UNIT 6 COORDINATE GEOMETRY AND STRAIGHT LINE

Structure

6.1 INTRODUCTION

The geometry you have been studying thus so far, is called *Euclidean Geometry*. Its approach was to start with certain *concepts*, like the concepts of points, lines and planes; attribute certain properties to them which we called *axioms* or *postulates* (suggested by physical experience); and then using the methods of deductive logic to derive a number of *theorems* which formed the main fruit of our mathematical activity and revealed to us the interesting and useful properties of the geometric figures under consideration.

This was the only approach to geometry for some two thousand years till the French philosopher and mathematician Rene Descartes (1596-1665) published *La Geometrie* in 1637 wherein he introduced the analytic approach by systematically using algebra in his study of geometry. This was achieved by representing points in the plane by ordered pairs of real numbers (called *Cartesian Coordinates* named after Rene Descartes), and representing lines and curves by algebraic equations. This wedding of algebra and geometry is known as *analytic* or *coordinate geometry*, and this is what we propose to study here.

Objectives

After studying this unit, you should be able to

- represent points by ordereal points of real numbers and define the co-ordinates of a point,
- find distance between two points,

- derive the co-ordinates of a point dividing the line segment between two given points,
- derive the equation of a line in different forms,
- find the angle between two given lines,
- the point of intersection of two given lines,
- have an idea when three lines are concurrent,
- find the distance of a point from a line, and
- derive the equation of bisections of angles between two given lines.

6.2 CARTESIAN COORDINATE SYSTEM

In this section, we shall establish a 1-1 correspondence between points on a straight line and the real numbers, and subsequently a 1-1 correspondence between points in the Euclidean plane and ordered pairs of real numbers. This would make it possible to apply the methods of algebra to study problems in geometry.

This can be done by defining what is called a *Cartesian Coordinate System* on the Euclidean plane, which we do as under :

In the Euclidean plane draw a horizontal line $X'OX$, a vertical line $Y'OY$ intersecting at *O*, the *origin*. We select a convenient unit of length and starting from the origin as zero, mark off a number scale on the horizontal line, positive to the right and negative to the left. We mark off the *same scale* on the vertical line, positive upwards and negative downwards of the origin *O*.

The horizontal line thus marked is called the *x-axis* and the vertical line the *y-axis*, and collectively they are called the *coordinate axes*.

Let *P* be any point in the plane. Draw perpendiculars from *P* to the coordinate axes, meeting the *x-axis* in *M* and the *y-axis* in *N* (Figure 6.1). Let *x* be the length of the *directed line segment OM* in the units of the scale chosen. This is called the *x-coordinate* or *abscissa* of *P*. Similarly, the length of the directed line segment *ON* in the same scale is called the *y-coordinate* or *ordinate* of *P*. The position of the point *P* in the plane with respect to the coordinate axis is represented by the *ordered pair* (x, y) of real numbers. The pair (x, y) is called the *coordinates* of *P*, and this system of coordinating an ordered pair (x, y) with every point of the plane is called the (Rectangular) *Cartesian Coordinate System*.

Figure 6.1

We thus see that to every point P in the Euclidean plane there corresponds a unique ordered pair (x, y) of real numbers called its Cartesian Coordinates. Conversely, given an ordered pair (x, y) and a Cartesian coordinate system, we mark off a directed line segment $OM = x$ on the *x-axis* and another directed line segment $ON = y$ on *y-axis*, draw perpendiculars at M and N to x and *y-axes* respectively, and their point of intersection shall uniquely locate the corresponding point P in the Euclidean plane. This establishes a 1-1 correspondence between the set of all ordered pairs (x, y) of real numbers and the points in the Euclidean plane. The set of all ordered pairs (x, y) of real numbers is called Cartesian plane. Finally, we observe (Figure 3.2) that the two axes divide the plane into four regions called the *quadrants*. The ray OX is taken as positive x-axis, OX' as negative x-axis, OY as positive y-axis and OY' as negative y-axis. The quadrants are thus characterised by the following signs of abscissa and ordinate.

I	quadrant	$x > 0, y > 0$	or (+, +)
II	quadrant	$x < 0, y > 0$	or (–, +)
III	quadrant	$x < 0, y < 0$	or (–, –)
IV	quadrant	$x > 0, y < 0$	or (+, –)

Further if the abscissa of a point is zero, it would lie on the Y-axis and if its ordinate is zero it would lie on X-axis.

Theorem 1

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Proof

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two points in the plane, and let d be the distance between them (Figure 6.2). Draw lines parallel to y-axis from the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ which will meet the x-axis in points $A(x_1, 0)$ and $B(x_2, 0)$ respectively. Now draw a line through $P(x_1, y_1)$ parallel to x-axis which will meet the vertical through Q in $R(x_2, y_1)$. The length of the segment between P and R, which we shall denote by $|PR|$, is equal to $|AB|$.

Figure 6.2

As in the figure, $|AB| = |OB| - |OA| = (x_2 - x_1)$. If, however, x_2 were to the left of x_1 (i.e. $(x_2 < x_1)$, this length were $(x_1 - x_2)$. In other case, since the length has got to be positive, we take the absolute value of $(x_2 - x_1)$, viz.

$|x_2 - x_1|$ as the length $|AB|$. Hence, $|PR| = |AB| = |x_2 - x_1|$. In passing we observe that when the ordinates of two points (in this case P and R) are the same, the distance between them is the absolute value of the difference between their abscissa (in this case $|x_2 - x_1|$).

Repeating the same argument for the points Q and R, by drawing lines parallel to x-axis and meeting the y-axis, we shall find that the length $|RQ| = |y_2 - y_1|$.

Now applying Pythagoras theorem, we get

$$|PQ|^2 = |PR|^2 + |RQ|^2 \text{ or } d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

which can also be written as

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Since the distance is always positive, taking the positive square root, we get the *distance formula*

$$d = |PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This proves the theorem in case the line PQ is parallel to neither X-axis nor Y-axis.

If PQ is parallel to x-axis, then obviously $y_1 = y_2$ and $PQ = |x_2 - x_1|$.

Also, $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_2 - x_1)^2} = |x_2 - x_1|$. Hence, the theorem is proved if PQ is parallel to x-axis. A similar proof can be given when PQ is parallel to Y-axis.

Corollary 1

The distance of any point P (x, y) from the origin is $\sqrt{(x^2 + y^2)}$.

In the above formula, take the point P as (x, y) and Q as (0, 0), i.e. the origin, to get the result.

Corollary 2 : Area of a Triangle

We now proceed to find the area of a triangle ABC, the coordinates of whose vertices are given to be $A\ (x_1, y_1)$, $B\ (x_2, y_2)$ and $C\ (x_3, y_3)$. Draw perpendiculars from A, B and C to x-axis meeting it in L, M and N respectively. As we have seen earlier, $|LM|$ or simply $ML = |x_1 - x_2| = x_1 - x_2$.

Figure 6.3

Similarly, $LN = (x_3 - x_1)$ and $MN = x_3 - x_2$. Now, the area of Δ ABC = area of trapezium BMLA + area of trapezium ALNC – area of trapezium BMNC,

$$= \frac{1}{2}\,(MB + AL)\,.\,MN + \frac{1}{2}\,(AL + CN)\,.\,LN - \frac{1}{2}\,(BM + CN)\,.\,MN$$

$$= \frac{1}{2}\,(y_2 + y_1)\,(x_1 - x_2) + \frac{1}{2}\,(y_1 + y_3)\,(x_3 - x_1) - \frac{1}{2}\,(y_2 + y_3)\,(x_3 - x_2)$$

$$= \frac{1}{2}\,[x_1\,(y_2 - y_3) + x_2\,(y_3 - y_1) + x_3\,(y_1 - y_2)]\,.$$

Note

(i) Sometimes the above expression for area may turn out to be negative. But we take the absolute value of the expression as the area.

(ii) The above proof uses the fact that all vertices of the triangle are on one side of the *y*-axis, it can be shown that even if some vertex or vertices are on the other side of the *y*-axis the same expression will give us the area of the triangle.

(iii) We have given the above proof by drawing perpendiculars from vertices on the *x*-axis. A proof can also be given by drawing perpendiculars on the *y*-axis.

Section Formula

Theorem 2

Given two points $A\,(x_1, y_1)$, and $B\,(x_2, y_2)$. The coordinates of the point P on AB which divides the line segment AB in the ratio $l : m$ (internally) are given by

$$x = \frac{mx_1 + lx_2}{l + m},\; y = \frac{my_1 + ly_2}{l + m}$$

Proof

Draw lines parallel to *y*-axis from *A*, *B* and *P* meeting *x*-axis in *C*, *D* and *Q* respectively. Draw lines parallel to *x*-axis from *A* and *P* meeting *PQ* and *BD* in *E* and *R* respectively. This being given that $\frac{AP}{PB} = \frac{l}{m}$. It is easily seen that the two right angled triangles APE and PBR are similar, and hence,

$$\frac{AP}{PB} = \frac{AE}{PR} = \frac{PE}{BR} = \frac{l}{m}$$

Now, $AE = CQ = |OQ - OC| = |x - x_1| = x - x_1$

and $PR = QD = |OD - OQ| = |x_2 - x| = x_2 - x$

Using

$$\frac{AP}{PB} = \frac{AE}{PR} = \frac{PE}{BR} = \frac{l}{m}$$

$$\frac{l}{m} = \frac{AE}{PR} = \frac{x - x_1}{x_2 - x} \text{ or } l\,(x_2 - x) = m\,(x - x_1)$$

i.e. $$x = \frac{mx_1 + lx_2}{l + m}$$

Again, $PE = |PQ - QE| = |PQ - AC| = |y - y_1| = y - y_1$

and $BR = |BD - RD| = |BD - PQ| = |y_2 - y| = y_2 - y$

Using $$\frac{l}{m} = \frac{PE}{BR} = \frac{y - y_1}{y_2 - y},$$

we have $l\,(y_2 - y) = m\,(y - y_1)$

i.e. $$y = \frac{my_1 + ly_2}{l + m}$$

Figure 6.4

Corollary 3

External Division

If the line *AB* is divided externally by a point *P* in the ratio $l : m$, then it is easy to see that $AP = l$ and $BP = m$, for a suitably chosen unit where *P* lies on *AB* produced.

Figure 6.5

Thus the point $B\,(x_2, y_2)$ divides the line *AP* *internally* in the ratio $(l - m) : m$. Our formula for internal division therefore implies that

$$x_2 = \frac{(l - m)x + mx_1}{(l - m) + m},\; y_2 = \frac{(l - m)y + my_1}{(l - m) + m}$$

so that $lx_2 = (l - m)x + mx_1$ and $ly_2 = (l - m)y + my_1$

giving $$x = \frac{lx_2 - mx_1}{l - m},\ y = \frac{ly_2 - my_1}{l - m}$$

Note that this is the same formula as for the internal division except that m is replaced by $-m$. If P divides AB externally in the ratio $l : m$ and $l < m$, then the coordinates of P will be given by

$$x = \frac{-lx_2 + mx_1}{-l + m},\ y = \frac{-ly_2 + my_1}{-l + m}$$

Mid-Point Formula

To find the coordinates of the *mid-point* of a line segment with end points $A(x_1, y_1)$, and $B(x_2, y_2)$, we put $l = m$ in the formula of Theorem 6.2 and obtain

$$x = \frac{x_1 + x_2}{2},\ y = \frac{y_1 + y_2}{2}$$

6.3 STRAIGHT LINE

6.3.1 The Equation of a Straight Line Parallel to the Axis of Coordinates

(i) *Line Parallel to the y-axis*

Let AB be a straight line parallel to the y-axis at a distance a from it. If this line meets x-axis at A, then $OA = a$. Take any point $P(x, y)$ on the line.

Figure 6.6

Then $x = OA = a$. Hence $x = a$ is the equation of this line which is parallel to y-axis at a distance a from it.

In particular $x = 0$ is the equation of the y-axis.

(ii) Similarly it can be proved that $y = b$ is the equation of a line parallel to x-axis at a distance b from it, and $y = 0$ is the equation of x-axis.

(a) **(b)** **(c)**

(d) **(e)**

Figure 6.7

Definition 1

The *slope* of a line is the tangent of the angle which the part of the line above the x-axis makes with the positive direction of x-axis. Thus the slope of the line AB in Figure 6.7(a) is ($\tan 60^{o}$) and in Figure 6.7(b) is ($\tan 135^{o}$).

Definition 2

If a line AB meets the x-axis in A and the y-axis in B then OA and OB are called the intercepts of the line AB on x-axis and y-axis respectively.

The intercept on x-axis is positive if A is to the right of the origin as in Figures 6.7(c) and (e) and negative if A lies on the left of the origin as in Figure 6.7(d).

Also intercept on y-axis can be positive or negative accordingly as B is above the origin or below the origin.

Now, we will find equation of a line determined by a given set of conditions.

6.3.2 The Point Slope Form

Let $P(x_1, y_1)$ be a fixed point on a line and m be the slope of the line. Let $Q(x, y)$ be any other point on the line.

Then $\dfrac{y - y_1}{x - x_1}$ is the slope of the line through P and Q. But this is given to be m

$$\therefore \qquad \frac{y - y_1}{x - x_1} = m$$

or
$$y - y_1 = m(x - x_1) \qquad \ldots (6.1)$$

Conversely, if a point $R(x, y)$ in a plane satisfies the condition (6.1), then as $\dfrac{y - y_1}{x - x_1}$ is the slope of the line RP. Eq. (6.1) expresses the fact that the slope of the line RP is m, i.e. R is on a line through P with slope m.

Thus if l is a line through $P(x_1, y_1)$ with slope m, we can say that Eq. (6.1) is the equation of the line l. The equation of a line in the Eq. (6.1) is called the point-slope form.

Note that the slope m is undefined for the lines parallel to y-axis.

6.3.3 Two Point Form

Let $P_1(x_1, y_1), P_2(x_2, y_2)$ be two points and *l* be the line passing through these points.

Figure 6.8

Let $Q(x, y)$ be another point on *l*. Then $P_1\,Q$ and $P_1\,P_2$ are the same lines, hence they have the same slope, i.e.

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \qquad ...(6.2)$$

Conversely, if $Q(x, y)$ satisfies Eq. (6.2), then Eq. (6.2) shows that the slope of the line $P_1Q =$ slope of the line $P_1\,P_2$, i.e. either P_1Q and P_1P_2 are the same lines or parallel lines. But these lines have a common point P_1. Therefore, they cannot be parallel.

Hence Eq. (6.2) above is the equation of a line *l* through $P_1\ (x_1, y_1)$ and $P_2\ (x_2, y_2)$.

Example 6.1

Determine the equation of a line

(i) Passing through the point $(-1, -2)$ and with slope $\frac{4}{7}$.

(ii) Passing through the points (3, 4) and 2, − 1).

Solution

(i) The equation of a line is

$$y - (-2) = \frac{4}{7}[x - (-1)] \quad \{x_1 = -1,\ y_1 = -2,\ m = \frac{4}{7}\}$$

i.e. $\quad y + 2 = \frac{4}{7}(x + 1)$

i.e. $\quad 7y + 14 = 4x + 4$

i.e. $\quad 7y = 4x - 10$

i.e. $\quad y = \frac{4}{7}x - \frac{10}{7}$

(ii) The equation of the line is

$$\frac{y - 4}{x - 3} = \frac{-1 - 4}{2 - 3} \quad (x_1 = 3,\ y_1 = 4,\ x_2 = 2,\ y_2 = -1)$$

i.e. $$\frac{y-4}{x-3} = \frac{-5}{-1}$$

i.e. $$y - 4 = 5\,(x - 3)$$

i.e. $$y = 5x - 15 + 4 = 5x = -11$$

i.e. $y = 5x - 11$ is the equation of the line.

6.3.4 Intercept Form

To find the equation of a line which cuts off given intercepts *a* and *b* from the axis.

Let *l* be the line which intersects *x*-axis at *A* and *y*-axis at *B* such that $OA = a$, and $OB = B$.

Then $A \equiv (a, 0)$ and $B \equiv (0, b)$.

Figure 6.9

As *l* passes through these points, equation of *l* is

$$\frac{y-0}{x-a} = \frac{b-0}{0-a} \quad \text{i.e.} \quad \frac{y}{x-a} = \frac{b}{-a}$$

i.e. $$\frac{y}{b} = \frac{x-a}{-a} = -\frac{x}{a} + 1$$

i.e. $$\frac{x}{a} + \frac{y}{b} = 1 \qquad \ldots (6.3)$$

The Eq. (6.3) is called the equation of a line in the intercept form.

6.3.5 Slope-Intercept Form

Let *l* be a line with slope *m* and *y*-intercept as *b* then $OC = b$, i.e. $C = (o, b)$.

Figure 6.10

$\therefore$ Equation of the line l is $y - b = m\,(x - o)$ {As it is a line passing through c with slope m}.

i.e. $$y = m\,x + b \qquad \ldots (6.4)$$

Example 6.2

Determine the equation of a line with slope 3 and y-intercept 2.

Solution

Here $m = 3$, $b = 2$

$\therefore$ Equation of the line is $y = 3\,x + 2$.

6.3.6 Normal Form

Let l be a line, p is the length of the perpendicular drawn from the origin to the line and θ is the angle which this perpendicular makes with the x-axis.

(a) **(b)**

(c) **(d)**

Figure 6.11

Draw OP perpendicular to the line l and let θ be the angle between OP and the positive x-axis.

Then the coordinates of p are (p cos θ, p sin θ).

$\therefore$ Slope of the line l is $-\dfrac{1}{\tan\theta} = -\cot\theta$.

If (x, y) is any other point on the line, the equation of the line is

$$y - p \sin \theta = - \cot \theta \, (x - p \cos \theta)$$

i.e. $$x \cos \theta + y \sin \theta = p \qquad \ldots (6.5)$$

Example 6.3

Find the equation of a line which has a perpendicular segment of length 4 from the origin and the inclination of the perpendicular segment with positive direction of x-axis is 30^o.

Solution

Here $p = 4$, $\theta = 30^o$

$\therefore$ Equation of the line is $x \cos 30^o + y \sin 30^o = 4$

i.e. $$x \frac{\sqrt{3}}{2} + y \, . \frac{1}{2} = 4$$

i.e. $$\sqrt{3}\, x + y = 8$$

6.3.7 General Form

We have seen that all forms in which we have found the equation of a straight line are of the first degree in x and y. The converse is also true. The most general form of any equation of the first degree in x and y is $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$ where A, B, C are constants, and A and B are not zero simultaneously.

Case I

$\mathrm{A} \neq 0, \mathrm{B} = 0$ then $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$, i.e. $\mathrm{A}x + \mathrm{C} = 0$, i.e. $x = -\frac{\mathrm{C}}{\mathrm{A}}$ which is the equation of a line parallel to y-axis. Similarly, if $\mathrm{B} \neq 0$ and $\mathrm{A} = 0$, then again $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$ will represent a line parallel to x-axis.

Case II

$\mathrm{A} \neq 0, \mathrm{B} \neq 0$, then $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$ can be written as $y = -\frac{\mathrm{A}}{\mathrm{B}} x - \frac{\mathrm{C}}{\mathrm{B}}$ which again represents a straight line with slope $-\frac{\mathrm{A}}{\mathrm{B}}$ and y-intercept as $-\frac{\mathrm{C}}{\mathrm{B}}$.

Hence $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$ represents the equation of a straight line.

The general equation of the straight line is reducible to the normal form in the following way.

General equation of the straight line is

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0,$$

and Normal Form is

$$x \cos \theta + y \sin \theta - p = 0.$$

Comparing the two we have

$$\frac{\mathrm{A}}{\cos \theta} = \frac{\mathrm{B}}{\sin \theta} = \frac{\mathrm{C}}{-p}$$

i.e. $$\cos \theta = -\frac{\mathrm{A}\, p}{\mathrm{C}}, \; \sin \theta = \frac{-\mathrm{B}\, p}{\mathrm{C}}$$

$$\therefore \quad \left(-\frac{A\,p}{C}\right)^2 + \left(-\frac{B\,p}{C}\right)^2 = \cos^2\theta + \sin^2\theta$$

i.e. $\quad 1 = \frac{p^2}{C^2}(A^2 + B^2)$ i.e. $p^2 = \frac{C^2}{A^2 + B^2}$

i.e. $\quad p = \pm \frac{C}{\sqrt{A^2 + B^2}}$

As p is the perpendicular segment it has to be positive.

$\therefore$ If $C \geq 0$, $p = \frac{C}{\sqrt{A^2 + B^2}}$

$$\cos\theta = \frac{-A}{\sqrt{A^2 + B^2}}, \quad \sin\theta = \frac{-B}{\sqrt{A^2 + B^2}}$$

$\therefore$ The general form is

$$-\frac{A}{\sqrt{A^2 + B^2}}\,x - \frac{B}{\sqrt{A^2 + B^2}}\,y = \frac{C}{\sqrt{A^2 + B^2}}$$

6.3.8 Angle between Two Lines

Let l_1 and l_2 be two non-perpendicular lines neither of which is parallel to the y-axis with slopes m_1 and m_2 and θ be the angle between the two.

(a) **(b)**

Figure 6.12

$\theta = \alpha_2 - \alpha_1$ in Figure 6.12(a).

$$\therefore \quad \tan\theta = \tan(\alpha_2 - \alpha_1) = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_2 \tan\alpha_1}$$

$$= \frac{m_2 - m_1}{1 + m_1 m_2}$$

$\alpha_1 = \alpha_2 + (\pi - \theta)$ in Figure 6.12(b)

$$\therefore \quad \theta = \alpha_2 - \alpha_1 + \pi$$

Hence

$$\tan\theta = \tan(\alpha_2 - \alpha_1) = \frac{m_2 - m_1}{1 + m_2 m_1}$$

Hence the positive angle θ between two lines l_1 and l_2 with slopes m_1 and m_2 is given by

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

If the two lines are perpendicular to each other then

$$m_1 = \frac{-1}{m_2} \qquad \therefore\ 1 + m_1 m_2 = 0$$

Thus tan θ is not defined in that case.

Notice that in numerical examples, the value of tan θ may sometimes found to be negative. This would only mean that instead of getting actual angle of intersection, its supplement which too is the angle of intersection is being obtained.

When two lines are parallel, θ the angle of intersection is 0, i.e. tan θ = 0

i.e. $$\frac{m_2 - m_1}{1 + m_1 m_2} = 0 \text{ i.e. } m_2 = m_1.$$

Hence two lines are parallel iff $m_1 = m_2$, and two lines are perpendicular iff $m_1 m_2 = -1$.

6.3.9 Line Parallel to a given Line

The equation of any line parallel to the line $ax + by + c = 0$ is $ax + by + k = 0$.

If m is the slope of the line $ax + by + c = 0$ then $m = -\frac{a}{b}$.

∴ Slope of the line parallel to the line $a\,x + b\,y + c = 0$ is $-\frac{a}{b}$, i.e. its equation is $y = -\frac{a}{b}x + d$.

i.e. $by = -ax + bd$

i.e. $ax + by - bd = 0$

i.e. $ax + by + k = 0$

6.3.10 Line Perpendicular to a given Line

The equation of any line perpendicular to the line $ax + by + c = 0$ is $bx - ay + k = 0$.

m, the slope of the line $ax + by + c = 0$ is $m = -\frac{a}{b}$.

∴ m_1, the slope of the line perpendicular to the above line is

$$m_1\left(-\frac{a}{b}\right) = -1 \text{ i.e. } m_1 = \frac{b}{a}$$

∴ Equation of perpendicular line is

$$y = \frac{b}{a}x + d$$

i.e. $ay - bx = ad$ i.e. $bx - ay + k = 0$.

Example 6.4

Find the equation of a line perpendicular to the line $3x - 4y + 7 = 0$ and passes through the point (– 3, 2).

Solution

The equation of any line perpendicular to the line $3x - 4y + 7 = 0$ is

$$4x + 3y + k = 0,$$

As it passes through (– 3, 2).

$$\therefore \quad 4(-3) + 3(2) + k = 0$$

i.e.
$$k = 12 - 6$$
$$= 6$$

$\therefore$ Equation of the line is

$$4x + 3y + 6 = 0$$

6.3.11 Point of Intersection of Two Straight Lines

Let

$$a_1x + b_1y + c_1 = 0$$

and
$$a_2x + b_2y + c_2 = 0$$

be equation of two lines.

Since the point of intersection of the above two lines lies on both the lines, its coordinates satisfy both the equations. Hence the coordinates of the point of intersection can be found by solving the equation.

$$a_1x + b_1y + c_1 = 0$$

and
$$a_2x + b_2y + c_2 = 0$$

simultaneously.

Condition of Concurrency of Three Lines

Three lines are concurrent if the point of intersection of two lines lies on the third line. Thus to show that the lines are concurrent, we have to show that the coordinates of the point of intersection of two lines satisfy the third equation.

Example 6.5

Show that the lines

$$3x + 2y - 5 = 0$$

$$4x + 3y + 7 = 0$$

$$21x + 13y - 76 = 0$$

are concurrent.

Solution

Solving the first two equations simultaneously, we get

$$\frac{x}{14+15} = \frac{y}{-20-21} = \frac{1}{9-8}$$

i.e. $\frac{x}{29} = \frac{y}{-41} = 1 \;\Rightarrow\; (29, -41)$ is the point of intersection of the first two lines.

Substituting the co-ordinates of this point in the third equation, we have

$$21 \times 29 + 13 \times (-41) - 76 = 0\,.$$

$\therefore$ The three lines are concurrent.

Distance of a Point from a Line

Case I

Let the equation of a line AB be

$$x \cos \alpha + y \sin \alpha = p$$

and $P\,(x', y')$ be the point and d is the length of the perpendicular PN from P on the line AB.

Figure 6.13

Draw $A'PB'$ a line parallel to the line AB passing through P and draw the common perpendicular $\mathrm{OTT'}$ from O to AB and $\mathrm{A'B'}$.

Now $\mathrm{OT} = p$ and $\mathrm{OT'} = \mathrm{OT} + \mathrm{TT'} = p + \mathrm{NP} = p + d$.

$\therefore$ The equation of the line $\mathrm{A'B'}$ is

$$x \cos \alpha + y \sin \alpha = p + d$$

[$\because$ α is the angle which the perpendicular from O is making with the line $\mathrm{A'B'}$].

$P\,(x', y')$ lies on $\mathrm{A'B'}$.

$\therefore \quad x' \cos \alpha + y' \sin \alpha = p + d$

i.e. $\quad d = x' \cos \alpha + y' \sin \alpha - p$

Case II

Let the equation of the line be in the general form as

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$$

Reducing this to normal form, we have

$$\frac{A}{\sqrt{A^2 + B^2}}\, x + \frac{B}{\sqrt{A^2 + B^2}}\, y = \frac{-C}{\sqrt{A^2 + B^2}}$$

Now length of the perpendicular from (x', y') is

$$\frac{Ax' + Ay' + C}{\sqrt{A^2 + B^2}}$$

We will take this to be

$$\left| \frac{Ax' + Ay' + C}{\sqrt{A^2 + B^2}} \right|$$

as length of a segment is always positive.

Example 6.6

Find the distance between the line $3x - 4y + 12 = 0$ and the point (4, 1).

Solution

Perpendicular distance of (4, 1) from the line is

$$\left| \frac{3.4 - 4.1 + 12}{\sqrt{9 + 16}} \right| = \left| \frac{12 - 4 + 12}{5} \right| = \left| \frac{20}{5} \right| = \frac{20}{5} = 4 .$$

SAQ 1

(a) Find the equation of a line

(i) Passing through (4, 3) and slope 2.

(ii) Passing through (–1, – 2) and (– 5, – 2).

(iii) That has y-intercept 4 and is parallel to the line $2x - 3y = 7$.

(iv) That has x-intercept – 3 and is perpendicular to the line $3x + 5y = 4$.

(v) Passing through the point (2, 2) and sum of the intercepts on the axis is 9.

(b) Find the angle between the lines

$$3x + y - 7 = 0$$

$$x + 2y + 9 = 0$$

(c) Find the equations of the lines which pass through (4, 5) and make an angle of 45^{o} with the line $2x + y + 1 = 0$.

(d) Find the distance between the lines

$$9x + 40y - 20 = 0$$

$$9x + 40y + 103 = 0$$

(**Hint :** The lines are parallel.)

(e) Find the equations of straight lines which are perpendicular to the line $3x + 4y - 7 = 0$ and at a distance 3 units from (2, 3).

6.4 FAMILY OF LINES

Let

$$A_1x + B_1y + C_1 = 0 \qquad \text{. . . (6.6)}$$

and

$$A_2x + B_2y + C_2 = 0 \qquad \text{. . . (6.7)}$$

be two given non-parallel lines. Then for any real number k

$$(A_1x + B_1y + C_1) + k\,(A_2x + B_2y + C_2) = 0 \qquad \text{. . . (6.8)}$$

for any values of x and y for which Eqs. (6.6) and (6.7) are true represents a line through the point of intersection of lines in Eqs. (6.6) and (6.7) for every k. Hence Eq. (6.8) represents the family of lines passing through the point of intersection of the two lines.

Note : If there exists no point common to the lines in Eq. (6.6) and (6.7), then they are parallel and Eq. (6.8) gives the family of lines parallel to them.

Example 6.7

Find the equation of a line parallel to the y-axis and drawn through the point of intersection of $x - 7y + 5 = 0$ and $3x + y - 7 = 0$.

Solution

The equation of any line passing through the point of intersection of the given lines is

$$(x - 7y + 5) + k\,(3x + y - 7) = 0\,.$$

i.e. $$(1 + 3\,k)\,x + (k - 7)\,y + (5 - 7\,k) = 0$$

If the line is parallel to the y-axis, then the co-efficient of y is zero if $k = 7$.

Hence the equation of the line is

$$(1 + 3 \times 7)\,x + (5 - 7 \times 7) = 0$$

i.e. $$22x - 44 = 0$$

i.e. $$x - 2 = 0$$

i.e. $$x = 2$$

6.4.1 Bisectors of Angles between Two Lines

To find the equation of the bisectors of angles between the lines

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

Let $P\,(x, y)$ be any point on the bisector, then the length of the perpendicular drawn through P on the two lines is equal.

i.e. $$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}\,.$$

Hence the above is the equations of the bisectors.

6.5 SUMMARY

In this unit, we have covered the following points :

(i) A general equation of first degree in x and y.

i.e. $Ax + By + C = 0$ represents a straight line.

(ii) Tangent of the angle which a line makes with the positive direction of x-axis is called slope and is denoted by m.

(iii) $x = a$ and $y = b$ are equations of lines parallel to y-axis and x-axis respectively.

(iv) $y = mx + c$, $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$ and $x \cos \theta + y \sin \theta = p$ are equations of a line in the slope, intercept, two points form and perpendicular form.

(v) Angle between two lines with slopes m_1 and m_2 is $\tan^{-1} \frac{m_1 - m_2}{1 + m_1 m_2}$.

(vi) $(A_1 x + B_1 y + C_1) + k(A_2 x + B_2 y + C_2) = 0$ is the equation of lines passing through the point of intersection of $A_1 x + B_1 y + C_1 = 0$ and $A_2 x + B_2 y + C_2 = 0$.

6.6 ANSWERS TO SAQs

SAQ 1

(a) (i) $y = 2x - 5$

(ii) $y = -2$

(iii) $3y = 2x + 12$

(iv) $3y = 5(x + 3)$

(v) $2x + y = 6$ or $x + 2y = 6$

(b) 45^o

(c) $x + 3y - 19 = 0$ and $3x - y - 7 = 0$

(d) $\frac{123}{\sqrt{168}}$

(e) $4x - 3y + 16 = 0$ and $4x - 3y - 14 = 0$