UNIT 7 CIRCLES AND CONIC SECTIONS

Structure

7.1 INTRODUCTION

In Unit 6, we have studied the Cartesian coordinate system and representing lines by algebraic equations. In this unit, we will be studying curves of intersection of a double right circular cone by a plane, commonly known as a conic section. These curves are called parabola, ellipse and hyperbola. We shall define these curves algebraically. We will also be studying circle in different forms.

Objectives

After studying this unit, you should be able to

- derive the equations of a circle in different forms,
- derive the points of intersection of a line and a circle,
- find the equation of a tangent and a normal to a circle at a given point,
- define a conic section,
- obtain the equations of different forms of conic section namely parabola, ellipse and hyperbola, and
- have an idea of the different forms of conic section.

7.2 CIRCLE

Definition 1

The set of all points in a plane each of which is at a constant distance from a fixed point in that plane is called a **circle**.

In other words, a circle is the locus of a point which moves in a plane so that it remains at a constant distance from a fixed point in the plane.

Definition 2

The fixed point is called the **centre** and the constant distance is called the **radius**. Radius of a circle is always positive and a circle is generally denoted by *S*.

7.2.1 Equations of a Circle in Different Forms

(1) **Central Form**

To find the equation of a circle whose centre is $C(h, k)$ and radius is *r*.

Figure 7.1

Let S be the circle and let $P(x, y)$ be any point on the circle.

Then $$|CP| = r$$

i.e. $$\sqrt{(x-h)^2 + (y-k)^2} = r$$

i.e. $$(x-h)^2 + (y-k)^2 = r^2$$

(2) **Simplest (Standard) Form**

If centre is the origin, then $h = 0, k = 0$.

$\therefore$ Equation of the circle is $x^2 + y^2 = r^2$.

(3) **General Form**

Equation of the circle is

$$(x-h)^2 + (y-k)^2 = r^2$$

i.e. $$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2$$

i.e. $$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$$

Let $h = -g, k = -f$ and $h^2 + k^2 - r^2 = c$

then the equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots (7.1)$$

$(-g, -f)$ is the center and r the radius $\sqrt{g^2 + f^2 - c}$

i.e. $$g^2 + f^2 - c > 0.$$

Thus Eq. (7.1) is the said to be the **general form**.

Remark

If $g^2 + f^2 - c = 0$ then the equation represents a single point $(-g, -f)$. It is called a point circle.

(4) **Diameter Form**

Find the equation of a circle having AB as the diameter where A is (x_1, y_1) and B is (x_2, y_2).

Let $P(x, y)$ be any point on the circle. Joint AP and PB.

Then $\angle APB = 90^o$, i.e. AP and PB are perpendicular lines.

Figure 7.2

Slope of AP is $= \dfrac{y - y_1}{x - x_1}$

Slope of BP is $= \dfrac{y - y_2}{x - x_2}$

$\therefore \qquad \dfrac{y - y_1}{x - x_1} \cdot \dfrac{y - y_2}{x - x_2} = -1$

i.e. $\qquad (y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$

i.e. $\qquad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Example 7.1

Determine the equation of a circle if its center is (8, – 6) and which passes through the point (5, – 2).

Solution

Since the circle passes through A (5, – 2) and C (8, – 6) is the centre

$\therefore \qquad \text{radius} = |CA| = \sqrt{(8-5)^2 + (-6+2)^2}$

$= \sqrt{9 + 16} = 5$

The equation of the circle is

$$(x - 8)^2 + (y + 6)^2 = 25$$

or $\qquad x^2 + y^2 - 16x + 12y + 75 = 0$

Example 7.2

Does $x^2 + y^2 - 12x + 6y + 45 = 0$ represent a circle? If yes find the centre and the radius.

Solution

$$x^2 + y^2 - 12x + 6y + 45 = 0$$

represents a circle as it is comparable to the general form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Now the equation of the circle can be written as

$$(x^2 - 12x) + (y^2 + 6y) + 45 = 0$$

i.e. $$(x-6)^2 + (y+3)^2 - 36 - 9 + 45 = 0$$

i.e. $$(x-6)^2 + (y+3)^2 = 0$$

i.e. the equation represents a point circle, centre being (6, – 3) and radius is zero.

Example 7.3

Find the equation of a circle which passes through the points (2, – 2) and (3, 4) and has its centre on the line $2x + 2y = 7$. Find the centre and the radius.

Solution

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

As (2, – 2) and (3, 4) lie on the circle

$\therefore$ $$4 + 4 + 4g - 4f + c = 0$$

i.e. $$4g - 4f + c + 8 = 0 \qquad \text{. . . (7.2)}$$

and $$9 + 16 + 6g + 8f + c = 0$$

i.e. $$6g + 8f + c + 25 = 0 \qquad \text{. . . (7.3)}$$

Since the centre $(-g, -f)$ lies on $2x + 2y = 7$.

$\therefore$ $$-2g - 2f - 7 = 0 \Rightarrow 2g + 2f + 7 = 0 \qquad \text{. . . (7.4)}$$

Solving Eqs. (7.2), (7.3) and (7.4), we have

$$g = -\frac{5}{2},\ c = -2,\ f = -1$$

$\therefore$ $\left(\frac{5}{2}, 1\right)$ is the centre and radius $= \sqrt{\frac{25}{4} + 1 + 2} = \frac{\sqrt{37}}{2}$.

7.2.2 Intersection of a Line and a Circle

To find the condition that the line $y = mx + c$ may intersect the circle $x^2 + y^2 = a^2$.

The equation of the line is

$$y = mx + c \qquad \text{. . . (7.5)}$$

and the equation of the circle is

$$x^2 + y^2 = a^2 \qquad \text{. . . (7.6)}$$

Figure 7.3

A point will lie on the intersection of the line and the circle iff its co-ordinates satisfy both the Eqs. (7.5) and (7.6) simultaneously.

Substituting the value of y from Eq. (7.5) into Eq. (7.6), we have

$$x^2 + (mx + c)^2 = a^2$$

i.e. $$x^2 + m^2x^2 + 2mc\,x + c^2 - a^2 = 0$$

i.e. $$(1 + m^2)\,x^2 + 2mc\,x + (c^2 - a^2) = 0 \qquad \text{. . . (7.7)}$$

This being quadratic in x has two roots, say x_1 and x_2.

$\therefore$ Line in Eq. (7.5) will meet the circle in Eq. (7.6) iff the roots of Eq. (7.7) are real

i.e. iff $4m^2c^2 - 4(1 + m^2)\,(c^2 - a^2) \geq 0$

i.e. iff $m^2c^2 - (c^2 - a^2 + m^2c^2 - a^2m^2) \geq 0$

i.e. iff $-c^2 + a^2 + a^2m^2 \geq 0$

i.e. iff $a^2(1 + m^2) \geq c^2$.

$\therefore$ The line in Eq. (7.5) will meet the circle in Eq. (7.6) in **two distinct points** iff $a^2(1 + m^2) > c^2$. The line will meet the circle in one and only one point, i.e. the line will be a tangent iff $a^2(1 + m^2) = c^2$ and the line will not meet the circle iff $a^2(1 + m^2) < c^2$.

Corollary 1

Line in Eq. (7.5) is a tangent line to the circle if $a^2(1 + m^2) = c^2$, i.e. $c = \pm\, a\sqrt{1 + m^2}$ i.e. $y = mx \pm a\sqrt{1 + m^2}$ are the equation of tangents to the circle $x^2 + y^2 = a^2$ with slope m.

Generally we write $y = mx + a\sqrt{1 + m^2}$ to be the equation of the tangent with slope m to the circle $x^2 + y^2 = a^2$.

Theorem 1

Find the equation of a tangent at a point $P\,(x_1, y_1)$ to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0, \quad g^2 + f^2 - c > 0 \qquad \text{. . .}$$

(i)

Let $Q\,(x_2, y_2)$ be another point on the circle close to the point $P\,(x_1, y_1)$.

As P and Q lie on the circle (i)

then

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad \text{. . . (ii)}$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \qquad \text{. . . (iii)}$$

i.e. $$(x_2^2 - x_1^2) + (y_2^2 - y_1^2) + 2g\,(x_2 - x_1) + 2f\,(y_2 - y_1) = 0$$

i.e. $$(x_2 - x_1)\,(x_2 + x_1) + (y_2 - y_1)\,(y_2 + y_1) + 2g\,(x_2 - x_1) + 2f\,(y_2 - y_1) = 0$$

$$\Rightarrow \ (y_2 - y_1)\,[y_2 + y_1 + 2f] = -\,(x_2 - x_1)\,(x_2 + x_1 + 2g)$$

$$\Rightarrow \ \frac{y_2 - y_1}{x_2 - x_1} = -\,\frac{x_2 + x_1 + 2g}{y_2 + y_1 + 2f}$$

$\dfrac{y_2 - y_1}{x_2 - x_1}$ is the slope of the line PQ and as Q $\to$ P i.e. $x_2 \to x_1, y_2 \to y_1$, then

$$\frac{y_2 - y_1}{x_2 - x_1} = -\,\frac{2x_1 + 2g}{+\,2y_1 + 2f} = -\,\frac{x_1 + g}{y_1 + f}$$

$\therefore$ Slope of the tangent is $-\dfrac{x_1 + g}{y_1 + f}$.

$\therefore$ Equation of tangent at $P\,(x_1, y_1)$ and with slope $-\dfrac{x_1 + g}{y_1 + f}$ is

$$y - y_1 = -\,\frac{x_1 + g}{y_1 + f}\,(x - x_1)$$

$$\Rightarrow \ (y - y_1)\,(y_1 + f) + (x_1 + g)\,(x - x_1) = 0$$

$$\Rightarrow \ x\,x_1 + y\,y_1 + g(x + x_1) + f\,(y + y_1) + c = 0$$

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \ \text{using (ii)}$$

$\therefore$ Equation of the tangent is

$$xx_1 + yy_1 + g(x + x_1) + f\,(y + y_1) + c = 0$$

Definition 3

Let P be a point on a given curve C. Then a line through P, perpendicular to the tangent at P, is called normal to the curve C at P.

Figure 7.4

Since a circle has a unique tangent at each point, it will have a unique normal at each point.

Theorem 2

Find the equation of a normal at $P\ (x_1, y_1)$ to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots$$

(i)

The equation of the tangent to the circle (i) is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

or $$(x_1 + g)x + (y_1 + f)y + gx_1 + fy_1 + c = 0$$

$\therefore$ Slope of the tangent at $P = -\dfrac{x_1 + g}{y_1 + f}$.

$\therefore$ Slope of normal at P is $\dfrac{y_1 + f}{x_1 + g}$.

$\therefore$ Equation of the normal at $P\ (x_1, y_1)$ to the circle is

$$y - y_1 = -\frac{y_1 + f}{x_1 + g}(x - x_1)$$

or $$(x_1 + g)(y - y_1) = (y_1 + f)(x - x_1) \qquad \ldots \text{(ii)}$$

Corollary 2

Centre of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(-g, -f)$ and the coordinates of the centre satisfy the equation (ii)

$\therefore$ The normal to a circle at any point passes through the centre.

Example 7.4

Find the equations of the tangent to the circle $x^2 + y^2 = 9$ which are parallel to $3x + 4y = 0$.

Solution

Centre of the circle $x^2 + y^2 = 9$ is (0, 0).

Equation of the line is $3x + 4y = 0$ i.e. $4y = -3x$. The slope of this line is $-\dfrac{3}{4}$. Hence the slope of any line parallel to the line will be $-\dfrac{3}{4}$.

Equations of the tangents are

$$y = mx \pm a\sqrt{1 + m^2},$$

i.e. $$y = -\frac{3}{4}x \pm 3\sqrt{1 + \frac{9}{16}} = -\frac{3}{4}x \pm \frac{3}{4} \cdot 5$$

i.e. $$4y = -3x \pm 15$$

i.e. $$3x + 4y = \pm 15$$

Example 7.5

Find the equations of tangent and normal to the circle

$$x^2 + y^2 - 2x - 10y + 1 = 0$$

at the point (– 3, 2).

Solution

The equation of the circle is

$$x^2 + y^2 - 2x - 10y + 1 = 0$$

The equation of the tangent at the point (– 3, 2) to the circle is

$$x(-3) + y(2) - \frac{2}{2}(x + (-3)) - \frac{10}{2}(y + 2) + 1 = 0$$

i.e. $\quad -3x + 2y - x + 3 - 5y - 10 + 1 = 0$

i.e. $\quad -4x - 3y - 6 = 0$

i.e. $\quad 4x + 3y + 6 = 0$

Slope of the tangent at $P = -\frac{4}{3}$

$\therefore$ Slope of the normal at $P = \frac{3}{4}$

$\therefore$ Equation of the normal at (– 3, 2) to the circle is

$$y - 2 = \frac{3}{4}(x + 3)$$

or $\quad 4y - 8 = 3x + 9$

or $\quad 3x - 4y + 17 = 0$

SAQ 1

(a) Obtain the equation of a circle which passes through the intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$ and whose centre is the point (2, – 3).

(b) Find the co-ordinates of the centre and radius of the circles

(i) $x^2 + y^2 - 8x + 10y = 0$

(ii) $4(x^2 + y^2) + 12ax - 6ay - a^2 = 0$

(c) Find the equation of the circle which is concentric with $x^2 + y^2 - 8x + 12y + 43 = 0$ and which passes through (6, 2).

(d) Show that the points (1, – 6), (5, 2), (7, 0), (– 1, – 4) are the vertices of a cyclic quadrilateral.

(e) Find the equation of the circle which passes through (4, 1) and (6, 5) and has its centre on the line $4x + y = 16$.

SAQ 2

(a) Find the equation of a circle passing through origin and making intercepts 4 and 5 on the co-ordinates axis.

(b) Find the equation of the circle circumscribing the Δ formed by the lines $3x + 4y = 24$ and the axes of x and y.

(c) One end of a diameter of the circle $x^2 + y^2 - 3x + 5y - 4 = 0$ is (2, 1). Find the co-ordinates of the other end.

(d) Find the equation of the circle with centre (1, 1) and which touches the line $x + y = 1$.

(e) Find the equation of the normal to the circle $x^2 + y^2 + 4x + 6y - 39 = 0$ at the point (2, 3). Show that the normal meets the circle again at the point (– 6, – 9).

(f) Show that the tangents drawn at the points (12, – 5) and (5, 12) to the circle $x^2 + y^2 = 169$ are perpendicular to each other.

7.3 CONIC SECTION

Definition 4

A conic is the locus of a point which moves so that its distance from a fixed point is in a constant ratio to its distance from a fixed straight line.

Definition 5

The fixed point is called the **focus** and is denoted by *S*.

The constant ratio is called the **eccentricity** and is denoted by *e*.

The fixed straight line is called the **directrix**.

Types of Conic

There are three types of conics.

(i) Parabola If *e* the eccentricity is = 1.

(ii) Ellipse If *e* the eccentricity is < 1.

(iii) Hyperbola If *e* the eccentricity is > 1.

Physical Interpretation

The section of a double cone with common vertex cut by a plane is known as conic section. The shape of the conic section depends upon the position of the plane.

(i) If the plane is parallel to the base, the section is the circle.

(ii) If the plane is inclined to the base, the section is ellipse.

(iii) If the plane is inclined to the base and cuts the base, the section is parabola.

(iv) If the plane is parallel to the axis of the cone, the section is hyperbola.

(v) If the plane contains the axis of the cone, the section is a pair of lines.

Equation of Conics in Standard Form

7.3.1 Parabola

A parabola is the locus of a point which moves so that its distance from a fixed point is equal to its distance from a fixed line.

Let S be the focus and ZM the directrix of the parabola. Let P be a point on the parabola. Draw $PM \perp ZM$.

$$\therefore \quad SP = PM$$

From S draw $SZ \perp ZM$. Bisect SZ at A.

i.e. $SA = AZ$.

Then A lies on the parabola. Let $SZ = 2a$, then $AS = AZ = a$.

Take A as origin, AS as the x-axis and $Ay \perp$ to it as the y-axis.

Figure 7.5

Then the co-ordinates of S are $(a, 0)$ and $PM = SP$.

i.e. $$\sqrt{(x-a)^2 + (y-0)^2} = PM = ZA + AS$$
$$= a + x$$

i.e. $$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$$

i.e. $$y^2 = 4ax$$

This is the equation of the parabola in the **Standard Form**.

Note

(i) Focus is $S \equiv (a, o)$

(ii) Directrix is $x = -a$ i.e. $x + a = 0$

(iii) Vertex is A and $A \equiv (0, 0)$.

(iv) AS a called the axis of the parabola and AS is $y = 0$.

(v) LL' is called the latus-rectum.

Let $SL = l$

Then $L \equiv (a, l)$ and L lies on the parabola

$$\therefore \quad l^2 = 4a \times a$$

i.e. $$l = \sqrt{4a^2} = \pm 2a$$

As l is positive $\therefore \quad l = 2a$

Hence latus-rectum $= LL' = 2(LS) = 4a$.

Figure 7.6

Other Shapes of the Parabolas

Find the equation of a parabola with

(i) focus $S(-a, 0), a > 0$ and the line $x - a = 0$ as directrix.

(ii) focus $S(0, a), a > 0$ and the line $y + a = 0$ as directrix.

(iii) focus $S = (0, -a), a > 0$ and the line $y - a = 0$ as directrix.

(i) Let $P(x, y)$ be any point on the parabola.

Then $|PS| = |PM|$ where PM is the length of the perpendicular drawn from P on the directrix,

i.e. $$\sqrt{(x + a)^2 + (y - 0)^2} = \frac{|x - a|}{1}$$

i.e. $$(x + a)^2 + y^2 = (x - a)^2$$

i.e. $$x^2 + 2ax + a^2 + y^2 = x^2 - 2ax + a^2$$

i.e. $$y^2 = -4ax; \; a > 0$$

Sometimes it is called left hand parabola.

(ii) Proceed as above. It will be found that the equation of the parabola is $x^2 = 4ay$; $a > 0$.

(iii) In this case the equation of the parabola is $x^2 = -4ay$; $a > 0$.

Table 7.1 : Main Facts about the Parabola

Equation	$y^2 = 4ax$, $a > 0$	$y^2 = -4ax$, $a > 0$	$x^2 = 4ay$, $a > 0$	$y^2 = -4ay$, $a > 0$
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Length of Latus-Rectum	$4a$	$4a$	$4a$	$4a$
Equation of Latus-Rectum	$x = a$	$x = -a$	$y = a$	$y = -a$
Focal distance of the point (x, y)	$x + a$	$a - x$	$y + a$	$a - y$

Example 7.6

Find the equation of the parabola whose focus is (3, 0) and the directrix is $3x + 4y = 1$.

Solution

Let $P(x, y)$ be any point on the parabola. Then distance of P from (3, 0) is

$$\sqrt{(x-3)^2 + y^2}$$

Distance of $P(x, y)$ from the line $3x + 4y - 1 = 0$ is

$$\frac{3x + 4y - 1}{\sqrt{3^2 + 4^2}} = \frac{3x + 4y - 1}{5}$$

$\therefore$ Equation of the parabola is

$$\sqrt{(x-3)^2 + y^2} = \frac{3x + 4y - 1}{5}$$

or $\quad 25[x^2 - 6x + 9 + y^2] = (3x + 4y - 1)^2$

$$= 9x^2 + 16y^2 + 1 + 24xy - 6x - 8y$$

i.e. $\quad 16x^2 + 9y^2 - 24xy - 144x + 8y + 224 = 0$

Example 7.7

Find the vertex, focus and directerix of the parabola $4y^2 + 12x - 12y + 39 = 0$.

Solution

The equation can be written as $4y^2 - 12y = -12x - 39$

i.e. $$y^2 - 3y = -3x - \frac{39}{4}$$

i.e. $$\left(y - \frac{3}{2}\right)^2 = -3x - \frac{39}{4} + \frac{9}{4}$$

$$= -3x - \frac{15}{2}$$

$$= -3\left(x + \frac{5}{2}\right)$$

write $Y = y - \frac{3}{2}$, $X = x + \frac{5}{2}$, , the equation can be written as $Y^2 = -3X$.

With respect to $Y^2 = -4\left(\frac{3}{4}X\right)$

$\therefore$ Vertex is $\left(-\frac{5}{2}, \frac{3}{2}\right)$ $\{\because$ Vertex is $(0, 0)\}$

Focus is $\left(-\frac{13}{4}, \frac{3}{2}\right)$ $\{\because$ Focus is $\left(-\frac{3}{4}, 0\right)\}$

Directrix is $x = -\frac{7}{4}$ $\{\because$ Directrix is $X = \frac{3}{4}\}$

SAQ 3

(a) Find the equation of a parabola with

(i) Focus is (1, 1) and the directrix is $x + y + 1 = 0$.

(ii) Focus is (0, – 3) and the vertex is (0, 0).

(iii) Focus is $(a, 0)$ and the vertex is $(a', 0)$.

(b) Find focus, vertex, length of latus rectum, equation of the directrix of the following parabolas :

(i) $y^2 = -4x$

(ii) $3x^2 = 8y$

(c) Find the vertex, focus and directrix of the parabola

(i) $(y + 3)^2 = 2(x + 2)$

(ii) $y^2 = 4x + 4y$

(iii) $x^2 + y = 9x - 14$

(iv) $3x^2 + 12x - 8y = 0$

7.3.2 Ellipse

Definition 6

Ellipse is the locus of a point which moves so that its distance from a fixed point is in a constant ratio (less than one) to its distance from a fixed straight line.

Standard Equation

Let S be the focus and ZM the directrix.

Figure 7.7

Let P be the point on the ellipse and $PM \perp ZM$, then

$$\frac{SP}{PM} = e$$

Draw $SZ \perp ZM$. Divide SZ internally and externally at A and A' in the ratio of $e : 1$ so that

$$\frac{SA}{AZ} = \frac{e}{1}, \frac{SA'}{A'Z} = \frac{e}{1}$$

i.e. $$SA = e\ AZ \qquad \text{. . . (i)}$$

$$SA' = e\ A'Z \qquad \text{. . . (ii)}$$

Bisect AA' at C. Let $AA' = 2a$.

Then $AC = CA' = a$

Let C be the origin and CA as x-axis,

$Cy \perp$ to it as y-axis.

Adding (i) and (ii) we have

$$SA + SA' = e\,[AZ + A'Z]$$

$$(CA - CS) + (CS + CA') = e\,[(CZ - CA) + (CA' + CZ)]$$

$$2a = e \,.\, 2 \,.\, CZ \quad \{\because \quad CA = CA' = a\}$$

i.e. $$CZ = \frac{a}{e} \qquad \text{. . . (iii)}$$

Subtracting (ii) from (i) we have

$$SA' - SA = e\,[A'Z - AZ]$$

i.e. $(CS + CA') - (CA - CS) = e\,[(CA' + CZ) - (CZ - CA)]$

i.e. $2CS = e\,[2CA]$ as $CA' = CA$

i.e. $CS = ae$ . . . (iv)

$\therefore$ Co-ordinates of S and Z are $(ae, 0)$ and $\left(\frac{a}{e}, 0\right)$ respectively.

Let P be (x, y), then

$$(x - ae)^2 + (y - 0)^2 = e^2\left(\frac{a}{e} - x\right)^2$$

i.e. $$x^2 - 2aex + a^2e^2 + y^2 = e^2\left(\frac{a^2}{e^2} - \frac{2ax}{e} + x^2\right)$$

i.e. $$x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

i.e. $$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

i.e. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{where} \quad b^2 = a^2(1 - e^2)$$

This is the standard equation of the ellipse.

B lies on the y-axis $\therefore$ x-coordinate of B is zero. Suppose $B \equiv (0, h)$.

Then as B lies on the ellipse

$$\frac{0}{a^2} + \frac{h^2}{a^2} = 1 \quad \text{i.e.} \quad h^2 = b^2 \quad \text{i.e.} \quad h = \pm b$$

$$\therefore \quad B \equiv (0, b) \text{ and } B' \equiv (0, -b)$$

Points to Note

(i) A and A' are called vertices.

(ii) AA' is called the major axis.

(iii) BB' is called the minor axis.

(iv) C is called the centre.

(v) LSL' is called the latus rectum and the length of the latus rectum can be found as follows.

Let $L = (ae, l)$

L lies on the ellipse

$$\therefore \quad \frac{a^2e^2}{a^2} + \frac{l^2}{b^2} = 1, \quad \text{i.e.} \quad \frac{l^2}{b^2} = 1 - e^2$$

i.e. $l^2 = b^2(1 - e^2)$

i.e. $l = b\sqrt{1 - e^2}$

$$LL' = 2b\sqrt{1-e^2} = \frac{2b^2}{a} \text{ as } a\sqrt{1-e^2} = b$$

(vi) It is symmetrical about x as well as y-axis.

$A \equiv (a, 0)$ and $A' \equiv (-a, 0)$ and $2a$ is called the length of the major axis.

$B \equiv (0, b)$, $B' \equiv (0, -b)$ and $BB' = 2b$ is the length of the minor axis.

(vii) Since $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ represents ellipse with focus $S(ae, 0)$, directrix $x - \frac{a}{e} = 0$ and as the ellipse is symmetrical about y-axis.

It follows that the ellipse with focus $S(-ae, 0)$ and $x + \frac{a}{e} = 0$ as directrix and e as eccentricity coincide with the given ellipse, i.e. the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has two focii (ae, o) and $(-ae, o)$ and the two corresponding directrices are $x - \frac{a}{e} = 0$ and $x + \frac{a}{e} = 0$.

Theorem 3

The sum of the focal distances of any point on an ellipse is constant and is equal to the length of the major axis.

i.e. $SP + S'P = 2a$

Figure 7.8

Let $P(x, y)$ be any point on the ellipse. Join $SP, S'P$. From P draw $PM \perp$ on ZM, $PM' \perp$ on $Z'M'$ and $PN \perp$ on the x-axis. Then

$$SP = e\,PM = e\,ZN = e\,(CZ - CN) = e\left(\frac{a}{e} - x\right) = a - ex \qquad \text{. . . (i)}$$

$$S'P = e\,PM' = e\,NZ' = e\,(CZ' + CN) = e\left(\frac{a}{e} + x\right) = a + ex \qquad \text{. . . (ii)}$$

Adding (i) and (ii) we have

$$SP + S'P = a - ex + a + ex = 2a$$

Example 7.8

Find the equation of an ellipse whose focus is (1, 0), the directrix is $x + y + 1 = 0$ and eccentricity is equal to $\frac{1}{\sqrt{2}}$.

Solution

Let $P(x, y)$ be any point on the ellipse.

Then $$(x-1)^2 + (y-0)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 \left[\frac{x+y+1}{\sqrt{1+1}}\right]^2$$

i.e. $$x^2 - 2x + 1 + y^2 = \frac{1}{2}\left[\frac{x^2 + y^2 + 2xy + 2x + 2y + 1}{2}\right]$$

i.e. $$4x^2 - 8x + 4 + 4y^2 = x^2 + y^2 + 2xy + 2x + 2y + 1$$

i.e. $$3x^2 - 2xy + 3y^2 - 10x - 2y + 3 = 0$$

Example 7.9

Find the centre, the length of the axes, eccentricity and foci of the ellipse

$$x^2 + 2y^2 - 2x + 12y + 10 = 0$$

Solution

$$x^2 + 2y^2 - 2x + 12y + 10 = 0$$

$$(x^2 - 2x + 1) + 2(y^2 + 6y + 9) = -10 + 19$$

$$(x-1)^2 + 2(y+3)^2 = 9$$

i.e. $$\frac{(x-1)^2}{9} + \frac{(y+3)^2}{\frac{9}{2}} = 1$$

i.e. $$\frac{(x-1)^2}{3^2} + \frac{(y+3)^2}{\left(\frac{3}{\sqrt{2}}\right)^2} = 1$$

i.e. $$\frac{X^2}{3^2} + \frac{Y^2}{\left(\frac{3}{\sqrt{2}}\right)^2} = 1$$

where $X = x - 1,\ Y = y + 3$

Now $$b^2 = \frac{9}{2} = a^2 (1 - e^2) = 9(1 - e^2)$$

or $$\frac{9}{2} = 9(1 - e^2)$$

i.e. $$1 - e^2 = \frac{1}{2}$$

i.e. $e^2 = 1 - \frac{1}{2} = \frac{1}{2}$

i.e. $e = \frac{1}{\sqrt{2}}$.

Now

Equation	$\frac{X^2}{(3)^2} + \frac{Y^2}{\left(\frac{3}{\sqrt{2}}\right)^2} = 1$	$\frac{(x-1)^2}{(3)^2} + \frac{(y+3)^2}{\left(\frac{3}{\sqrt{2}}\right)^2} = 1$
Centre	(0, 0)	(1, – 3)
Length of major axis	6	6
Length of minor axis	$3\sqrt{2}$	$3\sqrt{2}$
Eccentricity	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
Foci	$(\pm \frac{3}{\sqrt{2}}, 0)$	$(\pm \frac{\sqrt{3}}{2} + 1, -3)$

SAQ 4

(a) Find the equation of the ellipse

(i) whose centre is (1, 2), the directrix is $3x + 4y = 5$ and eccentricity $= \frac{1}{2}$.

(ii) whose foci are $(\pm 2, 0)$ and eccentricity is $= \frac{1}{2}$.

(iii) whose centre is (2, – 3), one focus is (3, – 3) and one vertex is (4, – 3).

(iv) whose axis are the axes of co-ordinates and which passes through the points (– 3, 1) and (2, – 2).

(b) Find the eccentricity, foci, length of the latus rectum of the ellipse

(i) $3x^2 + 4y^2 - 12x - 8y + 4 = 0$

(ii) $25x^2 + 16y^2 = 400$

(c) Find the centre, the length of the axes, eccentricity and the foci of the ellipse

(i) $x^2 + 4y^2 - 4x + 24y + 31 = 0$

(ii) $4x^2 + y^2 - 8x + 2y + 1 = 0$

7.3.3 Hyperbola

Definition 7

A hyperbola is the locus of a point, which moves so that its distance from a fixed point is in a constant ratio (greater than one) of its distance from a fixed line.

The Standard Equation of a Hyperbola

Let S be the focus, ZM be the directrix.

Let $P(x, y)$ be a point on the hyperbola and $PM \perp ZM$.

Then $$\frac{SP}{PM} = e$$

Draw $SZ \perp ZM$ and divide SZ internally and externally at A and A' in the ratio of $e : 1$.

Figure 7.9

$$\frac{SA}{AZ} = \frac{e}{1}, \quad \frac{SA'}{A'Z} = \frac{e}{1} \qquad \ldots$$

(i)

Bisect AA' at C. Let $AA' = 2a$,

then $$CA = CA^1 = a$$

Let C be the origin, CA as the x-axis and $CY \perp CA$ as the y-axis.

From (i) we have

$$SA + SA' = e\,(AZ + A'Z)$$

$$(CS - CA) + (CS + CA') = e\,[(CA - CZ) + (CZ + CA)]$$

$$2CS = 2e\,CA$$

$$CS = e\,CA$$

$$\therefore \quad S \equiv (ae, 0)$$

Again from (i) we have

$$SA' - SA = e(A'Z - AZ)$$

$$(CS + CA') - (CS - CA) = e\,[(CZ + CA') - (CA - CZ)]$$

$$2CA = e\,.\,2CZ$$

i.e. $$CZ = \frac{a}{e}$$

$$\therefore \quad Z \equiv \left(\frac{a}{e}, 0\right)$$

Now $$(SP)^2 = e^2\,(PM)^2$$

i.e. $$(x - ae)^2 + (y - 0)^2 = e^2\left(x - \frac{a}{e}\right)^2$$

i.e. $$x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2$$

or $$x^2\,(e^2 - 1) - y^2 = a^2\,(e^2 - 1)$$

or $$\frac{x^2}{a^2} - \frac{y^2}{a^2\,(e^2 - 1)} = 1$$

or $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{where} \quad b^2 = a^2\,(e^2 - 1)$$

This is the standard equation of the hyperbola.

Some Facts about the Hyperbola

(i) It is symmetric about x-axis. x-axis is called the transverse axis and $2a$ is the length of the transverse axis.

(ii) It is symmetric about y-axis, y-axis does not meet the curve. The line containing $B'\,(0, -b)$ and $B\,(0, b)$, i.e. y-axis is called the conjugate axis and is of length $2b$, called the length of the conjugate axis.

(iii) It is symmetric about origin, so $O\,(0, 0)$ is called the center.

(iv) The points $A'\,(-a, 0)$ and $A\,(a, 0)$ are called the vertices.

(v) Since $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ represents hyperbola with Focus $(ae, 0)$ and directrix $x - \frac{a}{e} = 0$ and eccentricity e, and hyperbola is symmetric about y-axis it follows that the hyperbola with Focus $F'\,(-ae, 0)$, directrix $x + \frac{a}{e} = 0$ and eccentricity e coincides with the given hyperbola. Thus the hyperbola has two foci and two directrixes.

(vi) A chord passing through focus F (or F') and perpendicular to transverse axis is called a latus rectum and its length as the length of the latus rectum.

(vii) Length of latus rectum. Let $L'L$ be a latus rectum of the hyperbola, then $L'L$ passes through $F(ae, 0)$ and is $\perp$ to the x-axis. Let $|FL| = l > 0$, then $L \equiv (ae, l)$ $L' = (-ae, -l)$.

Since L lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$$\therefore \quad \frac{a^2e^2}{a^2} - \frac{l^2}{b^2} = 1 \quad \Rightarrow \quad \frac{l^2}{b^2} = e^2 - 1$$

$$\Rightarrow \quad l^2 = b^2\,(e^2 - 1) \quad \Rightarrow \quad l = b\sqrt{e^2 - 1} \quad \because \quad l > 0$$

$$\therefore \quad L \equiv \left(ae, \frac{b^2}{a}\right), \; L' \equiv \left(ae, \frac{-b^2}{a}\right)$$

and length of the latus rectum is $2l = \frac{2b^2}{a}$.

(viii) There are two latra-recta and the extremities of latra recta are

$$L \equiv \left(ae, \frac{b^2}{a}\right), \; L' \equiv \left(ae, \frac{-b^2}{a}\right)$$

$$L_1 \equiv \left(-ae, \frac{b^2}{a}\right), \; L_1' \equiv \left(-ae, \frac{-b^2}{a}\right)$$

(ix) The distances of any points on the hyperbola from the two foci are called focal distances of the point.

(x) The difference of the focal distances of a point on a hyperbola is constant and equal to $2a$.

Let S, S^1 be the foci, ZM and $Z'M'$ be the corresponding directrix and $P(x, y)$ be any point on the hyperbola. Join $SP, S'P$.

Draw PM' and $PM \perp$ on ZM and $Z'M'$ respectively.

$$SP = e\,PM = e\,ZN = e\left(x - \frac{a}{e}\right) = ex - a \qquad \text{. . . (i)}$$

$$S'P = e\,PM' = e\,Z'N = e\left(x + \frac{a}{e}\right) = ex + a \qquad \text{. . . (ii)}$$

Figure 7.10

On subtracting (i) and (ii) we have

$$S'P - SP = (ex + a) - (ex - a)$$

i.e. $$S'P - SP = 2a$$

Example 7.10

Find the equation of hyperbola whose directrix is $2x + y = 1$, focus (1, 2) and eccentricity $\sqrt{3}$.

Solution

Let $P(x, y)$ be any point on the hyperbola.

Then $PM \perp$ on the directrix is

$$\frac{2x + y - 1}{\sqrt{2^2 + 1^2}}.$$

$$\therefore \quad (x-1)^2 + (y-2)^2 = e^2 \left(\frac{2x + y - 1}{\sqrt{4+1}} \right)^2$$

i.e. $5\,[x^2 - 2x + 1 + y^2 - 4y + 4] = e^2\,[4x^2 + y^2 + 1 + 4xy - 4x - 2y]$

i.e. $$7x^2 + 12xy - 2y^2 - 2x + 14y - 22 = 0$$

SAQ 5

(a) Find the equation of the hyperbola

(i) whose vertices are (0, 0), (10, 0) and one of foci is (18, 0).

(ii) the distance between foci is 16, and whose eccentricity is $\sqrt{2}$.

(iii) whose directrix is $x + y - 1 = 0$, focus is (0, 3) and eccentricity is 2.

(b) Find the eccentricity and the co-ordinates of the foci of the hyperbola

(i) $3x^2 - y^2 = 4$

(ii) $x^2 - y^2 + 4x = 0$

7.4 SUMMARY

(i) Equation of a circle is $(x-h)^2 + (y-k)^2 = r^2$ and a line intersects the circle in two points (They can be distinct, same or imaginary i.e. does not exist).

(ii) $y^2 = \pm 4ax$ or $x^2 = \pm 4ay$ are general equations of a parabola.

(iii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the equation of the ellipse where $b^2 = a^2(1 - e^2)$.

(iv) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the equation of the hyperbola where $b^2 = a^2(e^2 - 1)$.

(v) **Main Facts about the Parabola**

Equation	$y^2 = 4ax$ **Right-hand**	$y^2 = -4ax$ **Left-hand**	$x^2 = 4ay$ **Upward**	$x^2 = -4ay$ **Downward**
	($a > 0$)	**($a > 0$)**	**($a > 0$)**	**($a > 0$)**
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Vertex	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Tangent at the vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
Length of latus rectum	$4a$	$4a$	$4a$	$4a$
Equation of latus-spectrum	$x = a$	$x = -a$	$y = a$	$y = -a$
Focal distance of the point (x, y)	$x + a$	$a - x$	$y + a$	$a - y$

(vi) **Main Facts about the Ellipse**

Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b > 0$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b > 0$
Equation of Major axis	$y = 0$	$x = 0$
Length of Major axis	$2a$	$2a$
Equation of Minor axis	$x = 0$	$y = 0$
Length of Minor axis	$2b$	$2b$
Vertices	$(\pm a, 0), (0, \pm b)$	$(\pm b, 0), (0, \pm a)$
Foci	$(\pm ae, 0)$	$(0 \pm ae)$
Directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{a}{e}$
Equations of Latera recta	$x = \pm ae$	$y = \pm ae$
Length of a latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Centre	(0, 0)	(0, 0)
Focal distances of any	$a - ex, a + ex$	$a - ey, a + ey$

point (x, y)		

(vii) **Main Facts about the Hyperbola**

Equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, a > 0, b > 0$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, a > 0, b > 0$
Equation of Transverse axis	$y = 0$	$x = 0$
Length of Transverse axis	$2\,a$	$2\,a$
Equation of Conjugate axis	$x = 0$	$y = 0$
Length of Conjugate axis	$2\,b$	$2\,b$
Vertices	$(\pm\, a, 0)$	$(0, \pm\, a)$
Foci	$(\pm\, a\,e, 0)$	$(0, \pm\, a\,e)$
Directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{a}{e}$
Centre	$(0, 0)$	$(0, 0)$
Equation of Latera-recta	$x = \pm\, a\,e$	$y = \pm\, a\,e$
Length of a latus rectum	$\frac{2\,b^2}{a}$	$\frac{2\,b^2}{a}$

7.5 ANSWERS TO SAQs

SAQ 1

(a) $(x - 2)^2 + (y + 3)^2 = 109$

(b) (i) $(4, -5), \sqrt{41}$

(ii) $\left(\frac{-3a}{2}, \frac{3a}{4}\right), \frac{7a}{4}$

(c) $x^2 + y^2 - 8x + 12y - 16 = 0$

(e) $x^2 + y^2 - 6x - 8y + 15 = 0$

SAQ 2

(a) $x^2 + y^2 - 4x - 5y = 0$

(b) $x^2 + y^2 - 8x - 6y = 0$

(c) $(1, -6)$

(d) $2\,(x^2 + y^2) - 4x - 4y + 3 = 0$

(e) $3x - 2y = 0$

SAQ 3

(a) (i) $x^2 - 2xy + y^2 - 6x - 6y + 3 = 0$

(ii) $x^2 = -12y$

(iii) $y^2 = -4\,(a' - a)\,(x - a')$

(b) (i) Focus (– 1, 0), vertex (0, 0), latus rectum = 4, directrix $x - 1 = 0$.

(i) Focus $\left(0, \frac{2}{3}\right)$, vertex (0, 0), latus rectum $= \frac{8}{3}$, directrix $3y + 2 = 0$.

(c) (i) $(-2, -3)$; $\left(-\frac{3}{2}, -3\right)$; $x = -\frac{5}{2}$

(ii) $(-1, 2)$; $(0, 2)$; $x = -2$

(iii) $\left(\frac{9}{2}, \frac{25}{4}\right)$; $2x - 9 = 0$ or $\left(\frac{9}{2}, 6\right)$; $2y - 13 = 0$

(iv) $\left(-2, -\frac{3}{2}\right)$; $\left(-2, -\frac{5}{6}\right)$; $x + 2 = 0$

SAQ 4

(a) (i) $91x^2 + 84y^2 - 24xy - 170x - 360y + 475 = 0$

(ii) $\frac{x^2}{36} + \frac{y^2}{32} = 1$

(iii) $3x^2 + 4y^2 - 12x + 24y + 36 = 0$

(iv) $3x^2 + 5y^2 = 32$

(b) (i) $\frac{1}{2}$, (3, 1), (1, 1), 3

(ii) $\frac{3}{5}$, $(0, \pm 3)$, $\frac{32}{5}$

(c) (i) $(2, -3)$, 6, 3, $\frac{\sqrt{3}}{2}$, $\left(2 \pm \frac{\sqrt{3}}{2}, -3\right)$

(ii) $(1, -1)$, $\frac{\sqrt{3}}{2}$, $(1, \pm\sqrt{3} - 1)$, 1

SAQ 5

(a) (i) $\frac{(x-5)^2}{25} - \frac{y^2}{144} = 1$

(ii) $x^2 - y^2 = 32$

(iii) $x^2 + 4xy + y^2 - 4x + 2y - 7 = 0$

(b) (i) 2, $\left(\pm \frac{4}{\sqrt{3}}, 0\right)$

(ii) $\sqrt{2}$, $(-2 \pm 2\sqrt{2}, 0)$