UNIT 1 SURDS, LOGARITHMS AND QUADRATIC EQUATIONS

Structure

1.1 INTRODUCTION

In this unit, we shall be introducing the notion of surds, logarithms and quadratic equations. We shall confine ourselves to the study of basic laws of surds and logarithms and the various techniques of solving quadratic equations. The topics discussed in this unit represent the essential prerequisites for clear and better understanding of the course of science and engineering.

Objectives

After studying this unit, you should be able to

- introduce the notion of surds, and the laws of indices applicable to surds,
- reduce expression involving surd to an expression with rational denominator,
- introduce the notion of logarithms, antilogarithms and laws of logarithms,
- use the logarithms and antilogarithm tables,
- solve a given quadratic equation, and
- form a quadratic equation with given roots.

1.2 SURDS

Definition 1

Let *a* be a rational number and *n* be a positive integer such that $\sqrt[n]{a}$ is irrational then $\sqrt[n]{a}$ is called a surd or a radical of order *n* and *a* is called the radicand ($\sqrt[n]{a}$ is the n^{th} root of a, i.e. this is a number when raised to the power n gives a.)

A surd of order 2 is called a quadratic surd

A surd of order 3 is called a cubic surd

A surd of order 4 is called a biquadratic surd.

Remark

(i) $\sqrt[n]{a}$ is a surd when a is rational but $\sqrt[n]{a}$ is irrational.

(ii) When a is irrational, or $\sqrt[n]{a}$ is rational then $\sqrt[n]{a}$ is not a surd.

Example 1.1

Verify that $\sqrt{3}$ is a surd.

Solution

We know that, $\sqrt{3} = 3^{\frac{1}{2}}$

Clearly 3 is a rational, 2 is a positive integer, $\sqrt{3}$ is a quadratic surd.

Example 1.2

Verify that $\sqrt[4]{6}$ is a surd.

Solution

We know that, $\sqrt[4]{6} = 6^{\frac{1}{4}}$

In this example, 6 is a rational number, 4 is a positive integer, $\sqrt[4]{6}$ is irrational.

$\therefore$ $\sqrt[4]{6}$ is a bi-quadratic surd.

Example 1.3

Prove that $\sqrt[3]{8}$ is not a surd.

Solution

$\sqrt[3]{8}$ is not a surd as $\sqrt[3]{8} = 2$, which is rational.

Note that, $\sqrt{\pi}$ is not a surd as π is irrational.

Laws of Radicals

As surds can be expressed with fractional powers, the laws of indices are applicable to surds. Thus we have the following laws of radicals.

(i) $(\sqrt[n]{a})^n = a$

(ii) $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$

(iii) $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

(iv) $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$

(v) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$

Definition 2

A surd of order *n* is said to be in the simplest form if

(i) It has no fraction under the radical sign.

(ii) The radicand has no factor with exponent *n*.

(iii) The given surd is not equal to any surd of order lower than *n*.

Example 1.4

Show that the following surds are not in the simplest form.

(i) $\sqrt[3]{\frac{2}{3}}$ (ii) $\sqrt[6]{8}$

Solution

(i) $\sqrt[3]{\frac{2}{3}}$ is not in the simplest form as it has a fraction under the radical sign.

(ii) $\sqrt[6]{8}$ is also not in the simplest form, because,

$$\sqrt[6]{8} = (2^3)^{\frac{1}{6}} = 2^{\frac{1}{2}} = \sqrt{2}$$

$\therefore$ $\sqrt[6]{8}$ is a surd of order 6 and is equal to a surd of order 2. So it is also not in the simplest form.

Definition 3

A surd which has unity only as rational factor, the other being irrational is called a pure surd. If a surd has a rational factor other than unity, the other factor being irrational is called a mixed surd.

Example 1.5

$\sqrt[4]{3}$, $\sqrt[3]{5}$ are pure surds, but $2\sqrt{3}$, $\frac{4}{3}\sqrt[3]{35}$ are mixed surds.

Example 1.6

Express each of the following surd in the simplest form

(i) $\sqrt[3]{128}$

(ii) $\sqrt[4]{\frac{16}{27}}$

Solution

(i) $\sqrt[3]{128} = \sqrt[3]{64 \times 2} = \sqrt[3]{64}\,\sqrt[3]{2} = 4\sqrt[3]{2}$

(ii) $\sqrt[4]{\frac{16}{27}} = \frac{\sqrt[4]{16}}{\sqrt[4]{27}} = \frac{2}{\sqrt[4]{3^3}} = \frac{2\,\sqrt[4]{3}}{\sqrt[4]{3^3}\,\sqrt[4]{3}} = \frac{2\,\sqrt[4]{3}}{\sqrt[4]{3^4}} = \frac{2}{3}\sqrt[4]{3}$

Example 1.7

Convert $2\sqrt[3]{4}$ as a pure surd of order 6.

Solution

$$2\sqrt[3]{4} = 2.4^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} \,.\, 4^{\frac{1}{3}} = (2^3 \,.\, 4)^{\frac{1}{3}}$$

$$= (8 \times 4)^{\frac{1}{3}} = 32^{\frac{1}{3}} = \sqrt[3]{32}$$

Now $$\sqrt[3]{32} = (32)^{\frac{1}{3}} = \left[(32)^{\frac{6}{3}}\right]^{\frac{1}{6}} = \left[(32)^2\right]^{\frac{1}{6}}$$

$$= (1024)^{\frac{1}{6}} = \sqrt[6]{1024}$$

SAQ 1

(a) State, give reasons, which of the following are surds.

(i) $\sqrt{12}$; (ii) $\sqrt[5]{243}$; (iii) $\sqrt{3 + \sqrt{2}}$

(b) Express each of the following as a mixed surd in the simplest form

(i) $\sqrt{50}$; (ii) $\sqrt[4]{\dfrac{81}{8}}$; (iii) $\sqrt{192}$

(c) Express each of the following as a pure surd

(i) $5\sqrt[3]{4}$; (ii) $3\sqrt[4]{5}$

(d) Express $\dfrac{3}{2}\sqrt{8}$ as a pure surd of order 4.

1.2.1 Rationalisation of Surds

If the denominator of an expression is a surd it can be reduced to an expression with rational denominator. The process is known as rationalizing of the denominator. When an expression involving surds is required to be simplified, the denominator should be reduced to an integer.

Definition 4

$a + \sqrt{b}$ is called the conjugate of $a - \sqrt{b}$ and $a - \sqrt{b}$ is called the conjugate of $a + \sqrt{b}$.

$\therefore$ If the denominator in the surd is of the form $a \pm \sqrt{b}$, we multiply the numerator and the denominator by $a \mp \sqrt{b}$ respectively which reduces the denominator to a rational number.

Example 1.8

Simplify $\dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}$

Solution

$$\frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{\sqrt{3}+1} \cdot \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{(\sqrt{3}-1)^2}{3-1}$$

$$\frac{3+1-2\sqrt{3}}{2} = 2-\sqrt{3}$$

Example 1.9

If $$\frac{\sqrt{5}-1}{\sqrt{5}+1} + \frac{\sqrt{5}+1}{\sqrt{5}-1} = a + b\sqrt{5}\ .$$

Find *a* and *b*.

Solution

$$\text{L. H. S.} = \frac{(\sqrt{5}-1)\,(\sqrt{5}-1)}{(\sqrt{5}+1)\,(\sqrt{5}-1)} + \frac{(\sqrt{5}+1)\,(\sqrt{5}+1)}{(\sqrt{5}-1)\,(\sqrt{5}+1)}$$

$$= \frac{(\sqrt{5}-1)^2}{5-1} + \frac{(\sqrt{5}+1)^2}{5-1} = \frac{5+1-2\sqrt{5}+5+1+2\sqrt{5}}{4}$$

$$= \frac{12}{4} = 3$$

$$\therefore \quad a = 3, \quad b = 0$$

SAQ 2

(a) Simplify each of the following

(i) $\dfrac{\sqrt{5}-2}{\sqrt{5}+2} - \dfrac{\sqrt{5}+2}{\sqrt{5}-2}$; (ii) $\dfrac{\sqrt{7}+\sqrt{2}}{9+2\sqrt{14}}$

(iii) $\dfrac{4}{\sqrt{7}+\sqrt{3}}$; (iv) $\dfrac{4}{2+\sqrt{3}+\sqrt{7}}$

(v) $\dfrac{1}{\sqrt{3}+\sqrt{2}-\sqrt{5}}$; (vi) $\dfrac{\sqrt[6]{12}}{\sqrt{3}\,.\,\sqrt[3]{2}}$

(vii) $\dfrac{9\sqrt[3]{4}}{3\sqrt[3]{2}}$; (viii) $(3\sqrt{5}-5\sqrt{2})\,(4\sqrt{5}+3\sqrt{2})$

(b) If $$x = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}, \quad y = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$$

find the value of

(i) $x^2 + y^2$;

(ii) $x^2 + xy + y^2$; and

(iii) $x^3 + y^3$

Example 1.10

If $$x = \frac{\sqrt{a + 2\,b} + \sqrt{a - 2\,b}}{\sqrt{a + 2\,b} - \sqrt{a - 2\,b}}$$

prove that $b\,x^2 - a\,x + b = 0$, provided $b \neq 0$.

Solution

$$x = \frac{\sqrt{a + 2b} + \sqrt{a - 2b}}{\sqrt{a + 2b} - \sqrt{a - 2b}} \times \frac{\sqrt{a + 2b} + \sqrt{a - 2b}}{\sqrt{a + 2b} + \sqrt{a - 2b}}$$

$$= \frac{(\sqrt{a + 2b} + \sqrt{a - 2b})^2}{(a + 2b) - (a - 2b)} = \frac{a + 2b + a - 2b + 2\,\sqrt{a + 2b}\,\sqrt{a - 2b}}{4b}$$

$$= \frac{2\left[a + \sqrt{(a + 2b)\,(a - 2b)}\right]}{4b} = \frac{a + \sqrt{a^2 - 4b^2}}{2b}$$

$$2\,bx = a + \sqrt{a^2 - 4b^2}$$

$$2\,bx - a = \sqrt{(a^2 - 4b^2)}$$

$$\Rightarrow \quad (2bx - a)^2 = a^2 - 4b^2$$

$$\Rightarrow \quad 4b^2x^2 - 4abx + a^2 = a^2 - 4b^2$$

$$\Rightarrow \quad 4b^2x^2 - 4abx + 4b^2 = 0$$

$$\Rightarrow \quad 4b\,(bx^2 - ax + b) = 0$$

$$\Rightarrow \quad bx^2 - ax + b = 0$$

$$[\because\ b \neq 0]$$

1.3 LOGARITHMS

Definition 5

If $a > 0$ and $a \neq 1$, then logarithm of a positive number m to the base 'a' is defined as the index 'x' of that power of 'a' which equals m

i.e. $\log_a^m = x \quad \text{iff} \quad a^x = m$

where $m > 0, \quad a > 0, \quad a \neq 1$.

This, in turn, implies $a^{\log_a m} = m$.

The number 'a' is called the *base*. Logarithms to the base 10 are called **common logarithms** and the logarithms to the base 'e' are called ***natural logarithms***, where e is the number equal to

$$1 + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \frac{1}{1 \times 2 \times 3 \times 4} + \ldots$$

to infinity and is nearly equal to 2.7183.

The function f defined by $f(x) = \log_e x$ is called ***(natural) logarithmic function***. Usually, it is denoted by $\ln x$ or $\log x$.

The domain of the function defined by

$$f(x) = \log_a x, \quad a > 0, \quad a \neq 1$$

is the set of all positive reals and its range is the set of all reals. Logarithmic function is one-one. Its graph in the two cases (i) when $0 < a < 1$, (ii) when $a > 1$, is shown in Figure 1.1.

Figure 1.1

From the graph, it is clear that when $0 < a < 1$, $\log_a x$ is a decreasing function and when $a > 1$, $\log_a x$ is an increasing function. This means that

For $\quad 0 < a < 1,\ x > y > 0 \quad \text{iff} \quad \log_a x < \log_a y$

and for $\quad a > 1,\ x > y > 0 \quad \text{iff} \quad \log_a x > \log_a y$

Note : In all theoretical problems, when base is not mentioned, it is usually taken as '*e*' and in all numerical problems, when no base is mentioned, it is taken as 10.

Remark : For $a > 0,\ a \neq 1,\ a^0 = 1$ and $a^1 = a,$ therefore, $\log_a 1 = 0$ and $\log_a a = 1$.

Thus, $\quad \boldsymbol{\log_a 1 = 0}$ **and** $\boldsymbol{\log_a a = 1}$

where *a* is any positive number except unity.

1.3.1 Laws of Logarithms

Theorem 1

(i) **Prove that $\boldsymbol{\log_a (mn) = \log_a m + \log_a n}$, where $\boldsymbol{m, n > 0, a > 0, a \neq 1}$.**

Proof

Let $\quad \log_a m = x \Rightarrow a^x = m \quad \ldots (1.1)$

and $\quad \log_a n = y \Rightarrow a^y = n \quad \ldots (1.2)$

From Eqs. (1.1) and (1.2), we get

$$a^x a^y = mn \Rightarrow a^{x+y} = mn$$

$$\boxed{\Rightarrow \log_a (mn) = \log_a m + \log_a n}$$

This is known as **product formula**.

Cor.

The above result is capable of extension to the product of more than two numbers i.e.

$$\log_a (m_1 m_2 \ldots m_n) = \log_a m_1 + \log_a m_2 + \ldots + \log_a m_n$$

where $m_1, m_2, \ldots, m_n$ are positive real numbers and n is any natural number.

Proof

We prove the result by induction. For $n = 1$, the result is obviously true. Let the result be true for a positive integer k, i.e.

$$\log_a (m_1 m_2 \ldots m_k) = \log_a m_1 + \log_a m_2 + \ldots + \log_a m_k \quad \ldots (1.3)$$

Then $\quad \log_a (m_1 m_2 \ldots m_k m_{k+1}) = \log_a \{m_1 m_2 \ldots m_k\} + \log_a m_{k+1}$

$$= (\log_a m_1 + \log_a m_2 + \ldots + \log_a m_k) + \log_a m_{k+1}$$

Using Eq. (1.3)

$$= \log_a m_1 + \log_a m_2 + \ldots + \log_a m_k + \log_a m_{k+1}$$

So, the result is true for $n = k + 1$. Hence by induction, the result is true for all $n \in N$.

(ii) **Prove that $\boldsymbol{\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n}$, where $\boldsymbol{m, n > 0, a > 0, a \neq 1}$.**

Proof

Let $\quad \log_a m = x \Rightarrow a^x = m \quad \ldots (1.4)$

and $\quad \log_a n = y \Rightarrow a^y = n \quad \ldots (1.5)$

From Eqs. (1.4) and (1.5), we get

$$\frac{a^x}{a^y} = \frac{m}{n} \Rightarrow a^{x-y} = \frac{m}{n}$$

$$\Rightarrow \log_a \left(\frac{m}{n}\right) = x - y$$

$$\boxed{\Rightarrow \log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n}$$

This is known as **quotient formula**.

(iii) **Prove that $\boldsymbol{\log_a (m^n) = n \log_a m}$, where $\boldsymbol{m > 0, a > 0, a \neq 1}$.**

Proof

Let $\quad \log_a m = x \Rightarrow a^x = m$

$$\Rightarrow (a^x)^n = m^n \Rightarrow a^{nx} = m^n$$

$$\Rightarrow \log_a (m^n) = nx$$

$$\boxed{\Rightarrow \log_a (m^n) = n \log_a m}$$

This is known as **power formula**.

(iv) **Prove that $\boldsymbol{\log_{a^n} m = \frac{1}{n} \log_a m}$, where $\boldsymbol{m > 0, a > 0, a \neq 1}$.**

Proof

Let $\log_{a^n} m = x$

$$\Rightarrow (a^n)^x = m$$

$$\Rightarrow a^{nx} = m$$

$$\Rightarrow \log_a m = nx$$

$$\Rightarrow \log_{a^n} m = \frac{1}{n}\log_a m$$

(v) **Prove that $\log_a m = \dfrac{\log_b m}{\log_b a}$, where $m > 0$, $a, b > 0$, $a, b \neq 1$.**

Proof

Let $\log_a m = x \Rightarrow m = a^x$ . . . (1.6)

$\log_b m = y \Rightarrow m = b^y$ . . . (1.7)

and $\log_b a = z \Rightarrow a = b^z$ . . . (1.8)

From Eqs. (1.6) and (1.7), we have

$$a^x = b^y$$

$$\Rightarrow (b^z)^x = b^y \qquad \text{Using Eq. (1.8),}$$

$$\Rightarrow b^{zx} = b^y$$

$$\Rightarrow zx = y \Rightarrow x = \frac{y}{z}$$

(Note that $z = \log_b a \neq 0$ as $a \neq 1$)

$$\boxed{\Rightarrow \log_a m = \frac{\log_b m}{\log_b a}}$$

This is known as the **base changing formula**.

Cor.

$$\log_a m = \frac{\log_e m}{\log_e a}, \text{ i.e. } \log_a m = \frac{\log m}{\log a} \qquad \text{(Taking } b = e\text{)}$$

Cor.

Taking $m = b$ in the base changing formula, we have

$$\log_a b = \frac{\log_b b}{\log_b a}$$

or $$\boxed{\log_a b = \frac{1}{\log_b a}} \qquad (\because \log_b b = 1)$$

This is known as the reciprocal formula.

Cor.

$$\boxed{\log_a m \log_b a = \log_b m}$$

(vi) **Prove that if a and b are any two positive reals then $a^{\log b} = b^{\log a}$.**

Proof

$$a^{\log b} = b^{\log a}$$

Iff $\log (a^{\log b}) = \log (b^{\log a})$

i.e. iff $\log b \log a = \log a \log b$, which is true.

(a)	$\log_a m = x$ iff $m = a^x$, $m > 0$, $a > 0$, $a \neq 1$
(b)	$a^{\log_a m} = m$, in particular $e^{\log m} = m$
(c)	$\log_a a = 1$ and $\log_a 1 = 0$ where $a > 0$
(d)	$\log_a (mn) = \log_a m + \log_a n;\ m, n, a > 0, a \neq 1$
(e)	$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n;\ m, n, a > 0, a \neq 1$
(f)	$\log_a (m^n) = n \log_a m;\ m, a > 0, a \neq 1$
(g)	$\log_{a^n} m = \frac{1}{n} \log_a m;\ m, a > 0, a \neq 1$
(h)	$\log_a m = \frac{\log_b m}{\log_b a}$ and $\log_b a \log_a b = 1$
(i)	$a^{\log b} = b^{\log a};\ a, b > 0$
(j)	If a > 1, then $x > y > 0 \Leftrightarrow \log_a x > \log_a y$ and if $0 < a < 1$, then $x > y > 0 \Leftrightarrow \log_a x < \log_a y$

1.3.2 Common Logarithms

Characteristic and Mantissa

$$\begin{array}{ll}
\because\ 10^0 = 1, & \therefore\ \log 1 = 0 \\
\because\ 10^1 = 10, & \therefore\ \log 10 = 1 \\
\because\ 10^2 = 100, & \therefore\ \log 100 = 2 \\
\ldots\ \ldots\ \ldots\ \ldots & \ldots\ \ldots\ \ldots\ \ldots \\
\ldots\ \ldots\ \ldots\ \ldots & \ldots\ \ldots\ \ldots\ \ldots \\
\ldots\ \ldots\ \ldots\ \ldots & \ldots\ \ldots\ \ldots\ \ldots
\end{array}$$

Also,

$$\begin{array}{ll}
\because\ 10^{-1} = \frac{1}{10} = .1, & \therefore\ \log .1 = -1 \\
\because\ 10^{-2} = \frac{1}{100} = .01, & \therefore\ \log .01 = -2 \\
\ldots\ \ldots\ \ldots\ \ldots & \ldots\ \ldots\ \ldots\ \ldots \\
\ldots\ \ldots\ \ldots\ \ldots & \ldots\ \ldots\ \ldots\ \ldots \\
\ldots\ \ldots\ \ldots\ \ldots & \ldots\ \ldots\ \ldots\ \ldots
\end{array}$$

Thus we observe that if a number lies

(i) between 1 and 10, its logarithm lies between 0 and 1.

(ii) between 10 and 100, its logarithm lies between 1 and 2, . . . and so on.

(iii) between .1 and 1, its logarithm lies between – 1 and 0.

(iv) between .01 and .1, its logarithm lies between – 2 and – 1 . . . and so on.

Hence, we find that the logarithm of a number is not always integral. In fact, it consists of two parts, one integral and the other fractional.

Definition 6

The integral part of the logarithm of a number, after expressing the decimal part as **non-negative**, is called the **characteristic** and the **non-negative** decimal part is called **mantissa**.

Remember that mantissa is always non-negative, characteristics may be positive, negative or zero.

Rule to find Characteristic

(i) The characteristic of the logarithm of a number greater than 1 is **non-negative** and one less than the number of digits before the decimal point.

(ii) The characteristic of the logarithm of a number less than 1 is –ve and numerically one more than the number of zeros immediately after the decimal point.

Rule to find Mantissa

To find the mantissa of logarithm of a number (positive), consider the number without decimal point and take its first four significant figures. To find mantissa read the logarithmic table in a horizontal line lead by first two figures and in the vertical column headed by the third figure. Note down the number written at this place, then read, in the same horizontal line the number written in the mean difference column headed by the fourth figure and add it to the previous number. Sum of these two gives the required mantissa.

For example, to find manitssa of log (.1205), we shall take the number as 1205 and read in the horizontal line lead by 12 (first two figures) and headed by 0 (third figure), we find 0792 at that place. Now in the **same horizontal line**, we find the number 18 in the mean difference column headed by 5 (fourth figure), therefore, mantissa of log (.1205) is 0810 (0792 + 18).

Hence, $\log(.1205) = \bar{1}.0810$.

Note that log .01205, log 1.205, log 12.05, log 12050 etc. will have same mantissa and differ only in characteristic.

Remark

In case there are more than four figures, we shall consider only first four and approximate the fourth with the help of fifth figure. This means that

$$\log 1.5625 = \log 1.563$$

and $$\log 2353457 = \log 2353000.$$

Example 1.11

Find the characteristics and mantissa when the logarithm of a number is (i) 3.0784, and (ii) – 3.0784.

Solution

(i) The logarithm of a number is 3.0784, here decimal is positive.

$\therefore$ Characteristics = 3 and mantissa = 0.0784

(ii) The logarithm of a number is – 3.0784 = – 3 + (– 0.0784), here decimal part is negative and in order to make it positive, we subtract and add 1, so that

$$-3.0784 = -3 + (-0.0784) = -3 - 1 + (1 - 0.0784)$$

$$= -4 + 0.9216 = \bar{4}.9216.$$

The 'bar' placed over 4 denotes that only 4 is negative while the decimal part .9216 is positive.

$\therefore$ Characteristics = – 4 and mantissa = 0.9216.

1.3.3 Antilogarithms

We know that $y = \log_a x \;\Leftrightarrow\; x = a^y$, in particular $y = \log_{10} x \;\Leftrightarrow\; x = 10^y$.

We write 10^y as antilog$_{10}$, y or simply as antilog y. So, when y = log x is known, then to find x, we need to calculate antilog y.

Before we find antilog y, note that fractional part of y is non-negative. So we can find antilog 3.54 or antilog $\bar{3}.5482$, but we cannot find antilog (– 2.9634) directly. In such cases we proceed as

$$\text{Antilog}(-2.9634) = \text{antilog}(-3 + 3 - 2.9634)$$

$$= \text{antilog}(-3 + .0366)$$

$$= \text{antilog}(\bar{3}.0366).$$

To write antilog, we make use of table of antilogarithms and look for only the part after decimal using four places after decimal, in the same manner as we read mantissa.

To find antilog $\bar{3}.0366$, we read the line of .03 in the column headed by 6 (the third figure) and add to it the number in the mean difference column in the same horizontal line and headed by 6 (the fourth figure). We shall get 1086 + 1 = 1087 and then we place decimal point after the first digit and multiply by 10^{-3} (10 raised to the power equal to characteristics).

Hence

$$\text{Antilog } \bar{3}.0366 = 1.087 \times 10^{-3} = .001087.$$

Example 1.12

Evaluate $\log_{81} 27$.

Solution

Let $\log_{81} 27 = x \Rightarrow (81)^x = 27 \Rightarrow (3^4)^x = 3^3$

$$\Rightarrow 3^{4x} = 3^3 \Rightarrow 4x = 3 \Rightarrow x = \frac{3}{4}$$

$$\therefore \log_{81} 27 = \frac{3}{4}$$

Example 1.13

Prove that

$$\frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc} = 1$$

Solution

$$\text{L. H. S.} = \frac{1}{\frac{\log_c abc}{\log_c a}} + \frac{1}{\frac{\log_c abc}{\log_c b}} + \frac{1}{\frac{\log_c abc}{\log_c c}}$$

$$= \frac{\log_c a}{\log_c abc} + \frac{\log_c b}{\log_c abc} + \frac{\log_c c}{\log_c abc}$$

$$= \frac{\log_c a + \log_c b + \log_c c}{\log_c abc} = \frac{\log_c abc}{\log_c abc} = 1 = \text{R. H. S.}$$

Example 1.14

If $\log_{10} 2 = 0.30103$, evaluate $\log_{10}\left(\frac{1000}{256}\right)$.

Solution

$$\log_{10}\left(\frac{1000}{256}\right) = \log_{10}(1000) - \log_{10}(256) = \log_{10}(10^3) - \log_{10} 2^8$$

$$= 3\log_{10} 10 - 8\log_{10} 2 = 3 \times 1 - 8\,(.30103)$$

$$= 3 - 2.40824 = .59176$$

Example 1.15

Find the seventh root of (.0043).

Solution

Let $$x = (.0043)^{\frac{1}{7}}$$

$$\Rightarrow \log x = \frac{1}{7}\log(.0043)$$

$$= \frac{1}{7}[\bar{3}.6335]$$

$$= \frac{1}{7}[-3 + .6335] = \frac{1}{7}[-7 + 4.6335]$$

$$= -\frac{7}{7} + \frac{4.6335}{7} = -1 + .6619 = \bar{1}.6619$$

$$\therefore \quad x = \text{anti}\log(\bar{1}.6619) = .4591$$

SAQ 3

(a) Evaluate $\log_{.5} 256$.

(b) If $a^2 + b^2 = 7ab$, prove that $2\log(a+b) = \log 9 + \log a + \log b$

(c) Prove that $x^{\log y - \log z} \times y^{\log z - \log x} \times z^{\log x - \log y} = 1$

(d) Prove that $\log_{10}^2 + 16\log_{10}\frac{16}{15} + 12\log_{10}\frac{25}{24} + 7\log_{10}\frac{81}{80} = 1$

SAQ 4

(a) Find the characteristics of the logarithms of

(i) 0.003741; and (ii) 0.3741

(b) Given log 2 = 0.3010, log 3 = .4771

(i) Find $\log (.0036)^{\frac{1}{4}}$

(ii) Solve the equation $5^{x-1} = 6^{2-x}$

(c) Evaluate

(i) $(3.142)^3 \; (.078)^{\frac{1}{3}}$

(ii) $\sqrt{\dfrac{.0075 \times .014}{80.35}}$

SAQ 5

(a) If $a = \log_a \; yz$, $b = \log_b \; zx$, $c = \log_c \; xy$, show that

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 1$$

(b) If x, y, z are three distinct numbers and $\dfrac{\log x}{y - z} = \dfrac{\log y}{z - x} = \dfrac{\log z}{x - y}$,

show that

(i) $x \, y \, z = 1$, and (ii) $x^x \; y^y \; z^z = 1$

(c) Without using log tables, prove that

$$3 \log_{10} (1.5) + \log_{10} (240) - 2 \log_{10} 9 = 1$$

(d) If log 3 = .4771213, find the position of the first significant figure in 3^{-32} .

1.4 QUADRATIC EQUATIONS

Definition 8

An expression of the type $ax^2 + bx + c$, where $a, b, c \in R, a \neq 0$ is called a quadratic polynomial with real coefficients and an equation of the form $ax^2 + bx + c = 0$, where $a, b, c \in R, a \neq 0$ is called a quadratic equation.

Definition 9

A number α (real or complex) is said to be a root of the equation $ax^2 + bx + c = 0$ iff $a\alpha^2 + b\alpha + c = 0$. The set of all roots of an equation is called the solution set of the equation.

1.4.1 Solution of General Quadratic Equation

Let the general equation be

$$ax^2 + bx + c = 0, \quad a \neq 0$$

Case 1

If $ax^2 + bx + c$ can be factorised into two linear factors say $A_1x + B_1$, $A_2x + B_2$ then we have $(A_1x + B_1)(A_2x + B_2) = 0$ i.e. $A_1x + B_1 = 0$, $A_2x + B_2 = 0$, i.e. $\frac{-B_1}{A_1}, \frac{-B_2}{A_2}$ are the two roots ($\because \; ab = 0$ iff $a = 0$ or $b = 0$).

Case II

$ax^2 + bx + c = 0, \quad a \neq 0$ can be written as

$$ax^2 + bx = -c$$

i.e. $$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

i.e. $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$ (Adding $\left(\frac{b}{2a}\right)^2$ to both the sides).

i.e. $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

i.e. $$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

i.e. $$x = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$$

$\therefore$ the roots of the quadratic equation are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Let these be denoted by α and β.

Then $$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The nature of roots will depend upon $b^2 - 4ac$. $b^2 - 4ac$ is called discriminant of the equation and is usually denoted by Δ. Δ is a real number.

Case I

If $\Delta > 0$, then the two roots are real and unequal as $\sqrt{\Delta}$ is real.

Case II

If $\Delta = 0$, then $\sqrt{\Delta} = 0$, the roots are real and equal, $\alpha = \beta = -\dfrac{b}{2a}$.

Case III

If $\Delta < 0$, then $\sqrt{\Delta}$ is non-real, hence the two roots are non-real and unequal.

Let $\Delta = -K$, $K > 0$, then the two roots are

$$\frac{-b \pm \sqrt{-K}}{2a} = \frac{-b \pm \sqrt{K}\, i}{2a} \quad (\because\ i = \sqrt{-1})$$

$\therefore$ the two roots are unequal and complex conjugates.

Remark

It is worth noticing that the nature of roots depend upon a, b, c and Δ. If a, b, c are not real numbers, nothing can be said about the nature of the roots even if $\Delta > 0$.

1.4.2 Relation between Roots and Coefficients of Quadratic Equation

(i) $$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = \frac{-b}{a}$$

i.e. sum of the roots $= -\dfrac{\text{Coefficients of } x}{\text{Coefficients of } x^2}$

(ii) $$\alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \left(\frac{-b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2$$

$$= \frac{b^2}{4a^2} - \frac{b^2 - 4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

i.e. product of the roots $= \dfrac{\text{Constant term}}{\text{Coefficients of } x^2}$.

Definition 10 : Symmetric Functions of Roots

If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0,\ a \neq 0$, then the expression of the form $\alpha + \beta,\ \alpha\beta,\ \alpha^2 \pm \beta^2 \ldots$ are called functions of α and β and is called symmetric if it remains unchanged by interchanging α and β. Otherwise it is called unsymmetric e.g. $\alpha + \beta,\ \alpha\beta,\ \alpha^2 + \beta^2$ are symmetric whereas $\alpha^2 - \beta^2$ is unsymmetric function.

We will now evolve a method to find the value of a symmetric function of roots of a quadratic equation.

(i) Express the symmetric function first in terms of $\alpha + \beta$ and $\alpha\beta$.

(ii) Then substitute the value of $\alpha + \beta$ and $\alpha\beta$.

Example 1.16

Solve the equation $3x^2 - 4x - 4 = 0$.

Solution

i.e. $(3x + 2)(x - 2) = 0$

i.e. $3x + 2 = 0$ or $x - 2 = 0$

i.e. $x = -\dfrac{2}{3}$ or $x = 2$

Hence $-\dfrac{2}{3}$ and 2 are the two roots.

Example 1.17

Discuss the nature of the roots of the equation

(i) $x^2 - 4x + 4 = 0$; (ii) $3x^2 + 5x + 7 = 0$

Solution

(i) $x^2 - 4x + 4 = 0$

$\Delta = (-4)^2 - 4 . 1 . 4 = 16 - 16 = 0$

$\therefore$ The roots are real and equal.

(ii) $3x^2 + 5x + 7 = 0$

$\Delta = (5)^2 - 4 . 3 . 7 = 25 - 84 = -59 < 0$

$\therefore$ The roots are a pair of complex conjugates.

Example 1.18

Find the roots of the equation

$$3x^2 - 2x - 6 = 0$$

Solution

The roots are $\dfrac{2 \pm \sqrt{4 - 4.3(-6)}}{2.3} = \dfrac{2 \pm \sqrt{4 + 72}}{6}$

$$= \frac{2 \pm \sqrt{76}}{6} = \frac{1 \pm \sqrt{19}}{3}$$

Example 1.19

If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, find the value of

(i) $\alpha^3\beta + \alpha\beta^3$; (ii) $\dfrac{\alpha^2}{\beta^2} + \dfrac{\beta^2}{\alpha^2}$

Solution

$$\left(\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.\right)$$

(i) $\alpha^3\beta + \alpha\beta^3 = \alpha\beta\,(\alpha^2 + \beta^2) = \alpha\beta\left\{(\alpha+\beta)^2 - 2\alpha\beta\right\}$

$$= \frac{c}{a}\left[\frac{b^2}{a^2} - \frac{2c}{a}\right] = \frac{c}{a}\left(\frac{b^2 - 2ac}{a^2}\right)$$

$$= \frac{c\,(b^2 - 2ac)}{a^3}$$

(ii) $\dfrac{\alpha^2}{\beta^2} + \dfrac{\beta^2}{\alpha^2} = \dfrac{\alpha^4 + \beta^4}{\alpha^2\beta^2} = \dfrac{(\alpha^2 + \beta^2)^2 - 2\,\alpha^2\,\beta^2}{\alpha^2\beta^2}$

$$= \frac{[(\alpha+\beta)^2 - 2\alpha\beta]^2 - 2\alpha^2\beta^2}{\alpha^2\beta^2}$$

$$= \frac{a^2}{c^2}\left[\left(\frac{b^2}{a^2} - \frac{2c}{a}\right)^2 - \frac{2c^2}{a^2}\right]$$

$$= \frac{a^2}{c^2}\left[\frac{b^4}{a^4} + \frac{4c^2}{a^2} - \frac{4b^2\,c}{a^3} - \frac{2c^2}{a^2}\right]$$

$$= \frac{a^2}{c^2}\left[\frac{b^4 + 4a^2c^2 - 4b^2ca - 2a^2c^2}{a^4}\right]$$

$$= \frac{1}{a^2c^2}\left[b^4 + 2a^2c^2 - 4b^2ac\right]$$

SAQ 6

(a) Solve the equation

$$3x^2 - 4x = 0$$

(b) If α, β are the roots of $ax^2 + bx + c = 0$, find the value of $(a\alpha + b)^{-3} + (a\beta + b)^{-3}$

[**Hint :** $a\alpha^2 + b\alpha + c = 0 \Rightarrow a\alpha + b = -\dfrac{c}{\alpha}$]

(c) If a and b are the roots of $x^2 - px + 2 = 0$ and c and d are the roots of $x^2 - p'x + q' = 0$, show that
$(a - c)\,(b - d) + (a - d)\,(b - c) = 2\,(q + q') - pp'$

(d) If one root of the equation $ax^2 + bx + c = 0$ is square of the other, prove that

$$b^3 + ac\,(c + a) = 3abc$$

SAQ 7

(a) If the equation $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ have both roots common, prove that

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

(b) Find the value of k so that both the equations $x^2 + kx + 64 = 0$ and $x^2 - 8x + k = 0$ may have real roots.

(c) If p and q are the roots of the equation $2x^2 - 6x + 3 = 0$, find the value of $p^3 + q^3 - 3pq\,(p^2 + q^2) + 3pq\,(p + q)$.

We can find the quadratic equation if the two roots are given in the following way.

Example 1.20

Form an equation whose roots are $\frac{-p^2}{q}, \frac{-q^2}{p}$ where p, q are the roots of $3x^2 + 6x + 2 = 0$.

Solution

Now $\qquad p + q = -\frac{6}{3} = -2 \qquad$ . . . (1.9)

and $\qquad pq = \frac{2}{3} \qquad$ . . . (1.10)

Let $\quad S$ = Sum of the roots of the equation

$$= \frac{-p^2}{q} - \frac{q^2}{p} = \frac{-(p^3 + q^3)}{pq} = -\frac{[(p+q)^3 - 3pq\,(p+q)]}{pq}$$

$$= -\frac{(-2)^3 - 3\,.\,\frac{2}{3}\,(-2)}{\frac{2}{3}} = 6$$

P = Product of the roots

$$= \left(\frac{-p^2}{q}\right)\left(-\frac{q^2}{p}\right) = pq = \frac{2}{3}$$

$\therefore$ The required equation is

$$x^2 - 6x + \frac{2}{3} = 0 \quad \text{i.e.} \quad 3x^2 - 18x + 2 = 0$$

Solution of equations which can be reduced to quadratic equations.

Example 1.21

Find the roots of the equation

$$(x^2 - 5x)^2 - 30\,(x^2 - 5x) - 216 = 0$$

Solution

Let $x^2 - 5x = y$ then the equation becomes

$$y^2 - 30y - 216 = 0$$

The two roots of this equation are 36, – 6.

If $\quad x^2 - 5x = 36 \Rightarrow x^2 - 5x - 36 \Rightarrow 0 \Rightarrow x = 9, -4$

and $\quad x^2 - 5x = -6 \Rightarrow x^2 - 5x + 6 = 0 \Rightarrow x = 3, 2$

$\therefore$ The four roots are 9, – 4, 3, 2.

SAQ 8

(a) Solve the equation

$$\sqrt{x^2 + 3x + 32} + \sqrt{x^2 + 3x + 5} = 9$$

(b) Find an equation whose roots are the squares of the reciprocals of the roots of the equation $ax^2 + bx + c = 0, \quad a \neq 0$.

1.5 SUMMARY

Surds

(i) $\sqrt[n]{a}$ is a surd if *a* is a rational number, *n* is a positive integer such that $\sqrt[n]{a}$ is irrational.

(ii) Law of indices are applicable to surds.

Logarithms

(i) $a > 0, a \neq 1, m > 0$ then $\log_a m = x$ iff $m = a^x$

(ii) $\log_a mn = \log_a m + \log_a n$

$$\log_a \frac{m}{n} = \log_a m - \log_a n$$

$$\log_a (m^n) = n \log_a m$$

(iii) $\log_a m = \dfrac{\log_b m}{\log_b a}\ a \neq 1, b \neq 1$

(iv) Logarithm to the base *e* are called natural and logarithms to the base 10 are called common logarithms.

Quadratic Equation

(i) Roots of a quadratic equation $ax^2 + bx + c = 0,\ a \neq 0$ are

$\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

(ii) Sum of the roots $= \dfrac{-\text{Coefficients of } x}{\text{Coefficients of } x^2}$, and

Product of the roots $= \dfrac{\text{Cons tan t term}}{\text{Coefficients of } x^2}$

(iii) Equation of a quadratic with given roots is

$x^2 - (\text{Sum of the roots})\, x +$ Product of the roots = 0

(iv) If $b^2 - 4ac =$ +ve, then roots are real and unequal.

= 0, then roots are equal and real.

= –ve, then roots are complex conjugate.

1.6 ANSWERS TO SAQs

SAQ 1

(a) (i) Yes,

(ii) No, since $\sqrt[5]{243} = 3$ (rational),

(iii) No, as $3 + \sqrt{2}$ is irrational.

(b) (i) $5\sqrt{2}$,

(ii) $\dfrac{3}{2}\sqrt[4]{2}$,

(iii) $8\sqrt{3}$

(c) (i) $\sqrt[3]{500}$

(ii) $\sqrt[4]{405}$

(d) $\sqrt[4]{324}$

SAQ 2

(a) (i) $-8\sqrt{5}$

(ii) $\dfrac{1}{5}(\sqrt{7} - \sqrt{2})$

(iii) $\sqrt{7} - \sqrt{3}$

(iv) $\dfrac{1}{3}(2\sqrt{3} + 3 - \sqrt{21})$

(v) $\dfrac{1}{12}(3\sqrt{2} + 2\sqrt{3} + \sqrt{30})$

(vi) $\sqrt[3]{\frac{1}{3}} = \frac{1}{\sqrt[3]{3}}$

(vii) $3\sqrt[3]{2}$

(viii) $30 - 11\sqrt{10}$

(b) (i) 98,

(ii) 99,

(iii) 970.

SAQ 3

(a) (– 8)

SAQ 4

(a) (i) $\bar{3}$,

(ii) $\bar{1}$

(b) (i) $\bar{1}.3891$,

(ii) x = 1.527 nearly.

(c) (i) 49.83,

(ii) 0.001143.

SAQ 5

(d) The first significant figure occurs in the 16th place.

SAQ 6

(a) $x = 0, \frac{4}{3}$

(b) $\left(\frac{b^3 - 3abc}{a^3c^3}\right)$

SAQ 7

(a) k = 16

(b) (0)

SAQ 8

(a) $x = -4, 1$