UNIT 3 BINOMIAL THEOREM AND COMPUTER MATHEMATICS

Structure

3.1 INTRODUCTION

In this unit, we shall be introducing Binomial Theorem and Computer Mathematics. Binomial Theorem for Positive Integral Index is discussed in detail. Special emphasis is given on how to remember Binomial expansion as well as special terms used in Binomial expression. Applications of Binomial Theorem are also given.

Computer is affecting practically every activity of human life. In this, we are confining ourselves to algorithms which are nothing but a sequence of instructions to solve specific problem. This unit also describes the methods of presenting the logic with the help of flowcharts.

Objectives

After studying this unit, you should be able to

- conceptualise Binomial Theorem,
- remember the Binomial expression correctly,
- apply Binomial Theorem in engineering areas, and
- develop algorithms and flow charts for any process or system.

3.2 BINOMIAL THEOREM

3.2.1 An Algebraic Expression

An algebraic expression consisting of two terms is called a Binomial expression,

e.g. $x + 3a, \frac{5x^2}{7} - \frac{3}{2x^2}, x^3 - \frac{3}{x}$ etc. are all Binomial expressions.

The Binomial theorem is a general algebraical formula by means of which any power of a binomial expression can be expressed as a series.

Theorem 1 : Binomial Theorem for a Positive Integral Index

If n is positive integer then

$$(x+a)^n = {}^nC_0\,x^n + {}^nC_1\,x^{n-1}a + {}^nC_2\,x^{n-2}a^2 + \ldots + {}^nC_r\,x^{n-r}a^r + \ldots + {}^nC_n\,a^n$$

where n and a are any numbers.

Proof

We will prove it by mathematical induction.

Step I

By actual multiplication

$$(x+a)^2 = x^2 + 2x\,a + a^2$$

$$= {}^2C_0\,x^2 + {}^2C_1\,x\,a + {}^2C_2\,a^2$$

$$(x+a)^3 = x^3 + 3x^2a + 3xa^2 + a^3$$

$$= {}^3C_0\,x^3 + {}^3C_1\,x^2a + {}^3C_2\,xa^2 + {}^3C_3a^3$$

This shows that the theorem is true for $n = 2$ and 3.

Step II

Let us assume that the theorem is true for $n = m$ where $m \in N$.

Then

$$(x+a)^m = {}^mC_0\,x^m + {}^mC_1\,x^{m-1}a + \ldots + {}^mC_{r-1}\,x^{m-r+1}a^{r-1} + {}^mC_r\,x^{m-r}a^r + \ldots + {}^mC_m\,a^m \qquad \ldots (3.1)$$

Now multiplying both the sides by $x + a$ we have

$$(x+a)^{m+1} = (x+a)\,[{}^mC_0\,x^m + {}^mC_1\,x^{m-1}a + {}^mC_2\,x^{m-2}a^2 + \ldots + {}^mC_{r-1}\,x^{m-r+1}a^{r-1} + {}^mC_r\,x^{m-r}a^r + \ldots + {}^mC_m\,a^m]$$

$$= {}^mC_0\,x^{m+1} + {}^mC_0\,x^m a + {}^mC_1\,x^m a + {}^mC_1\,x^{m-1}a^2 + {}^mC_2\,x^{m-1}a^2 + \ldots + {}^mC_{r-1}\,x^{m-r+1}a^r + {}^mC_r\,x^{m-r+1}a^r + \ldots + {}^mC_m\,a^{m+1}$$

$$= {}^mC_0\,x^{m+1} + ({}^mC_0 + {}^mC_1)\,x^m a + ({}^mC_1 + {}^mC_2)\,x^{m-1}a^2 + \ldots + ({}^mC_{r-1} + {}^mC_r)\,x^{m-r+1}a^r + \ldots + {}^mC_m\,a^{m+1} \qquad \ldots (3.2)$$

Since $\qquad {}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$

And $\qquad {}^mC_0 = {}^{m+1}C_0 = 1$

$$^mC_m = {}^{m+1}C_{m+1} = 1$$

From Eq. (3.2), we get

$$(x+a)^{m+1} = {}^{m+1}C_0\,x^{m+1} + {}^{m+1}C_1\,x^m a + {}^{m+1}C_2\,x^{m-1}a^2 + \ldots + {}^{m+1}C_r\,x^{m-r+1}a^r + \ldots + {}^{m+1}C_{m+1}\,a^{m+1}$$

This shows that the theorem is true for $n = m + 1$. But by actual multiplication, we have seen that the theorem is true for $n = 2, 3$.

$\therefore$ It must be true for $n = 3 + 1 = 4$ and therefore for $n = 4 + 1 = 5$ and so on.

Hence the theorem is true for all positive integral values of n.

How to Remember the Binomial Expansion

The following observations help us to remember the binomial expansion correctly :

(i) The expansion of $(x + a)^n$ has $n + 1$ terms. In other words, the number of terms is one more than n, the index of $x + a$.

(ii) In the successive terms of the expansion, the index of x goes on decreasing by unity, starting from n, then $n - 1$, . . . and ending with zero, on the contrary, the index of a goes on increasing by unity, starting from 0 then 1 . . . and ending with n.

(iii) In any term, the sum of indices of x and a is equal to n.

(iv) The binomial co-efficients can be remembered by observing the following known as Pascals Triangles.

Figure 3.1

Here we note that each row is bounded by 1 on both sides. Any entry in a row is the sum of two entries in the preceding row, one on the immediate left and the other on the immediate right.

Some Particular Cases of Binomial Expansion

(i) $(x - y)^n = {}^nC_0\, x^n - {}^nC_1\, x^{n-1}\, y + {}^nC_2\, x^{n-2}\, y^2 + \ldots + (-1)^n\, {}^nC_n\, y^n$

This is obtained by taking $a = (-y)$ in the expansion of $(x + a)^n$.

(ii) $(1 + x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + {}^nC_n\, x^n$

(iii) $(1 - x)^n = {}^nC_0 - {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + (-1)^n\, {}^nC_n\, x^n$

Some Special Terms in the Binomial Expansion

(i) In the expansion of $(x + a)^n$, the $(r + 1)^{\text{th}}$ term is ${}^nC_r\, x^{n-r}\, a^r$. This is called the general term.

(ii) In the expansion of $(x + a)^n$, the middle term is the $\left(\frac{n}{2} + 1\right)^{\text{th}}$ term if n is even. If n is odd, the two middle terms are $\left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+1}{2} + 1\right)^{\text{th}}$.

Example 3.1

Expand $(1 - x + x^2)^4$

Solution

Let $y = -x + x^2$

Then $(1 - x + x^2)^4 = (1 + y)^4$

$$= 1 + {}^4C_1\, y + {}^4C_2\, y^2 + {}^4C_3\, y^3 + {}^4C_4\, y^4$$

$$= 1 + 4\,(-x + x^2) + 6\,(-x + x^2)^2 + 4\,(-x + x^2)^3 + (-x + x^2)^4$$

$$= 1 + 4x\,(x-1) + 6x^2\,(x-1)^2 + 4x^3\,(x-1)^3 + x^4\,(x-1)^4$$

$$= 1 + 4x\,(x-1) + 6x^2\,(x^2 - 2x + 1) + 4x^3\,(x^3 - 3x^2 + 3x - 1)$$
$$+ x^4\,(x^4 - 4x^3 + 6x^2 - 4x + 1)$$

$$= 1 + 4x^2 - 4x + 6x^4 - 12x^3 + 6x^2 + 4x^6 - 12x^5 + 12x^4$$
$$- 4x^3 + x^8 - 4x^7 + 6x^6 - 4x^5 + x^4$$

$$= 1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$$

Example 3.2

(i) Find the co-efficient of x^{10} in the binomial expansion of $\left(2x^2 - \dfrac{3}{x}\right)^{11}$ where $x \neq 0$.

(ii) Prove that there is no term involving x^6.

Solution

(i) The general term in the expansion of $\left(2x^2 - \dfrac{3}{x}\right)^{11}$ is

$${}^{11}C_r\;(2x^2)^{11-r}\left(\frac{-3}{x}\right)^r$$

In this term the exponent of x is

$$2\,(11 - r) - r = 22 - 2r - r = 22 - 3r$$

This will be equal to 10 when $r = 4$.

$\therefore$ The co-efficient of that term is

$${}^{11}C_4\;2^{11-4}\;(-3)^4 = \frac{11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4} \times 2^7 \times 3^4$$

$$= 2^8 \times 3^5 \times 5 \times 11$$

(ii) The general term is of the form $k\,.\,x^{22-3r}$ and the index $22 - 3\,r$ never takes the value 6 for any integer value of r. Therefore, there is no term involving x^6.

Example 3.3

Use the Binomial Theorem to evaluate $(99)^4$.

Solution

$$(99)^4 = (100-1)^4 = (100)^4 - {}^4C_1\,(100)^3\,.1 + {}^4C_2\,(100)^2\,.1^2 - {}^4C_3\,(100).1^3 + {}^4C_4\,.1^4$$

$$= (100)^4 - 4\,(100)^3 + 6\,(100)^2 - 4\,(100) + 1$$

$$= 100000000 - 4000000 + 60000 - 400 + 1$$

$$= 100060001 - 4000400 = 96059601$$

Example 3.4

If P be the sum of the odd terms and Q the sum of the even terms in the expansion of $(x+a)^n$, prove that

$P^2 - Q^2 = (x^2 - a^2)^n$ and $4\,PQ = (x+a)^{2n} - (x-a)^{2n}$

Solution

$$(x+a)^n = x^n + {}^nC_1\,x^{n-1}a + {}^nC_2\,x^{n-2}a^2 + {}^nC_3\,x^{n-3}a^3 + {}^nC_4\,x^{n-4}a^4 + \dots$$

$$= (x^n + {}^nC_2\,x^{n-2}a^2 + {}^nC_4\,x^{n-4}a^4 + \dots)$$

$$+ ({}^nC_1\,x^{n-1}a + {}^nC_3\,x^{n-3}a^3 + \dots)$$

$$= P + Q \qquad \dots (3.3)$$

Changing a to $-a$ we have

$$(x-a)^n = (x^n + {}^nC_2\,x^{n-2}a^2 + {}^nC_4\,x^{n-4}a^4 + \dots)$$

$$- ({}^nC_1\,x^{n-1}a + {}^nC_3\,x^{n-3}a^3 + \dots)$$

$$= P - Q \qquad \dots (3.4)$$

Multiplying Eqs. (3.3) and (3.4), we have

$$(x+a)^n\,(x-a)^n = (P+Q)\,(P-Q)$$

i.e. $\quad [(x+a)\,(x-a)]^n = P^2 - Q^2$

i.e. $\quad (x^2 - a^2)^n = P^2 - Q^2$

and $\quad 4\,PQ = (P+Q)^2 - (P-Q)^2$

$$= (x+a)^{2n} - (x-a)^{2n}$$

Example 3.5

Using binomial theorem, prove that $6^n - 5n$ always leaves the remainder 1 when divided by 25.

Solution

Consider

$$6^n = (1+5)^n = 1 + {}^nC_1\,5 + {}^nC_2\,5^2 + {}^nC_3\,5^3 + \dots + {}^nC_n\,5^n$$

$$= 1 + 5n + {}^nC_2\,5^2 + {}^nC_3\,5^3 + \dots + 5^n$$

i.e. $\quad 6^n - 5n = 5^2\,[{}^nC_2 + {}^nC_3\,5 + \dots + 5^{n-2}] + 1$

$$= 25 \times \text{a positive integer} + 1$$

$\therefore\ 6^n - 5n$ leaves the remainder 1 when divided by 25.

Example 3.6

Prove that $\sum_{r=0}^{n} {}^nC_r\, 3^r = 4^n$

Solution

$$\sum_{r=0}^{n} {}^nC_r\, 3^r = {}^nC_0\, 3^0 + {}^nC_1\, 3^1 + {}^nC_2\, 3^2 + \ldots + {}^nC_r\, 3^r + \ldots + {}^nC_n\, 3^n$$

$$= (1+3)^n = 4^n$$

Example 3.7

Find the value of *r* if the co-efficient of $(2r+4)^{\text{th}}$ and $(r-2)^{\text{th}}$ terms in the expansion of $(1+x)^{18}$ are equal.

Solution

$$(1+x)^{18} = 1 + {}^{18}C_1\, x^{17} + {}^{18}C_2\, x^{16} + \ldots + x^{18}$$

$(2r+4)^{\text{th}}$ term in the expansion of $(1+x)^{18}$ is ${}^{18}C_{2r+3}\, x^{2r+3}$ and $(r-2)^{\text{th}}$ term is ${}^{18}C_{r-3}\, x^{r-3}$.

$\therefore$ The co-efficient of these terms are equal if either $2r+3 = r-3$ or $(2r+3)+(r-3) = 18$. The former is not possible since *r* is to be positive. The latter gives *r* = 6.

Example 3.8

Find the co-efficient of x^5 in the expansion of $(1+2x)^6\,(1-x)^7$.

Solution

$$(1+2x)^6 = 1 + {}^6C_1\,(2x) + {}^6C_2\,(2x)^2 + {}^6C_3\,(2x)^3$$

$$+ {}^6C_4\,(2x)^4 + {}^6C_5\,(2x)^5 + {}^6C_6\,(2x)^6$$

$$= 1 + 6\,(2x) + 15\,(4x^2) + 20\,(8x^3) + 15\,(16x^4) + 6\,(32x^5) + 64x^6$$

$$= 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$$

Also $\quad (1-x)^7 = 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7$

We have to find the co-efficient of x^5 in the product

$$(1 + 12x + 60x^2 + \ldots + 64x^6)\,(1 - 7x + 21x^2 + \ldots - x^7)$$

We need not carry out the full multiplication and write down all the 56 terms. It is enough to observe which terms in the product involve x^5. This arise as

$$1\,(-21x^5) + (12x)\,(35x^4) + (60x^2)\,(-35x^3) + (160x^3)\,(21x^2)$$

$$+ (240x^4)\,(-7x) + (192x^5)\,1$$

(**Explanation :** When a term involving x^r multiplied by a term involving x^{5-r}, we get a term involving x^5. Here *r* varies through 0, 1, 2, 3, 4, 5)

$\therefore$ The co-efficient of x^5 in the product is

$$(-21) + (12)\,(35) + (60)\,(-35) + (160)\,(21) + (240)\,(-7) + 192 = 171$$

SAQ 1

(a) Find the value of $(\sqrt{2}+1)^5 + (\sqrt{2}-1)^5$.

(b) Find the middle terms in the expansion of $\left(3\,x - \dfrac{x^3}{6}\right)^7$.

(c) The co-efficients of three consecutive terms in the expansion of $(1+a)^n$ are in the ratio 1 : 7 : 42. Find *n*.

(d) If the coefficient of a^{r-1}, a^r, a^{r+1} in the binomial expansion of $(1+a)^n$ are in arithmetic progression, prove that
$n^2 - n\,(4r+1) + 4r^2 - 2 = 0$.

SAQ 2

(a) Find the term independent of *x* in the expansion of $\left(2x - \dfrac{1}{x}\right)^{10}$.

(b) Show that the coefficient of the middle term of $(1+x)^{2n}$ is equal to the sum of the coefficients of the two middle terms of $(1+x)^{2n+1}$.

(c) Using binomial theorem, evaluate

(i) $(99)^5$

(ii) $(102)^6$

(d) Using binomial theorem, prove the following

(i) $C_0 + C_2 + C_4 + \ldots = 2^{n-1}$

(ii) $\dfrac{C_1}{C_0} + 2\,\dfrac{C_2}{C_1} + \ldots + n\,\dfrac{C_n}{C_{n-1}} = \dfrac{n\,(n+1)}{2}$

(iii) $C_0 + {}^2C_1 + {}^3C_2 + \ldots + {}^{(n+1)}C_n = 2^n + n\,2^{n-1}$

(iv) $C_1 + {}^2C_2 + {}^3C_3 + \ldots + {}^nC_n = n\,.\,2^{n-1}$

(v) $C_0\,C_1 + C_1\,C_2 + \ldots + C_{n-1}\,C_n = \dfrac{2^n\,.\,n\,.1\,.\,3\,.\,5\ldots(2n-1)}{n+1\,!}$

Here C_r denotes ${}^nC_r$.

(vi) Find the coefficient of x^{n-r} in the expansion of $(x+1)^n$.
Deduce that

$$C_0\,C_r + C_1\,C_{r+1} + \ldots + C_{n-r}\,C_n = \frac{(2n)\,!}{(n-r)\,!\,(n+r)\,!}$$

3.2.2 Binomial Theorem for Any Index

We know the expansion of $(1+x)^n$ where *n* is a positive integer. It is given below :

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_r x^r + \ldots + {}^nC_n x^n$$

$$= 1 + n\,x + \frac{n\,(n-1)}{2}\,x^2 + \ldots + \frac{n\,(n-1)\,(n-2)\ldots(n-r+1)}{r\,!}\,x^r + \ldots + x^n .$$

It contains $n + 1$ terms. But this is not so when n is either a negative integer or a fraction, whether positive or negative. Here ${}^nC_r$ also do not make any sense when n is not a positive integer. This difficulty may be overcome by writing $\frac{n\,(n-1)\ldots(n-r+1)}{1\,.\,2\,.\,3\ldots r}$ in place of ${}^nC_r$.

We now state without proof the more general theorem in which the index is not a whole number.

The formula

$$(1+x)^m = 1 + mx + \frac{m\,(m-1)}{1.2}\,x^2 + \ldots + \frac{m\,(m-1)\ldots(m-r+1)}{1\,.\,2\,.\,3\ldots r}\,x^r + \ldots$$

holds whenever $|\,x\,| < 1$.

Consider $$(a+b)^m = \left\{a\left[\left(1+\frac{b}{a}\right)\right]\right\}^m$$

$$= a^m\left(1+\frac{b}{a}\right)^m$$

$$= a^m\left[1 + m\,\frac{b}{a} + \frac{m\,(m-1)}{1\,.\,2}\left(\frac{b}{a}\right)^2 + \ldots\right]$$

$$= a^m + m\,a^{m-1}\,b + \frac{m\,(m-1)}{1\,.\,2}\,a^{m-2}\,b^2 + \ldots$$

This expression is valid when $\left|\frac{b}{a}\right| < 1$, i.e. when $|b| < |a|$.

Thus we have the formula

$$(a+b)^m = a^m + m\,a^{m-1}\,b + \frac{m\,(m-1)}{1\,.\,2}\,a^{m-2}\,b^2 + \ldots$$

holds whenever $|b| < |a|$.

These two theorems are known by the name ‘binomial theorem for general index’. This expansion is also known as binomial series.

The term $\frac{m\,(m-1)\,(m-2)\ldots(m-r+1)}{1\,.\,2\,.\,3\ldots r}\,x^r$ is called the general term in the binomial expansion of $(1+x)^m$ and the general term in the expansion of $(a+b)^m$ is $\frac{m\,(m-1)\ldots(m-r+1)}{1\,.\,2\,\ldots\ldots r}\,a^{m-r}\,b^r$.

Note that there are infinite number of terms in the expansion when m is a negative integer or a fraction.

Some important cases of the binomial series

(i) Taking $m = -1$, we have

$$\frac{1}{1+x} = 1 + (-1)\, x + \frac{(-1)(-2)}{1\,.\,2}\, x^2 + \ldots$$

$$= 1 - x + x^2 - x^3 + \ldots$$

(ii) Taking $x = -a$, in the above formula we have

$$\frac{1}{1-a} = 1 + a + a^2 + a^3 + \ldots$$

(iii) Taking $m = -2$, we have

$$\frac{1}{(1+x)^2} = 1 + (-2)x + \frac{(-2)(-3)}{1\,.\,2}\, x^2 + \frac{(-2)(-3)(-4)}{1\,.\,2\,.\,3}\, x^3 + \ldots$$

$$= 1 - 2x + 3x^2 - 4x^3 + \ldots$$

(iv) Taking $x = -a$ in the above formula we have

$$\frac{1}{(1-a)^2} = 1 + 2a + 3a^2 + 4a^3 + \ldots$$

In all these formulas, we assume that $|\, x \,| < 1$. So if the binomial expansion for $(1+x)^m$ is to be valid, there has to be a restriction either on x or on m. When m is a whole number, it is true for all values of x. When $|\, x \,| < 1$, it is true for all values of m.

Example 3.9

Prove that the coefficient of y^n in the expansion of $\frac{(1+y)^2}{(1-y)^2} = 4\,n$ for each $n = 1, 2, 3, \ldots$

Solution

$$(1+y)^2 = 1 + 2y + y^2$$

$$\frac{1}{(1-y)^2} = (1-y)^{-2} = 1 + 2y + 3y^2 + \ldots$$

$\therefore$ In the product $(1 - 2y + y^2)\,(1 + 2y + 3y^2 + \ldots)$, the terms involving y^n are

$$1\,.\,(n+1)\; y^n + (2y)\,(ny^{n-1}) + y^2\,\{(n-1)\; y^{n-2}\},\ \text{if}\ \ n \geq 2$$

The coefficient of y^n is

$$1\,.\,(n+1) + 2\,n + (n-1) = 4n$$

When $n = 1$, the coefficient of y is 4. Therefore the result holds for $n = 1$ also.

Example 3.10

Expand $\frac{1}{\sqrt[3]{6-3x}}$.

Solution

$$\frac{1}{\sqrt[3]{6-3x}} = (6-3x)^{-\frac{1}{3}}$$

$$= 6^{-\frac{1}{3}} + \left(-\frac{1}{3}\right) 6^{\left(-\frac{1}{3}-1\right)} (-3x) + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{1\,.\,2} 6^{\left(-\frac{1}{3}-2\right)} (-3x)^2 + \ldots$$

$$= 6^{-\frac{1}{3}} + 6^{-\frac{4}{3}} x + 2\,.\,6^{-\frac{7}{3}} x^2 + \ldots$$

$$= 6^{-\frac{1}{3}} \left[1 + \frac{x}{6} + \frac{2x^2}{6^2} + \ldots\right]$$

Example 3.11

Find the cube root of 127 upto four places of decimals.

Solution

$$(127)^{\frac{1}{3}} = \left[125\left(1 + \frac{2}{125}\right)\right]^{\frac{1}{3}}$$

$$= 5\left(1 + \frac{2}{125}\right)^{\frac{1}{3}}$$

$$= 5\left[1 + \left(\frac{1}{3}\right)\left(\frac{2}{125}\right) + \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{1\,.\,2}\left(\frac{2}{125}\right)^2 + \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{1\,.\,2\,.\,3}\left(\frac{2}{125}\right)^3 + \ldots\right]$$

$$= 5\left[1 + \frac{2}{375} + \left(-\frac{1}{9}\right)\left(\frac{2}{125}\right)^2 + \frac{5}{81}\left(\frac{2}{125}\right)^3 + \ldots\right]$$

$$= 5\left[1 + \frac{16}{3000} - \frac{4}{9}\cdot\frac{64}{10^6} + \frac{80}{81}\cdot\frac{256}{109} + \ldots\right]$$

$$= 5\,[1 + 0.0053 - 0.0000284 + 000000253 - \ldots]$$

$$= 5\,[1\,.\,005300253 - 0.0000284] = 5 \times 1.005271853$$

= 5.0264 upto four places of decimals.

Example 3.12

Assuming x to be so small that x^2 and higher powers of x can be neglected, find the value of $\dfrac{(1-2x)^{\frac{2}{3}}\ (4+5x)^{\frac{3}{2}}}{\sqrt{1-x}}$.

Solution

$\because\ x^2$ and higher powers of x are to be neglected we shall use $(1+x)^n = 1 + n\,x$.

$$\frac{(1-2x)^{\frac{2}{3}}\ (4+5x)^{\frac{3}{2}}}{\sqrt{1-x}} = \frac{(1-2x)^{\frac{2}{3}}\left[4\left(1+\frac{5x}{4}\right)\right]^{\frac{3}{2}}}{(1-x)^{\frac{1}{2}}}$$

$$= (1 - 2x)^{\frac{2}{3}} \cdot 4^{\frac{3}{2}} \left(1 + \frac{5x}{4}\right)^{\frac{3}{2}} (1 - x)^{-\frac{1}{2}}$$

$$= 8\,(1 - 2x)^{\frac{2}{3}} \left(1 + \frac{5x}{4}\right)^{\frac{3}{2}} (1 - x)^{-\frac{1}{2}}$$

$$= 8\left[1 + \frac{2}{3}(-\,2x)\right]\left[1 + \frac{3}{2}\left(\frac{5x}{4}\right)\right]\left[1 + \left(-\frac{1}{2}\right)(-\,x)\right]$$

$$= 8\left[1 - \frac{4x}{3}\right]\left[1 + \frac{15x}{8}\right]\left[1 + \frac{x}{2}\right]$$

$$= 8\left(1 + \frac{15x}{8} - \frac{4x}{3}\right)\left(1 + \frac{x}{2}\right) \qquad \text{(Neglecting } x^2\text{)}$$

$$= 8\left(1 + \frac{13x}{24}\right)\left(1 + \frac{x}{2}\right) = 8\left(1 + \frac{x}{2} + \frac{13x}{24}\right)$$

$$= 8 + \frac{25}{3}\,x$$

SAQ 3

(a) Prove that $\sqrt[3]{126} = 5.01330$ to five decimal places.

(b) Find $(0.98)^{-3}$ upto two decimal places.

(c) If $\dfrac{(1 - 3\,x)^{\frac{1}{2}} + (1 - x)^{\frac{5}{3}}}{\sqrt{4 - x}}$ is approximately equal to $a + bx$ for all small values of x, find a and b.

(d) Show that if x^3 and higher powers of x are neglected then

$$\frac{(4 + x)^{\frac{1}{2}}}{1 - x^2} = 2 + \frac{x}{4} + \frac{127}{64}\,x^2$$

3.3 COMPUTER MATHEMATICS

The advent of computer is affecting practically every activity of human life. Whether it is reservation in railways, managing a super market, or clearing cheques in banks, we have begun using computer as a tool everywhere.

A computer, however, is not a machine that can do more mathematical operations than what a human being can do. But it can do some operations much faster and can handle large amount of data. You may be surprised to know that the present day computer can perform a million additions per second. It can perform all operations such as addition, subtraction, multiplication, division and comparison.

There has been a variety of computers, depending on their purpose and capabilities. One common feature is that every computer is constituted of three major components :

(i) Input and Output component, (ii) Central Processing Unit, and (iii) Memory.

Depending on the nature of these three components, the computers vary. All the computers do the same functions with varying degrees of speed and limits.

However, when a problem is given to the computer, it cannot automatically decide the operations and the operands. It is necessary for a person, desirous to use a computer for solving a problem, to instruct the computer properly to choose the operations and the operands. This requires a mode of communication with a computer.

Such languages that computer understands are called Programming Languages, e.g. FORTRAN, BASIC, PASCAL, COBOL etc. Irrespective of language, what is equally important is to prepare the sequence of instructions to solve the specific problem. That is what we call an *algorithm*.

3.3.1 Algorithms

An algorithms is nothing but a sequence of simple steps to solve the problem at hand. It is a step-by-step procedure leading to the solution.

For example, to solve the equation $ax + b = c$, we may write :

Step 1 : Subtract b from c

Step 2 : Divide this answer by a, if $a \neq 0$

Step 3 : This answer represents the value of x as the solution.

Some characteristics features of an algorithm are :

(i) It is precise. There can be no ambiguity in a computer algorithm. Each step of the execution must be uniquely defined and may depend only on the inputs, the previous steps and the internal capabilities of the machine.

(ii) It is sufficiently detailed and precise to allow execution by the processor. For example, "find the area of the triangle", "solve the quadratic equation", etc. are instructions that require more details. In an algorithm, we have to incorporate how the area is to be found, or how the roots of the quadratic equation are to be determined. There is no room left for the creative imagination of the executor. But while planning an algorithm, some details may be suppressed.

(iii) It is in a definite order. The instructions in an algorithm are to be given in the order in which they are to be performed. The machine carries out these instructions, one after the other.

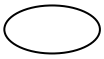
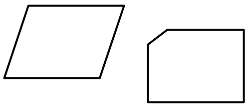
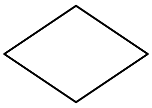
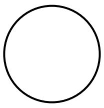
We say an algorithm, which is designed for a specific problem, is correct if it serves for all instances of the problem. Designing and studying computer algorithm occupies a unique and central position in the study of computer. Designing of algorithms is nothing but planning.

3.3.2 Flowcharts

Before beginning to solve a problem, it is frequently useful to do some preliminary planning. A graphic representation of such a plan is called a *flowchart*. The flowcharts constitute schematic pictures that we intend to implement in the program. The flowchart helps to break a big job down into many small pieces of the job and to represent them pictorially, showing the order of instructions.

A flowchart consists of some boxes, linked by arrows. In each box, certain action to be carried out is mentioned. Arrows on the lines connecting the boxes indicate the direction in which we should proceed.

The boxes are of different shapes as shown below.

Shape of the Box	**Meaning**
 A Stretched ellipse	Terminal box : To begin or end a program.
 A parallelogram	Input/Output box : The data fed into the computer and the print out given by the computer.
 A rhombus or a diamond	Decision box : Computer is to decide among alternative instructions.
 A rectangle	Assignment/Calculation box : Computer is to assign some values for the variables, or is to perform some operations like addition, etc.
 A circle	Connector box : To come from or to go to another part of the chart.

We start with an easy example.

Example 3.13

Find the modulus of a complex number given in the form (X, Y).

Solution

The modulus of (X, Y) is given by the formula $r = \sqrt{X^2 + Y^2}$.

Figure 3.2

Example 3.14

Given two complex numbers $Z_1 = (X_1, Y_1)$ and $Z_2 = (X_2, Y_2)$; compute $\frac{Z_1}{Z_2}$.

Solution

If $\frac{Z_1}{Z_2}$ is (X_3, Y_3), then $X_3 = \frac{X_1 X_2 + Y_1 Y_2}{X_2^2 + Y_2^2}$

And $Y_3 = \frac{Y_1 X_2 - Y_2 X_1}{X_2^2 + Y_2^2}$

When $Z_2 = 0$, that is when both X_2 and Y_2 are zeroes, we do not calculate $\frac{Z_1}{Z_2}$.

Figure 3.3

3.3.3 Loops

In this section, you will be taught how to assign different values for a variable, and create a loop while writing an algorithm. This is best explained by means of an example.

Example 3.15

Multiply 20 × 22 × 24 × 26 × 28.

Solution

First multiply 20 × 22.

Then multiply this answer by 24.

Then multiply this answer by 26.

Lastly multiply this answer by 28.

Print the answer and stop.

Now, we note that the factors, 20, 22, 24, 26 and 28 are progressively having an increment of 2. Therefore, we may write the algorithm as follows :

Note : In computer language the symbol '*' denotes multiplication.

$X = 20$

$A = 20$

$X \leftarrow X + 2$ [This means, the variable X is assigned a new value, equal to old value plus 2. Now $X = 22$]

$A \leftarrow A*X$ [Now $A = 20 \times 22$]

$X \leftarrow X + 2$ [Now $X = 22 + 2 = 24$]

$A \leftarrow A*X$ [Now $A = (20 \times 22) \times 24$]

$X \leftarrow X + 2$ [Now $X = 24 + 2 = 26$]

$A \leftarrow A*X$ [Now $A = (20 \times 22 \times 24) \times 26$]

$X \leftarrow X + 2$ [Now $X = 26 + 2 = 28$]

$A \leftarrow A*X$ [Now A is the required answer]

Print A and stop.

Figure 3.4

In this algorithm, we find that the two lines $X \leftarrow X + 2$ and $A \leftarrow A*X$ are repeated many times. There is a way to write this briefly, as shown in the flowchart below :

Explanation

The portion is a loop. It is traced again and again. When do we go out of the loop? Only when the answer to the decision box 'Is $X > 28$?' is 'YES'. At that time the value of A becomes $20 \times 22 \times 24 \times 26 \times 28$.

Figure 3.5

Conclusion

In Example 3.13, we saw an algorithm of the easiest type where the problem involved calculations only. In Example 3.14, we saw algorithm for a problem involving both calculations and decision-making. In Example 3.15, we saw an algorithm that involves calculations, decisions and also loops. In the examples that follow, we take some problems and write down algorithms for solving them. Examples 3.16 and 3.17 will be accompanied by actual working of the algorithm in a particular instance. In the later example, only the flowchart and its method will be given.

Example 3.16

To find the intersection of a finite number of finite sets.

Solution

Let the given sets be $A_1, A_2, \ldots, A_n$.

Arrange them in such a way that A_1 has least number to elements, when compared with $A_2, \ldots, A_n$ (this facilitates a quicker solution).

Let $A_1 = \{x_1, x_2, \ldots, x_n\}$

Is $x_1 \in A_2$?

If not, leave x_1 and go to x_2.

Else, is $x_1 \in A_3$?

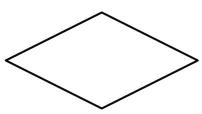
and so on.

If yes for all questions upto "Is $x_1 \in A_n$?", print x_1,. Otherwise, do not print x_1. Then do the same for x_2, x_3 and so on. The printed elements are the elements of the intersection.

Remark

After Print x_j we have the arrow towards 1

This means, we should go to the step marked 1 namely, the decision box



Is
$j = m$?

Let us see a particular case of this example.

Let $\quad A = \{1, 2, 3, 4, 5\}$

$$B = \{3, 4, 5, 8, 9\}$$

$$C = \{2, 3, 8, 9\}$$

$$D = \{1, 2, 3, 10\}$$

be four sets, for which we want to find the intersection. We rename them as

$$A_1 = C = \{2, 3, 8, 9\}$$

$$A_2 = A = \{1, 2, 3, 4, 5\}$$

$$A_3 = B = \{3, 4, 5, 8, 9\}$$

and $\quad A_4 = D = \{1, 2, 3, 10\}$

Figure 3.6

Here $m = 4$, $n = 4$, $x_1 = 2$, $x_2 = 3$, $x_3 = 8$, $x_4 = 9$

Is $x_1 \in A_2$?

Yes, because $2 \in A_2$.

Is $1 = n$? No, because $n = 4$.

New $i = 3$.

Is $x_1 \in A_3$?

No, because $2 \notin A_3$.

Is 3 = m? No, because m = 4.

New j = 2 and i = 2.

Is $x_2 \in A_2$? Yes, because $3 \in A_2$.

Is 2 = n? No, because n = 4.

New i = 3.

Is $x_2 \in A_3$? Yes, because $3 \in A_3$.

Is i = n? No, $3 \neq 4$.

New i = 4.

Is $x_2 \in A_4$? Yes, $3 \in A_4$

Is i = n? Yes i = n = 4.

Print x_2. Now 3 is printed.

Is j = m? No, $2 \neq 4$

New j = 3 and i = 2.

Is $x_3 \in A_2$? No, $8 \notin A_2$.

Is j = m? No, $3 \neq 4$.

New j = 4 and i = 2.

Is $x_4 \in A_2$? No, $9 \notin A_2$.

Is j = m? Yes, both = 4.

Stop.

After stopping, what are all the things already printed out? 3 only. Then we conclude that 3 is the only element of the intersection.

That is $A \cap B \cap C \cap D = \{3\}$.

Remark

When m and n are large, i.e. when too many big sets are given, finding the result by using this algorithm in a computer will be faster than finding the intersection without the computer.

Example 3.17

Check whether a given finite sequence of numbers is in geometric progression.

Solution

Let the given sequence be

$$a_1, a_2, \ldots, a_n$$

This sequence is in GP if and only if

$$a_i^2 = a_{i-1}a_{i+1} \text{ for } 2 \leq i \leq n-1$$

Step 1 : Are there at least three terms? If not, conclude that it is not a GP and stop.

Step 2 : Else, verify if $a_2^2 = a_1 a_3$. If not, conclude that it is not a GP and stop.

Step 3 : Else, verify if $a_3^2 = a_2 a_4$. If not, conclude that it is not in GP and stop. Else, proceed.

Step 4 : Go on, until $a_{n-1}^2 = a_{n-2}\, a_n$ is verified. If yes until that, conclude that it is in GP and stop.

Particular Instance

Is, 2, 6, 18, 36 in Geometric Progression?

Here, $n = 4$

$a_1 = 2$

$a_2 = 6$

$a_3 = 18$

$a_4 = 36$

Are there at least 3 terms? i.e., is $n > 3$?

Yes,

Initially $i = 2$

$A = 2$

$B = 6$

$C = 18$

Is $B^2 = AC$

$6^2 = 18 \times 2?$

Yes.

New $i = 3$

Is $i = n$?

No.

Set $A = 6$

$B = 18$

$C = 36$

Is $B^2 = AC$

i.e., is $18^2 = 6 \times 36?$

No.

PRINT : Not a Geometric Progression.

STOP.

Figure 3.7

3.4 SUMMARY

Binomial Theorem

(i) Binomial theorem for positive integral index n is

$$(x + a)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} a + \ldots + {}^nC_r\, x^{n-r} a^r + \ldots + {}^nC_n\, a^n$$

(ii) General term in the expansion is

$$T_{r+1} = {}^nC_r\, x^{n-r} a^r$$

(iii) If n is even then there is only one middle term in the expansion $(x+a)^n$ i.e. $\left(\frac{n+2}{2}\right)^{\text{th}}$ term and if n is odd then there are two middle terms namely $\left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+3}{2}\right)^{\text{th}}$ terms.

(iv) Binomial theorem for any rational index n and $|\,x\,| < 1$ is

$$(1+x)^n = 1 + nx + \frac{n\,(n-1)}{2!}\,x^2 + \frac{n\,(n-1)\,(n+2)}{3!}\,x^3 + \ldots$$

3.5 ANSWERS TO SAQs

SAQ 1

(a) $58\sqrt{2}$

(b) $-\,35\,(3\,x)^4\left(\frac{x^3}{6}\right)^3,\ 35\,(3\,x)^3\left(\frac{x^3}{6}\right)^4$

(c) 55

SAQ 2

(a) 6^{th} term

(c) (i) 9509900499

(ii) 1126162419264

SAQ 3

(b) 1.06

(c) $a = 1,\ b = -\,\frac{35}{24}$