UNIT 5 APPLICATION OF TRIGONOMETRY

Structure

5.1 INTRODUCTION

In Unit 4, we have studied trigonometric ratios of some standard angles, allied, multiple and sub-multiple angles. As already mentioned, trigonometry means the science of measuring triangles. In this unit, we will learn relations between sides and angles of triangle so as to be able to calculate sides and angles of a triangle when some of the sides and angles are given.

Objectives

After studying of this unit, you should be able to

- find relation between sides and angles of a triangle,
- find the unknown elements of a triangle when some of them are given, and
- solve problems on heights and distances especially in case of inaccessible objects.

5.2 PROPERTIES OF TRIANGLES

Theorem 1 : Law of Sines

Prove that if in any triangle ABC, the angles are denoted by A, B, C and the sides opposite to them are denoted by *a*, *b*, *c* respectively, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} .$$

Figure 5.1

Proof

From B, draw BD $\perp$ AC.

From the above figures we have

$$\frac{h}{c} = \sin\ A,\ \frac{h}{a} = \sin\ C$$

These two relations hold whether A is acute or obtuse. It obviously holds if A is a right angle.

Thus $$c \sin\ A = a \sin\ C$$

or, $$\frac{\sin\ A}{a} = \frac{\sin\ C}{c}$$

Similarly, drawing the altitude from the vertex C we have

$$\frac{\sin\ A}{a} = \frac{\sin\ B}{b}$$

i.e., $$\frac{\sin\ A}{a} = \frac{\sin\ B}{b} = \frac{\sin\ C}{c}$$

Theorem 2 : Law of Cosines

Prove that $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, $\cos B = \dfrac{c^2+a^2-b^2}{2ca}$, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$.

Proof

Again referring to Figure 5.1, we have

$$\begin{aligned} AB^2 &= BD^2 + AD^2 \\ &= BD^2 + (AC - CD)^2 \\ &= BD^2 + AC^2 + CD^2 - 2AC\ .\ CD \qquad \ldots (5.1) \end{aligned}$$

Now, $BD^2 + CD^2 = BC^2$ and $CD = BC \cos C$

$$\therefore\ AB^2 = AC^2 + BC^2 - 2AC\ .\ BC \cos C \ \text{(From Eq. (5.1))}$$

or, $$c^2 = a^2 + b^2 - 2ab \cos\ C$$

i.e. $$\cos\ C = \frac{a^2 + b^2 - c^2}{2ab}$$

Similarly, we have $\cos\ A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$$\cos\ B = \frac{c^2 + a^2 - b^2}{2ca}$$

Theorem 3

Prove that

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{bc}},\ \sin\frac{B}{2} = \sqrt{\frac{(s-c)\,(s-a)}{ca}},\ \sin\frac{C}{2} = \sqrt{\frac{(s-a)\,(s-b)}{ab}}$$

where $2s = a + b + c$.

Proof

We have $$\cos\ A = \frac{b^2 + c^2 - a^2}{2bc}$$

i.e. $$2\sin^2\frac{A}{2} = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{a^2 - (b-c)^2}{2bc} = \frac{(a+b-c)\,(a-b+c)}{2bc}$$

Let $a + b + c = 2s$

$$\therefore \quad \sin^2\frac{A}{2} = \frac{4(s-b)\,(s-c)}{4bc}$$

or $$\sin\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{bc}}$$

Similarly, $$\sin\frac{B}{2} = \sqrt{\frac{(s-c)\,(s-a)}{ca}}$$

$$\sin\frac{C}{2} = \sqrt{\frac{(s-a)\,(s-b)}{ab}}$$

Theorem 4

Prove that

(i) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

(ii) $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$

(iii) $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

where, $2s = a + b + c$

Proof

$$2\cos^2\frac{A}{2} = 1 + \cos A$$

$$= 1 + \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(b+c)^2 - a^2}{2bc} = \frac{(b+c+a)\,(b+c-a)}{2bc} = \frac{2s(2s-2a)}{4bc}$$

$$\therefore \quad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

Similarly, $$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$$

$$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

Cor.

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{s\,(s-a)}}$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-c)\,(s-a)}{s\,(s-b)}}$$

$$\tan\frac{C}{2} = \sqrt{\frac{(s-a)\,(s-b)}{s\,(s-c)}}$$

Theorem 5 : Area of a Triangle

Prove that the area of a triangle is $\sqrt{s(s-a)\,(s-b)\,(s-c)}$ where $2s = a+b+c$.

Proof

Referring to Figure 5.1, we have area of the triangle

ABC = $\Delta = \frac{1}{2}\,bh = \frac{1}{2}\,b\,(c \sin A)$.

$$= \frac{1}{2}\,bc \sin A$$

$$= bc \sin\frac{A}{2} \cos\frac{A}{2}$$

$$= bc\sqrt{\frac{(s-b)\,(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{s(s-a)\,(s-b)\,(s-c)}$$

This formula is known as **Hero's Formula**.

Cor. 1

Again referring Figure 5.1, we have

$$b = AD + DC,\ AD = c \cos A,\ DC = a \cos C$$

$$\therefore\quad b = c\cos A + a \cos C$$

If A is obtuse (Figure 5.1), we have

$$b = CD - AD,\ CD = a\cos C,\ AD = c\cos(\pi - A) = -\,c\cos A$$

$$\left.\begin{aligned} \therefore\quad b &= a\cos C + c\cos A \\ a &= b\cos C + c\cos B \\ c &= a\cos B + b\cos A \end{aligned}\right]$$

Similarly,

Cor. 2

By law of sines we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k\ \text{(say)}$$

$$\therefore\quad \frac{b-c}{b+c} = \frac{k\,(\sin B - \sin C)}{k\,(\sin B + \sin C)} = \frac{\sin B - \sin C}{\sin B + \sin C}$$

$$= \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$= \tan\left(\frac{B-C}{2}\right)\frac{\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)}{\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)} \quad \left(\because \frac{A+B+C}{2}=\frac{180^o}{2}=\frac{\pi}{2}\right)$$

$$= \tan\left(\frac{B-C}{2}\right)\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$

$$= \tan\left(\frac{B-C}{2}\right)\tan\frac{A}{2}$$

$$\therefore \ \tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\frac{A}{2}$$

Similarly, formulae for $\tan\left(\frac{C-A}{2}\right)$ and $\tan\frac{A-B}{2}$ can be found.

Theorem 6

Prove that the radius r of the incircle of a triangle ABC is $r=\frac{\Delta}{s}$, where Δ is the area of the triangle and $2s = a + b + c$.

Proof

In a triangle ABC, let AO, BO, CO be the angle bisectors. Then
OD = OE = OF = r (say) where r is the radius of the incircle.

Figure 5.2

We see that

$$\tan\frac{A}{2}=\frac{r}{AF}$$

$$\tan\frac{B}{2}=\frac{r}{BD}$$

$$\tan\frac{C}{2}=\frac{r}{CE}$$

Δ, the area of the triangle ABC is the sum of the areas of the triangles AOB, BOC and AOC.

Hence $$\Delta=\frac{1}{2}cr+\frac{1}{2}ar+\frac{1}{2}br$$

$$= \frac{1}{2} r\,(a + b + c) = r\,.\,s$$

i.e. $$r = \frac{\Delta}{s}$$

Cor.

As triangles AFO, AEO are congruent

$$\therefore \quad AF = AE$$

Similarly, $$BF = BD$$

$$CE = CD$$

Since the sum of these six segments is $a + b + c = 2s$,

$$\therefore \quad AF + BD + CD = s$$

i.e, $$AF = s - (BD + CD) = s - a$$

$$BD = s - b$$

$$CE = s - c$$

Hence $\tan \frac{A}{2} = \frac{r}{s-a}$, $\tan \frac{B}{2} = \frac{r}{s-b}$, $\tan \frac{C}{2} = \frac{r}{s-c}$

5.3 SOLUTIONS OF TRIANGLES

We will now see how we can solve a triangle if we know three of its parts at least one of which is a side with the help of these formulae. Different cases to be considered are

Case 1

Three sides are given.

Case 2

Two sides and the included angle is given.

Case 3

Two sides and the angle opposite to one of them are given.

Case 4

One side and two angles are given.

Case 5

The triangle is a right angled triangle.

Case 1

When *a, b, c* are given

When a, b, c are given then $s = \frac{a+b+c}{2}$ is known. Then half angles can be computed by using formulas for their sines, cosines or tangents (Theorems 3 or 4). Only two angles need be found. The third can be found as sum of all the angles is 180^{o}. The angles can also be found by using the cosine law but it is usually avoided unless a, b, c are small quantities, otherwise it involves huge calculations.

Case 2

Given *b, c* and the angle A

Let b denote the greater side. Then using the formula

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$

We can find the angle $\frac{B-C}{2}$. We can also find $\frac{B+C}{2}$ as $\frac{B+C}{2}=90^{o}-\frac{A}{2}$.

$\therefore$ We can find all the angles. The third side can be found by the sine law.

Case 3

Given *b, c* and angle B

By using cosine law we have

$$b^2=c^2+a^2-2ac\cos B$$

$$a^2-2ac\cos B+c^2\cos^2 B=b^2-c^2+c^2\cos^2 B$$

$$=b^2-c^2(1-\cos^2 B)$$

$$(a-c\cos B)^2=b^2-c^2\sin^2 B$$

or, $$a=c\cos B\pm\sqrt{b^2-c^2\sin^2 B}$$

This equation helps us to determine a. Other angles can be found by using the sine law.

Case 4

Given *a*, B, C

$$A=180^{o}-(B+C)\qquad\therefore\ \text{A is known.}$$

The other sides can be found by using the sine formula.

Case 5

Given a Right angle Triangle

If the triangle is a right triangle, it is simple to solve the triangle.

Suppose A = 90^{o}.

Suppose B and b are known then $\frac{b}{\sin B}=\frac{a}{\sin A}=\frac{a}{\sin 90^{o}}$ $\therefore$ a can be found.

Also $a=\sqrt{b^2+c^2}$ $\therefore$ c can be found.

Similarly, we can solve the triangle in other cases.

Example 5.1

Given a = 15, b = 36, c = 39, find $\sin\frac{A}{2}$.

Solution

$$2s=a+b+c=90$$

$$\therefore \quad \sin\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{bc}} = \sqrt{\frac{(45-36)\,(45-39)}{36\times 39}}$$

$$= \sqrt{\frac{9\times 6}{36\times 39}} = \sqrt{\frac{1}{26}}$$

Example 5.2

If the sides a, b, c of a triangle are in AP, prove that $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in AP.

Solution

We have to show that

$$\cot\frac{A}{2} + \cot\frac{C}{2} = 2\cot\frac{B}{2}$$

$$\cot\frac{A}{2} + \cot\frac{C}{2} = \frac{s-a}{r} + \frac{s-c}{r} = \frac{2s-(a+c)}{r}$$

$$= \frac{2s-2b}{r} = 2\left(\frac{s-b}{r}\right) = 2\cot\frac{B}{2}$$

($\because \;\; a + c = 2b$ as a, b, c are in AP).

SAQ 1

(a) Solve the triangle ABC given

(i) $a = 20$ cm, $b = 30$ cm, $c = 21$ cm

(ii) $a = 257$ cm, $b = 229.2$ cm, $c = 181.2$ cm

(iii) $c = 473$ cm, $A = 72^{o}\ 43'$, $B = 64^{o}\ 23'$

(iv) $a = 152$ cm, $B = 53^{o}$, $A = 80^{o}$

(v) $a = \sqrt{3} + 1$, $b = \sqrt{3} - 1$, $C = 60^{o}$

(vi) $a = 100$, $c = 100\sqrt{2}$, $A = 30^{o}$

(vii) $b = 50$, $c = 63$, and $B = 54^{o}$

(b) Given $a = 18$, $b = 24$, $c = 30$, find $\tan A, \sin B, \tan\frac{C}{2}$.

(c) Prove that

$$\frac{a+b}{c} = \frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}} \quad \text{and} \quad \frac{a-b}{c} = \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\frac{C}{2}}.$$

(d) Solve the right triangle where $\angle C = 90^{o}$

(i) $a = 43$, $A = 53^{o}$

(ii) $c = 6.5$, $A = 26^{o}$

(iii) $a = 50.4$, $b = 26.2$

(iv) $a = 412$, $c = 610$

(v) $b = 3.3$, $c = 4.4$.

5.4 HEIGHTS AND DISTANCES

Definition 1

Suppose P and Q are the two points, Q being at a higher level than P. Let PB be the horizontal line drawn through P to meet the vertical line through Q at B. The angle BPQ is called the **angle of elevation** of the point Q as seen from P. Draw QA parallel to PB, so that QA is a horizontal line passing through Q. The angle AQP is called the **angle of depression** of the point P as seen from Q.

Figure 5.3

Trigonometry enables us to find the heights of objects and the distances between points, the actual measurements of these being difficult or impossible.

Example 5.3

A pole is broken by the wind, the top struck the ground at an angle of 30^o and at a distance of 10 m from the foot of the pole. Find the height of the pole.

Figure 5.4

Solution

Let ABC be the pole. When broken at B by the wind, let its top A strike the ground at O so that $\angle$ BOC = 30^{o} and OC = 10 m and OB = AB.

Now, $$\frac{BC}{10} = \tan 30^{o} = \frac{1}{\sqrt{3}} \Rightarrow BC = \frac{10}{\sqrt{3}} \text{ m}.$$

$$\frac{OB}{10} = \sec 30^{o} \Rightarrow OB = \frac{20}{\sqrt{3}} \text{ m}$$

Height of the pole = AB + BC

$$= OB + BC = \frac{20}{\sqrt{3}} + \frac{10}{\sqrt{3}} = \frac{30}{\sqrt{3}} \text{ m}$$

Example 5.4

A person standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is 60^{o}, when he retires 100 m from the bank, he finds the angle to be 30^{o}. Find the height of the tree and the breadth of the river.

Figure 5.5

Solution

Let h = AB be the height of the tree and CB = x metres be the breadth of the river.

$$\therefore \angle BCA = 60^{o}.$$

Let D be the 2nd position of the man. Then $\angle ADB = 30^{o}$.

Now $$\frac{AB}{CB} = \frac{h}{x} = \tan 60^{o} = \sqrt{3}$$

$$\therefore h = \sqrt{3}\, x \qquad \text{. . . (5.2)}$$

Again from the triangle ADB we have

$$\frac{AB}{DB} = \frac{h}{100 + x} = \tan 30^{o} = \frac{1}{\sqrt{3}}$$

$$\sqrt{3}\, h = 100 + x \qquad \text{. . . (5.3)}$$

From Eqs. (5.2) and (5.3), we have

$$\sqrt{3}\,\sqrt{3}\, x = 100 + x$$

i.e. $3x = 100 + x$

i.e. $x = 50$, then $h = 50\sqrt{3}$ m

Example 5.5

From the top of a cliff 150 m high the angle of depressions of two boats which are due north of the observer are 60^{o} and 30^{o}. Find the distance between them.

Figure 5.6

Let AB = 150 m be the height of the cliff and let D and C be the position of the two boats.

$$\angle \text{BDA} = 30^{o}, \ \angle \text{BCA} = 60^{o}$$

Let $\text{BC} = y, \quad \text{CD} = x$

then $\dfrac{y}{150} = \cot 60^{o} = \dfrac{1}{\sqrt{3}}$ (From the right angled triangle ACB)

i.e. $y = \dfrac{150}{\sqrt{3}}$ m . . . (5.4)

and $\dfrac{x + y}{150} = \cot 30^{o} = \sqrt{3}$ (From the right angled triangle ADB)

i.e. $x + y = 150\sqrt{3}$

$$\therefore \quad x = 150\sqrt{3} - \frac{150}{\sqrt{3}} = 150\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) = 173.2 \text{ m}$$

Example 5.6

The angle of elevation of a tower from a point A due south of it is x and from a point B due east of A is y. If AB = l, show that h, the height of the tower is given by $h^2 (\cot^2 y - \cot^2 x) = l^2$.

Solution

Let OP be the tower of height h. In right angled triangle OAP, $\angle \text{OAP} = x$

$$\Rightarrow \quad \frac{\text{OA}}{h} = \cot x$$

$$\Rightarrow \quad \text{OA} = h \cot x$$

In right angled triangle OBP, $\angle \text{OBP} = y$

$$\therefore \quad \frac{\text{OB}}{h} = \cot y \Rightarrow \text{OB} = h \cot y$$

In right angled triangle OAB

$$AB^2 + OA^2 = OB^2$$

i.e. $$l^2 + h^2 \cot^2 x = h^2 \cot^2 y$$

i.e. $$h^2 (\cot^2 y - \cot^2 x) = l^2$$

Figure 5.7

SAQ 2

(a) A vertical tower stands on a horizontal plane and is surmounted by a vertical flag-staff of height h. At a point on the plane, the angle of elevation of the bottom of the flag-staff is α and that of the top of the flag-staff is β. Prove that the height of the tower is $\dfrac{h \tan \alpha}{\tan \beta - \tan \alpha}$.

(b) The angle of elevation of a ladder leaning against a house is 58^o and the foot of the ladder is 9.6 m from the house. Find the length of the ladder.

(c) At a point A, the angle of elevation of a tower is such that its tangent is $\dfrac{5}{12}$. On walking 240 m nearer the tower, the tangent of the angle of elevation is $\dfrac{3}{4}$. Find the height of the tower.

SAQ 3

(a) At the foot of a mountain the elevation of its summit is 45^o, after ascending 1000 m towards the mountain up a slope of 30^o inclination, the elevation is found to be 60^o. Find the height of the mountain.

(b) From the top of a cliff 60 m high, the angles of depression of the top and bottom of a tower are observed to be 30^o and 60^o respectively. Find the height of the tower.

(c) The elevation of a tower at a station A due north of it is α and at a station B due west of A is β. Prove that its altitude is

$$\frac{\text{A B} \sin \alpha \sin \beta}{\sqrt{\sin^2 \alpha - \sin^2 \beta}}$$

SAQ 4

(a) A 5 m pole placed vertically on a hill side casts a shadow of 7 m straight down the slope. At the top of the shadow, the angle subtended by the pole is 35^o. Find the elevation of the sun and the angle made by the hill side with the horizontal.

(b) The breadth of a street between two houses is 9 m and the angle of depression of the top of one as observed from the top of the other which is 12 m high is 30^o. Find the height of the other house.

(c) The horizontal distance between two towers is 60 m and the angular depression of the top of the 1^{st} as seen from the top of the 2^{nd} which is 150 m high is 30^o. Find the height of the first.

5.5 SUMMARY

- Relation between the Sides and Angles of a triangle

  $$\frac{a}{\sin \text{A}} = \frac{b}{\sin \text{B}} = \frac{c}{\sin \text{C}} = 2R \text{ where } R \text{ is the circumradius of the triangle.}$$

  $$\cos \text{A} = \frac{b^2 + c^2 - a^2}{2bc}, \text{ similarly for cos B, cos C}$$

  $$\sin \frac{\text{A}}{2} = \sqrt{\frac{(s-b)\,(s-c)}{bc}}, \cos \frac{\text{A}}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

- Area of the triangle is $\Delta = \sqrt{s(s-a)\,(s-b)\,(s-c)}$
- Techniques of solving the triangle.

  (i) If the triangle is right angled, then the triangle can be solved by using the sine formula.

(ii) If three sides of the triangle are given, then A and B can be found by using $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ and similar formula for $\tan \frac{B}{2}$ and $C = 180 - (A + B)$.

(iii) If two sides and included angle are given (say *b*, *c*, *A*) then the triangle can be solved by using the formula $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$, we can find $\frac{B-C}{2}$. Also $\frac{B+C}{2} = \frac{180^\circ - A}{2}$. $\therefore$ We can find all the angles. The third side *a* can be found by the sine law.

(iv) When two angles and one side are given, then the third angle is known by subtracting the sum of the two given angles from 180° and the other two sides are known by using the sine formula.

(v) When two sides and the angle opposite to the greater side is given, then we get a unique solution, two solutions or no solutions.

5.6 ANSWERS TO SAQs

SAQ 1

(a) (i) $A = 41^\circ\ 40'$, $B = 94^\circ$, $C = 44^\circ\ 20'$

(ii) $A = 76^\circ\ 32'$, $B = 60^\circ\ 10'$, $C = 43^\circ\ 18'$

(iii) $a = 663.4$, $b = 626.6$, $C = 42^\circ\ 54'$

(iv) $b = 123.2$, $c = 112.8$, $A = 47^\circ$

(v) $c = \sqrt{6}$, $A = 105^\circ$, $B = 15^\circ$

(vi) $B = 15^\circ$, $C = 135^\circ$, $b = 50\ (\sqrt{6} - \sqrt{2})$

or $B = 105^\circ$, $C = 45^\circ$, $b = 50\ (\sqrt{6} + \sqrt{2})$

(vii) There is no solution.

(b) $\frac{3}{4}, \frac{4}{5}, 1$

(d) (i) $B = 37^\circ$, $b = 32$ (nearly), $c = 54$ (nearly)

(ii) $B = 64^\circ$, $b = 5.64$, $a = 2.85$ (nearly)

(iii) $A = 62^\circ\ 32'$, $B = 27^\circ\ 28'$, $c = 56.8$

(iv) $A = 42^\circ\ 29'$, $B = 47^\circ\ 32$, $b = 450$

(v) $A = 41^\circ$ (nearly), $B = 49^\circ$ (nearly), $a = 2.9$ (nearly)

SAQ 2

(b) 18.12 m.

(c) 225 m.

SAQ 3

(a) 1366 m.

(b) 40 m.

SAQ 4

(a) Elevation of the sun $= 36^{o}\ .\ 35'$, Angle made by the hill $= 1^{o}\ .\ 35'$.

(b) 6.804 m.

(c) 115.359 m approx.