP}} \cdot \dfrac{\text{RP}}{\text{OQ}}} = \frac{\tan \text{A} + \dfrac{\text{RP}}{\text{OQ}}}{1 - \tan \text{A} \cdot \dfrac{\text{RP}}{\text{OQ}}} \quad \text{. . . (4.3)}$$

$$\left[\because \frac{\text{QN}}{\text{OQ}} = \tan \text{A} \text{ and } \frac{\text{RN}}{\text{RP}} = \tan \text{A}\right]$$

But Δ*s* PRN and ONQ are similar.

$\because\ \angle \text{A} = \angle \text{A}$, *rt* $\angle$ = *rt* $\angle$

$$\frac{\text{RP}}{\text{OQ}} = \frac{\text{NP}}{\text{ON}} = \tan \text{B}$$

Putting in Eq. (4.3), we have

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Cor. 1

$$\tan\left(\frac{\pi}{4} + A\right) = \tan(45^{\circ} + A)$$

$$= \frac{\tan 45^{\circ} + \tan A}{1 - \tan 45^{\circ} \tan A} = \frac{1 + \tan A}{1 - \tan A}$$

Cor. 2

$$\sin(A - B) = \sin[A + (-B)]$$

$$= \sin A \cos(-B) + \cos A \sin(-B)$$

$$= \sin A \cos B - \cos A \sin B$$

Similarly it can be proved that

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

and $$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Cor. 3

$$\tan\left(\frac{\pi}{4} - A\right) = \tan(45^{\circ} - A)$$

$$= \frac{\tan 45^{\circ} - \tan A}{1 + \tan 45^{\circ} \tan A} = \frac{1 - \tan A}{1 + \tan A}$$

Cor. 4

Similarly it can be proved that

$$\cot(A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

and $$\cot(A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

[**Hint :** Take $\cot(A + B) = \dfrac{\cos(A + B)}{\sin(A + B)}$].

Theorem 4

Prove that

(i) $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

(ii) $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$

(iii) $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

(iv) $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

Proof

(i) $\sin(A + B) + \sin(A - B) = (\sin A \cos B + \cos A \sin B)$

$$+ (\sin A \cos B - \cos A \sin B)$$

$$= 2 \sin A \cos B$$

The others can be proved on the same lines.

Cor.

$$\sin A + \sin B = \sin\left(\frac{A+B}{2} + \frac{A-B}{2}\right) + \sin\left(\frac{A+B}{2} - \frac{A-B}{2}\right)$$

$$= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

Similarly,

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}.$$

4.3.4 Trigonometric Ratios of Multiple Angles

Theorem 5

Prove that

(i) $\sin 2A = 2 \sin A \cos A = \dfrac{2 \tan A}{1 + \tan^2 A}$

(ii) $\cos 2A = \cos^2 A - \sin^2 A$

$$= 1 - 2 \sin^2 A = 2 \cos^2 A - 1$$

$$= \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

(iii) $\tan 2A = \dfrac{2 \tan A}{1 + \tan^2 A}$

Proof

We have

(i) $\sin 2A = \sin (A + A) = \sin A \cos A + \cos A \sin A$

($\because$ $\sin (A + B) = \sin A \cos B + \cos A \sin B$)

$$= 2 \sin A \cos A = \frac{2 \sin A \cos A}{1}$$

$$= \frac{2 \sin A \cos A}{\cos^2 A + \sin^2 A} = \frac{2 \tan A}{1 + \tan^2 A}$$

(Dividing num. and denom. by $\cos^2 A$)

(ii) $\cos 2A = \cos (A + A) = \cos A \cos A - \sin A \sin A$

($\because$ $\cos (A + B) = \cos A \cos B - \sin A \sin B$) $= \cos^2 A - \sin^2 A$

$$= \begin{cases} 1 - \sin^2 A - \sin^2 A = 1 - 2\sin^2 A \\ \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1 \end{cases}$$

$(\because\ \cos^2 A = 1 - \sin^2 A \text{ and } \sin^2 A = 1 - \cos^2 A)$

Also, $\cos 2A = \cos^2 A - \sin^2 A$

$$= \frac{\cos^2 A - \sin^2 A}{1}$$

$$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

(Dividing num. and denom. by $\cos^2 A$)

(iii) $\tan 2A = \tan (A + A)$

$$= \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2\tan A}{1 - \tan^2 A}$$

Cor. 1

Replacing A by $\frac{A}{2}$ and hence 2A, by A, we get

(i) $$\mathbf{\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = \frac{2\tan\frac{A}{2}}{1 + \tan^2\frac{A}{2}}}$$

(ii) $$\mathbf{\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}}$$

$$\mathbf{= \frac{1 - \tan^2\frac{A}{2}}{1 + \tan^2\frac{A}{2}} = \begin{Bmatrix} 2\cos^2\frac{A}{2} - 1 \\ 1 - 2\sin^2\frac{A}{2} \end{Bmatrix}}$$

(iii) $$\mathbf{\tan A = \frac{2\tan\frac{A}{2}}{1 + \tan^2\frac{A}{2}}}$$

Cor. 2

$$\cos A = 2\cos^2\frac{A}{2} - 1$$

$$\Rightarrow\ 2\cos^2\frac{A}{2} = 1 + \cos A$$

$$\Rightarrow\ \cos^2\frac{A}{2} = \frac{1 + \cos A}{2}$$

$$\Rightarrow\ \mathbf{\cos\frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}}}$$

and

$$\cos A = 1 - \sin^2\frac{A}{2}$$

$$\Rightarrow \quad 2\sin^2\frac{A}{2} = 1 - \cos A$$

$$\Rightarrow \quad \sin^2\frac{A}{2} = \frac{1-\cos A}{2}$$

$$\Rightarrow \quad \mathbf{\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}}$$

Consequently,

$$\tan^2\frac{A}{2} = \frac{\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}} = \frac{\frac{1-\cos A}{2}}{\frac{1+\cos A}{2}}$$

$$\Rightarrow \quad \tan^2\frac{A}{2} = \frac{1-\cos A}{1+\cos A}$$

$$\Rightarrow \quad \mathbf{\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}}$$

Theorem 6

Prove that

(i) $\sin 3A = 3\sin A - 4\sin^3 A$.

(ii) $\cos 3A = 4\cos^3 A - 3\cos A$.

(iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.

Proof

We have

(i) $\sin 3A = \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A$

$= 2\sin A\cos A\cos A + (1 - 2\sin^2 A)\sin A$

($\because$ $\sin 2A = 2\sin A\cos A$, $\cos 2A = 1 - 2\sin^2 A$)

$= 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A$

$\mathbf{= 3\sin A - 4\sin^3 A}$

(ii) $\cos 3A = \cos(2A + A) = \cos 2A\cos A - \sin 2A\sin A$

($\because$ $\cos(A + B) = \cos A\cos B - \sin A\sin B$)

$= \cos A(2\cos^2 A - 1) - \sin A(2\sin A\cos A)$

$= \cos A(2\cos^2 A - 1) - 2\sin^2 A\cos A$

$= \cos A(2\cos^2 A - 1) - 2(1 - \cos^2 A)\cos A$

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

$= 4\cos^3 A - 3\cos A$.

(iii) $\tan 3A = \tan (A + 2A)$

$$= \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$$

$$= \frac{\tan A + \dfrac{2 \tan A}{1 - \tan^2 A}}{1 - \tan A \cdot \dfrac{2 \tan A}{1 - \tan^2 A}}$$

$$= \frac{\tan A (1 - \tan^2 A) + 2 \tan A}{1 - \tan^2 A - 2 \tan^2 A}$$

$$= \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$$

Example 4.4

Prove that $\sin 75^{\circ} - \sin 15^{\circ} = \cos 105^{\circ} + \cos 15^{\circ}$

Solution

$$\cos 15^{\circ} = \cos (90^{\circ} - 75^{\circ}) = \sin 75^{\circ}$$

and

$$\cos 105^{\circ} = \cos (90^{\circ} + 15^{\circ}) = -\sin 15^{\circ}$$

$\therefore$ L. H. S. = R. H. S.

Example 4.5

Prove that $= \dfrac{\sin (x - y)}{\sin (x + y)} = \dfrac{\tan x - \tan y}{\tan x + \tan y}$

Solution

L. H. S. $= \dfrac{\sin (x - y)}{\sin (x + y)} = \dfrac{\sin x \cos y - \cos x \sin y}{\sin x \cos y + \cos x \sin y}$

$= \dfrac{\tan x - \tan y}{\tan x + \tan y}$ [Dividing by $\cos x \cos y$]

= R. H. S.

Example 4.6

Find the value of $\tan 22^{\circ} . 30'$

Solution

Let $\theta = 45^{\circ}$, then $22^{\circ}\ 30' = \dfrac{\theta}{2}$

$$\therefore \tan \frac{\theta}{2} = \frac{\sin \dfrac{\theta}{2}}{\cos \dfrac{\theta}{2}} = \frac{2 \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2}}{2 \cos^2 \dfrac{\theta}{2}} = \frac{\sin \theta}{1 + \cos \theta}$$

$$= \frac{\frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{\sqrt{2}-1} \cdot \frac{1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{1}$$

$$= \sqrt{2} - 1$$

SAQ 2

(a) If $\sin \alpha = \frac{15}{17}, \cos \beta = \frac{12}{13}$, find the value of $\tan (\alpha + \beta)$.

(b) Show that $\sin 105^{\circ} + \cos 105^{\circ} = \cos 45^{\circ}$.

(c) Prove the following

(i) $\frac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A$

(ii) $\tan 4\theta = \frac{4 \tan \theta (1 - \tan^2 \theta)}{1 - 6 \tan^2 \theta + \tan^4 \theta}$

(iii) $(\sin 3A + \sin A) \sin A + (\cos 3A - \cos A) \cos A = 0$

(iv) $\cos^2 A + \cos^2 B - 2 \cos A \cos B \cos (A + B) = \sin^2 (A + B)$

(v) $\tan (60^{\circ} + A) \tan (60^{\circ} - A) = \frac{2 \cos 2A + 1}{2 \cos 2A - 1}$

(vi) $(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 = 4 \cos^2 \frac{\alpha - \beta}{2}$

(vii) $\cos 6^{\circ} \cos 42^{\circ} \cos 66^{\circ} \cos 78^{\circ} = \frac{1}{16}$

(**Hint :** combine $\cos 6^{\circ} \cos 66^{\circ}$ and $\cos 42^{\circ} \cos 78^{\circ}$)

(viii) $\tan A + \tan (60^{\circ} + A) + \tan (120^{\circ} + A) = 3 \tan 3A$

(ix) $\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A$

(x) $\cos^2 A + \cos^2 (A + 120^{\circ}) + \cos^2 (A - 120^{\circ}) = \frac{3}{2}$

4.3.5 Graphs of Trigonometric Functions

Definition 4 : A function *f* is said to be periodic if their exists a real number *T* > 0 such that $f(x + T) = f(x)$ for all *x*.

Since $\sin (\theta + 2\pi) = \sin \theta$

$\cos (\theta + 2\pi) = \cos \theta$

Thus sine and cosine are periodic functions.

If a function f is periodic then the smallest $T > 0$ if it exists such that $f(x + T) = f(x)$ for all x is called the period of the function. tan x is a periodic function where period is π.

The graph of any periodic function with period *T* need to be sketched only in an interval of length *T* as once it is drawn in one such interval, it can easily be drawn by repeating it over other intervals of length *T*.

The graphs of all the *T*. functions are given in Figure 4.16.

Figure 4.16

4.4 INVERSE TRIGONOMETRIC FUNCTIONS

Definition 5 : The Inverse of a Function

Let $f : X \to Y$ be a function. *f* is said to be one-one if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ and *f* is said to be onto if for each $y \in Y,$ we can find a $x \in X$ such that $f(x) = y$. If $f : X \to Y$ is $1 - 1$ and onto, we can define a unique function $g : Y \to X$ such that $g(y) = x$ where $f(x) = y$. Thus the domain of *g* is the range of *f* and the range of *g* = domain of *f*. The function

g is called the inverse of f and is denoted by f^{-1}. Let us denote domain of f by D_f and range of f by R_f.

Definition 6 : Arc Sine Function

Consider the sine function f denoted by

$$f(x) = \sin x,\ D_f = \boldsymbol{R},\ R_f = [-1, 1].$$

If we restrict the domain from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ both inclusive, we observe that in this part of the domain, f is strictly increasing and is one-one. Therefore, the function $y = f(x) = \sin x$, with $D_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $R_f = [-1, 1]$ has an inverse function called the **arc sine function** or the **inverse sine function**, denoted by $\sin^{-1}$ and $y = \sin^{-1} x$ iff $x = \sin y$ and $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Figure 4.17

It has the following properties :

(i) Domain of $\sin^{-1} x$ is $[-1, 1]$ and its range is $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\sin(\sin^{-1} x) = x$ for $x \in [-1, 1]$, i.e. $|x| \le 1$

(iii) $\sin^{-1}(\sin y) = y$ for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, i.e. $|y| \le \frac{\pi}{2}$

(iv) $\sin^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is one-one.

The graph of $\sin^{-1} x$ is shown in Figure 4.17.

Remark

Besides $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, there exist other intervals where the sine function is one-one and, therefore, has an inverse function but for us $\sin^{-1} x$ shall always mean the function : $\sin^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ defined above (unless stated otherwise). The portion of the curve for which

$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ is known as the **principal value branch** of the function $y = \sin^{-1} x$ and these values of y are known as the **principal values** of the function $y = \sin^{-1} x$.

Definition 7 : Arc Cosine Function

Consider the cosine function f defined by

$$f(x) = \cos x,\ D_f = \boldsymbol{R} \text{ and } R_f = [-1, 1].$$

Obviously, f is not one-one but if we restrict the domain to $[0, \pi]$, f is one-one and so it has an inverse function called **arc cosine** or **inverse cosine**, denoted by $\cos^{-1}$.

and $\quad y = \cos^{-1} x$ *iff* $x = \cos y$ and $y \in [0, \pi]$.

It has the following properties :

(i) Domain of $\cos^{-1} x$ is $[-1, 1]$ and its range is $[0, \pi]$.

(ii) $\cos(\cos^{-1} x) = x$ for $x \in [-1, 1]$, i.e. $|x| \le 1$

(iii) $\cos^{-1}(\cos y) = y$ for all $y \in [0, \pi]$

(iv) $\cos^{-1} : [-1, 1] \to [0, \pi]$

is strictly decreasing and one-one. The graph of $\cos^{-1} x$ is shown in Figure 4.18.

Figure 4.18

The values of y satisfying $0 \le y \le \pi$ are known as the **principal values** of the function $y = \cos^{-1} x$.

Definition 8 : Arc Tangent Function

Consider the tangent function of defined by $f(x) = \tan x$, $D_f = \boldsymbol{R^*}$ and $R_f = \boldsymbol{R}$.

Obviously, f is not one-one but if we restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, f is one-one and so f has an inverse function called **arc tangent** or **inverse tangent**, denoted by $\tan^{-1}$

and $\quad y = \tan^{-1} x$ *iff* $x = \tan y, x \in R$ and $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

It has the following properties :

(i) Domain of $\tan^{-1} x$ is $\boldsymbol{R}$ and its range is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

(ii) $\tan(\tan^{-1} x) = x$ for all $x \in \boldsymbol{R}$.

(iii) $\tan^{-1}(\tan y) = y$ for all $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

(iv) $\tan^{-1} : \boldsymbol{R} \to \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ is strictly increasing and is one-one.

A portion of the graph of $\tan^{-1} x$ is shown in Figure 4.19. The values of *y* satisfying $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$ are known as the principal values of the function $y = \tan^{-1} x$.

Figure 4.19

Definition 9 : Arc Cotangent Function

Consider the cotangent function *f* defined by

$$f(x) = \cot x,\ D_f = \boldsymbol{R}^{**} \text{ and } R_f = \boldsymbol{R}.$$

Obviously, *f* is not one-one but if we restrict the domain to $(0, \pi)$, *f* is one-one and so it has an inverse function called **arc cotangent**, or **inverse cotangent**, denoted by $\cot^{-1}$.

and $\quad y = \cot^{-1} x$ *iff* $x = \cot y$, and $y \in [0, \pi]$.

It has the following properties :

(i) Domain of $\cot^{-1} x$ is $\boldsymbol{R}$ and its range is $[0, \pi]$.

(ii) $\cot(\cot^{-1} x) = x$ for all $x \in \boldsymbol{R}$

(iii) $\cot^{-1}(\cot y) = y$ for all $y \in [0, \pi]$

(iv) $\cot^{-1} : \boldsymbol{R} \to [0, \pi]$ is strictly decreasing and is one-one.

A portion of the graph of $\cot^{-1} x$ is shown in Figure 4.20. The values of y satisfying $0 < y < \pi$ are known as the **principal values** of the function $y = \cot^{-1} x$.

Figure 4.20

Definition 10 : Arc Secant Function

Consider the secant function *f* defined by

$$f(x) = \sec x,\ D_f = \boldsymbol{R^*} \text{ and range } = (-\infty, -1] \cup [1, \infty).$$

Obviously, *f* is not one-one but if we restrict the domain to $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, *f* is one-one and so it has an inverse function called **arc secant**, or **inverse secant**, denoted by $\sec^{-1}$.

and $\quad y = \sec^{-1} x$ *iff* $x = \sec y$, and $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

It has the following properties :

(i) Domain of $\sec^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$ and its range is $[0, \pi]$ except $\frac{\pi}{2}$, i.e. $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

(ii) $\sec(\sec^{-1} x) = x$ for $|x| \geq 1$

(iii) $\sec^{-1}(\sec y) = y$ for $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, i.e. $y \in [0, \pi],\ y \neq \frac{\pi}{2}$.

(iv) $\sec^{-1} x$ is strictly increasing (piece-wise) and is one-one.

The values of y in $[0, \pi]$ except $\frac{\pi}{2}$ are known as the **principal values** of the function $y = \sec^{-1} x$.

Definition 11 : Arc Cosecant Function

Consider the cosecant function *f* defined by

$$f(x) = \operatorname{cosec} x,\ D_f = \boldsymbol{R^{**}} \text{ and range } = (-\infty, -1] \cup [1, \infty).$$

Obviously, *f* is not one-one but if we restrict the domain to, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ except 0, i.e. $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, *f* is one-one and so it has an inverse function called **arc cosecant**, or **inverse cosecant**, denoted by $\operatorname{cosec}^{-1}$.

and $\quad y = \operatorname{cosec}^{-1} x$ *iff* $x = \operatorname{cosec} y$ and $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.

It has the following properties :

(i) Domain of $\operatorname{cosec}^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$ and its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ except 0, i.e. $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.

(ii) $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$ for $|x| \geq 1$

(iii) $\operatorname{cosec}^{-1}(\operatorname{cosec} y) = y$ for $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.

(iv) $\operatorname{cosec}^{-1} x$ is strictly decreasing (piecewise) and is one-one.

The values of y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ except 0 are known as the **principal values** of the function $y = \operatorname{cosec}^{-1} x$.

Theorem 7

Prove that

(i) $\sin^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1} x, |x| \leq 1$

(ii) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1} x, x \geq 0$

Proof

(i) Let $\tan^{-1} x = \theta \Rightarrow x = \tan\theta$

Since $|x| \leq 1$, therefore, $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$

$$\Rightarrow -\frac{2\pi}{4} \leq 2\theta \leq \frac{2\pi}{4} \Rightarrow 2\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Also, $\frac{2x}{1+x^2} = \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$

$$\Rightarrow \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\theta = 2\tan^{-1} x, |x| \leq 1$$

(ii) Let $\tan^{-1} x = \theta \Rightarrow x = \tan\theta$

Since $x \geq 0$, therefore, $0 \leq \theta < \frac{\pi}{2}$

$$\Rightarrow 2.0 \leq 2\theta, < \frac{2\pi}{2} \Rightarrow 0 \leq 2\theta \leq \pi$$

Also, $\frac{1-x^2}{1+x^2} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$

$$\Rightarrow \quad \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\theta = 2\tan^{-1} x,\ x \geq 0$$

Example 4.7

Find the principal values of

(i) $\operatorname{cosec}^{-1}(-1)$

(ii) $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

Solution

(i) Let $\operatorname{cosec}^{-1}(-1) = y$, then y must satisfy $-\frac{\pi}{2} \leq y < 0$ and $\operatorname{cosec} y = -1$. This is true only for $y = -\frac{\pi}{2}$.

$\therefore$ The principal value of $\operatorname{cosec}^{-1}(-1) = -\frac{\pi}{2}$.

(ii) $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right) = y$ (say). Then $\cot y = \frac{-1}{\sqrt{3}}$ or $\tan y = -\sqrt{3}$. Since $\tan\frac{\pi}{3} = \sqrt{3}$, the principal value of $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ is $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

Example 4.8

Prove that $\cos(\sin^{-1} x) = \sin(\cos^{-1} x) = \sqrt{1-x^2}$, $|x| \leq 1$

Solution

Let $\quad \sin^{-1} x = \theta$ so that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \Rightarrow x = \sin\theta$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2} \Rightarrow \cos(\sin^{-1} x) = \sqrt{1-x^2}$$

Let $\quad \cos^{-1} x = t$ so that $0 \leq t \leq \pi \Rightarrow x = \cos t$

$$\sin t = \sqrt{1-\cos^2 t} = \sqrt{1-x^2}$$

$$\therefore \quad \sin(\cos^{-1} x) = \sqrt{1-x^2}$$

Example 4.9

Show that

$$\tan^{-1}\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}} = \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$$

Solution

Let $\quad \theta = \frac{1}{2}\cos^{-1} x^2,\ x \neq 0$

Then $\quad x^2 = \cos 2\theta \qquad 0 \leq 2\theta \leq \frac{\pi}{2}$

Hence L. H. S. $= \tan^{-1} \dfrac{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}$

$$= \tan^{-1}\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$= \frac{\pi}{4} + \theta = \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2 = \text{R. H. S.}$$

Example 4.10

Write $\tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$ in the simplest term.

Solution

Let $\quad \tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right) = \theta$

Then $\quad \dfrac{\cos x}{1+\sin x} = \tan\theta$

$$\frac{\cos x}{1+\sin x} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \frac{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}$$

$$= \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}} = \frac{\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)}{\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)} = \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$$

$$\therefore \quad \tan^{-1}\frac{\cos x}{1+\sin x} = \theta = \frac{\pi}{4} - \frac{x}{2}$$

if $\quad \dfrac{-\pi}{2} < \dfrac{\pi}{4} - \dfrac{x}{2} < \dfrac{\pi}{2}$

i.e. if $\quad \dfrac{-\pi}{2} < x < \dfrac{3\pi}{2}$.

SAQ 3

(a) Find the principal values of

(i) $\sin^{-1}(-1)$

(ii) $\cos^{-1}\left(\frac{-1}{2}\right)$

(iii) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

(iv) $\cot^{-1}(\sqrt{3})$

(v) $\operatorname{cosec}^{-1}(-2)$

(vi) $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

(b) Prove the following

(i) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

(ii) $\sin\left(2\cos^{-1}\left(\frac{-3}{5}\right)\right) = -\frac{24}{25}$

(c) Prove the following

(i) $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{1}{2}\tan^{-1} x$

(ii) $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right) = \sin^{-1}\frac{x}{a}, |x| < a$

(iii) $\cot^{-1}(13) + \cot^{-1}(21) + \cot^{-1}(-8) = \pi$

(iv) $3\tan^{-1} x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

(v) $2\tan^{-1}\frac{1}{x} = \sin^{-1}\frac{2x}{x^2+1}, \quad |x| \geq 1$

(vi) $\tan^{-1}\left(\frac{x+\sqrt{x}}{1 - x\sqrt{x}}\right) = \tan^{-1} x + \tan^{-1}\sqrt{x}$

SAQ 4

(a) Write the following functions in the simplest term

(i) $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$

(ii) $\sec^{-1}\left(\frac{1}{2x^2 - 1}\right)$

(iii) $\sin^{-1} \sqrt{\dfrac{x}{1+x}}$

(iv) $\sin^{-1} (x \sqrt{1 - y^2} + y \sqrt{1 - y^2})$

(v) $\cos^{-1} \sqrt{1 - x^2}$

(b) Find x if $\sin^{-1} x + \sin^{-1} 2x = \dfrac{\pi}{3}$.

(c) Prove that $4 \tan^{-1} \dfrac{1}{5} - \tan^{-1} \dfrac{1}{70} + \tan^{-1} \dfrac{1}{99} = \dfrac{\pi}{4}$.

4.5 SUMMARY

(i) Measurement of an Angle.

- English System : 1 right angle = 90^o, 1^o = 60 minutes = $60'$, and $1' = 60 \text{ seconds} = 60''$
- Circular System : 2 right angles = 180^o.

(ii) Trigonometrical Ratios (circular functions)

$$\sin^2 x + \cos^2 x = 1, \quad 1 + \tan^2 x = \sec^2 x, 1 + \cot^2 x = \operatorname{cosec}^2 x$$

$$\tan x = \frac{\sin x}{\cos x}, \sec x = \frac{1}{\cos x}, \operatorname{cosec} x = \frac{1}{\sin x}$$

(iii) T Ratios of some standard angles.

t-ratio of the Angle	sin A	cos A	tan A	cot A	sec A	cosec A
0^o	0	1	0	…	1	…
30^o	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
45^o	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60^o	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$
90^o	1	0	…	0	---	1
180^o	0	-1	0	…	-1	…

(iv) Formulae for *t*-ratios of Allied Angles :

$$\sin (-A) = -\sin A = \sin (360 - A)$$

$\cos(-A) = \cos A = \cos(360 - A)$

$\sin(90 - A) = \cos A, \quad \sin(90 + A) = \cos A$

$\cos(90 - A) = \sin A, \quad \cos(90 + A) = -\sin A$

$\sin(180 - A) = \sin A, \quad \sin(180 + A) = -\sin A$

$\cos(180 - A) = -\cos A, \quad \cos(180 + A) = -\cos A$

(v) Standard Formulae involving *t*-ratios

$(\sin A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin C + \sin D = 2 \sin \frac{C + D}{2} \cos \frac{C - D}{2}$$

$$\sin C - \sin D = 2 \cos \frac{C + D}{2} \sin \frac{C - D}{2}$$

$$\cos C + \cos D = 2 \cos \frac{C + D}{2} \cos \frac{C - D}{2}$$

$$\cos C - \cos D = -2 \sin \frac{C + D}{2} \sin \frac{C - D}{2}$$

(vi) Formulae involving *t*-ratios of multiple and submultiple angles.

$$\sin 2A = 2 \sin A \cos A = \frac{2 \tan A}{1 + \tan^2 A}$$

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\sin 2A = \frac{1 - \cos 2A}{2}, \ \cos^2 A = \frac{1 + \cos 2A}{2}$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}, \ \tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

4.6 ANSWERS TO SAQs

SAQ 1

(a) (i) $\frac{\pi}{12}$,

(ii) $\frac{-\pi}{8}$

(b) (i) 420^o,

(ii) $14^o\ 19'$ (nearly)

(c) (i) $\sin\theta = \frac{\sqrt{3}}{2}, \tan\theta = -\sqrt{3}, \operatorname{cosec}\theta = \frac{2}{\sqrt{3}}, \sec\theta = -2, \cot\theta = \frac{-1}{\sqrt{3}}$

(ii) $\sin\theta = \frac{-3}{5}$, $\cos\theta = \frac{-4}{5}$, $\operatorname{cosec}\theta = \frac{-5}{3}$, $\sec\theta = \frac{-5}{4}$, $\cot\theta = \frac{4}{3}$

(iii) $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\operatorname{cosec}\theta = \frac{5}{3}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = \frac{4}{3}$

SAQ 2

(a) $\frac{220}{21}$

SAQ 3

(a) (i) $\frac{-\pi}{2}$, (ii) $\frac{2\pi}{3}$,

(iii) $\frac{\pi}{6}$, (iv) $\frac{\pi}{6}$,

(v) $\frac{-\pi}{6}$ (vi) $\frac{-\pi}{6}$

SAQ 4

(a) (i) $\frac{\pi}{4} - x$, (ii) $2\cos^{-1} x$,

(iii) $\tan^{-1}\sqrt{x}$, (iv) $\sin^{-1} x + \sin^{-1} y$,

(v) $\sin^{-1} x$

(b) $x = \sqrt{\frac{3}{28}}$