UNIT 2 EQUILIBRIUM : FREE BODY DIAGRAM

Structure

2.1 INTRODUCTION

In Unit 1, technique of reducing any system of forces to a force-couple system at some arbitrary point *O* was introduced. When this resultant force and the resultant couple (after the process of reduction) both are equal to zero, the system is said to be in equilibrium. Now, we study the necessary and sufficient conditions for the equilibrium of a rigid body. This will enable us to determine the unknown forces applied to the rigid body in equilibrium or unknown reactions exerted on it by its supports. This will help to analyze any structure, which is the first step in the design of structures.

Objectives

After studying this unit, you should be able to

- classify various types of supports and constraints,
- draw free-body diagrams,
- conceptualise conditions of equilibrium,
- determine the unknown forces acting on a body, and
- determine beam reactions for various types of loads.

2.2 TYPES OF SUPPORTS AND CONSTRAINTS

Bodies in a structural system are connected with the surroundings in a number of standard types. These connections and supports may be in the form of rollers, rockers, ball and socket joints, frictional surfaces, short links, cables, pins, hinges, etc. They restrict the movement of the bodies in particular direction by offering reactions. In some cases, it is easier to ascertain the direction of reactions by considering the direction of possible movement in absence of the support. Consider, for example, a rigid body kept on a table. If the table would not have been there, the rigid body would have fallen down due to gravity. The table is restricting the downward movement of the body. This is possible if the table is able to offer a force in upward direction. Thus, you can ascertain that the direction of reaction offered by the table will be upward.

2.2.1 Plane Structures

A plane structure lies in one plane, e.g. a thin flat plate, an assembly of straight bars lying in one plane, an arch, a simply supported beam, a cantilever, and the like. Some plane structures are constrained by supports that permit no rigid body movement upon application of loads. The reactions offered by various supports connecting plane structures can be grouped under three categories:

Reactions Equivalent to a Force with Known Line of Action

Figure 2.1(a) illustrates some of the supports in this category. They include rollers, rockers, frictionless surfaces, short links, cables, collars on frictionless rods and frictionless pins in slots. These supports prevent motion in one direction only. The direction of reaction can therefore be ascertained easily.

Figure 2.1(a)

Reactions Equivalent to a Force of Unknown Direction

A frictionless hinge or pin, knife-edge support, rough surfaces are some of the examples of this category shown in Figure 2.1(b). These supports prevent translation of the free body in all directions. However, rotation of the body about the connection is possible. The reactions offered by these supports involve two unknowns : magnitude and direction. They are usually represented by the components along two mutually perpendicular directions say horizontal and vertical.

Reactions Equivalent to a Force and a Couple

A beam built in a wall as shown in Figure 2.1(c) is an example of this category. Such a support offers full restraint against rotation and translational movement. Reactions of this group involve three unknowns. Generally these are represented by two components of the force along *x* and *y* directions and by the movement of the couple.

Figure 2.1(b)

Figure 2.1(c)

2.2.2 Space Structures

A space structure is a structure in a three-dimensional space. There are six fundamental motions possible viz. translation in *x*, *y*, and *z* directions and rotations about the *x*, *y*, and *z* axes. Depending on the constraints provided by the supports the number of unknown reactions vary from one to six (Figure 2.2).

Figure 2.2

Frictionless surfaces, cables and ball supports prevent translation in one direction only and thus exert a single force of known line of action. The magnitude of the reaction is the only unknown in such cases. Wheels on rails or rollers on rough surfaces involve two unknown reaction components as motion in two directions is restricted. In case of ball and socket joints or rough surfaces in direct contact there are three unknown as they prevent translation in three directions. A universal joint which does not allow rotation about one axis will involve 4 unknowns viz. three components along x, y and z directions and a couple. A fixed support does not allow any motion, neither translation nor rotation. This involves maximum number of reaction components : three force components and three couples.

2.3 FREE BODY DIAGRAMS

To study the balance of forces, viz-a-viz, any structure, you should be in a position to identify all the forces acting on it. A structure can be divided into various parts for simplicity. If you separate a body from its surrounding and draw a diagram to represent that body and indicate all the forces from the surroundings that act on it, such a diagram is called a **free-body diagram**.

Example 2.1

Consider the propped cantilever shown in Figure 2.3(a). To draw the free-body diagram for the beam, isolate the beam from its surroundings. The fixed and the hinge supports will replaced by constraining forces offered by them. All external loads will also be replaced by forces exerting on it, and then the resulting diagram is called free-body diagram of the beam.

(a) **(b)**

Figure 2.3

Solution

Free-body diagram for the beam in Figure 2.3(b) shows :

(i) beam AB,

(ii) constraining forces H_A, V_A and M_A offered by the fixed support at A,

(iii) constraining forces R_B offered by the roller support B, and

(iv) the external load acting at C.

Example 2.2

A 3 m long boom is held by a ball and socket joint at A and by two cables CD and BE. It carries a load of 20kN at B. Draw Free-body diagram for the boom AB (Figure 2.4(a)).

(a) **(b)**

Figure 2.4

Solution

Free-body diagram for the boom AB in Figure 2.4(b) shows

(i) boom AB,

(ii) constraining forces A_x, A_y and A_z due to ball and socket support at A,

(iii) tension T_{CD} in cable CD, and

(iv) tension T_{BE} in cable BE.

(v) the external load (20 kN) acting at B.

SAQ 1

(a) Draw the free-body diagram for the member AB in following cases :

(a) **(b)**

(c) **(d)**

In Figure 2.5(e), member AB is supported by a ball and socket at B and leans against a smooth wall at A. Cord CD is attached to the mid-point of AB.

In Figure 2.5(f), DF and CE are cables, joint A is ball and socket joint.

(e) **(f)**

Figure 2.5

(b) Identify the reaction components possible in case of following supports and connections :

(i) Frictionless pin in slot

(ii) Hinge Point

(iii) Wheel on rail

(iv) Hinge and bearing supporting axial thrust and radial load (Figure 2.6)

(v) Ball and socket joint

(vi) Fixed support

Figure 2.6

2.4 EQUILIBRIUM OF COPLANAR FORCES

Consider a rigid body acted upon by a number of coplanar forces. These forces will cause the rigid body either to

(i) move in a particular direction without rotating, or

(ii) rotate at its own place without moving, or

(iii) rotate about itself and at the same time move in any particular direction, or

(iv) remain completely at rest

The last case is that of the state of equilibrium. To move the body in a particular direction there must exist a resultant force and to rotate there must exist a resultant couple. If the resultant force and the resultant couple both are absent, the body can neither move nor rotate. It will remain at rest. Let us study some typical cases.

2.4.1 Concurrent Forces

If a number of coplanar concurrent forces are acting on a body then there may exist a single resultant force passing through the point of intersection of all the concurrent forces. But there will not exist any couple. The resultant force may have two components say R_x and R_y where x and y are any two mutually perpendicular co-ordinate axes. If the body is to be at rest then you must get both these components equal to zero. The conditions for equilibrium in case of coplanar concurrent forces are, therefore, as follow :

$$R_x = 0$$

$$R_y = 0$$

But $\quad R_x = \Sigma F_x \quad$ and $\quad R_y = \Sigma F_y$

$\therefore \quad \Sigma F_x = 0 \quad$ and $\quad \Sigma F_y = 0$

(i) The algebraic sum of the resolved components of all forces in any direction must be equal to zero.

(ii) The algebraic sum of the resolved components of all forces in a direction at right angles to the first direction must be equal to zero.

Example 2.3

Four coplanar concurrent forces act at a point and keep it at rest. These are shown in Figure 2.7. Determine the forces P and Q.

Figure 2.7

Solution

Let us assume that forces P and Q act away from point O. As the point O remains at rest we can apply equations of equilibrium, such as :

$$\Sigma F_x = 0$$

$$\therefore \quad 310 + P \cos 60^\circ - 1260 \cos 60^\circ = 0$$

$$\therefore \quad 310 + 0.5 P - 1260 \times 0.5 = 0$$

$$\therefore \quad P = \frac{630 - 310}{0.5} = 640 \text{ N}$$

As P here works out as a positive quantity, the assumed direction is therefore correct. Therefore, P acts away from O. Now, similarly,

$\Sigma F_y = 0$

$\therefore \quad P \sin 60^{o} + 1260 \sin 60^{o} - Q = 0$

$\therefore \quad Q = (640 + 1260) \sin 60^{o}$

$\therefore \quad Q = 1645.4$ N

As Q here comes out also as positive, the assumed direction of Q is correct. Therefore, Q acts away from O.

Example 2.4

A smooth sphere weighing 200 N is resting against smooth walls as shown in Figure 2.8. Determine the reactions at the supports.

Figure 2.8

Solution

Let us first ascertain the directions of reactions. As the wall (A) is vertical and smooth the reaction at A will be horizontal, i.e. normal to the wall. Similarly, the reaction at B will be normal to the line (wall) inclined at 60^{o} to the horizontal. Both these reactions will pass through O the centre of the sphere because these are normals drawn to the tangents of the sphere at A and B. The weight of the cylinder can be assumed concentrated at O. Thus, three concurrent forces keep the sphere at rest. Let us apply conditions of equilibrium.

$\Sigma F_x = 0$

$\therefore R_A - R_B \cos \theta = 0$ (θ = 30^{o} by geometry of the figure)

$R_A = 0.866 R_B$

$\Sigma F_y = 0$

$\therefore \quad R_B \sin 30^{o} - W = 0$

$\therefore \quad R_B (0.5) = 200$

$\therefore \quad R_B = \dfrac{200}{0.5} = 400\text{N}$

$\therefore \quad R_A = 0.866 \times 400 = 346.4\text{N}$

The same problem can be solved using Lami's Theorem which states, "If three forces (in a plane) acting on a body keep it at rest then each force is proportional to the sine of the angle between the other two forces" (Figure 2.9).

$$\frac{R_A}{\sin (90^\circ + \theta)} = \frac{R_B}{\sin 90^\circ} = \frac{W}{\sin (180^\circ - \theta)}$$

Figure 2.9

Putting the known values, we get

$$\frac{R_A}{\cos 30^\circ} = \frac{R_B}{1} = \frac{200}{\sin 30^\circ}$$

$$\therefore \qquad R_A = \frac{200 \cos 30^\circ}{\sin 30^\circ} = 346.4 \text{ N}$$

$$\therefore \qquad R_B = \frac{200}{\sin 30^\circ} = 400.0 \text{ N}$$

SAQ 2

Determine the minimum value of force P required just to start the wheel over the step 300 mm high. The diameter of the wheel is 1.2 m and the weight is 800 N. Also find the direction of P (Figure 2.10).

(Note that the reaction offered by the ground is zero when the wheel is just on the point of moving over the step.)

Figure 2.10

2.4.2 Non-concurrent Forces

If non-concurrent coplanar forces are acting on a body, their resultant may be in the form of a couple in addition to a resultant force. If a body is to remain at rest then there should neither be a resultant force nor a resultant moment, i.e.

$$R_x = \Sigma F_x = 0$$

$$R_y = \Sigma F_y = 0$$

$$M_0 = 0$$

Therefore, in addition to the two conditions of equilibrium as in case of concurrent forces there is one more condition of equilibrium which can be stated in words as :

"The algebraic sum of moments of all the forces about any point in their planes must be equal to zero."

Example 2.5

A board ABCD is held in position as shown in Figure 2.11 by a cable BE and hinge at A. If the weight of the board is 5 kN, determine the reactions at hinge A and the tension T in the cable.

Solution

Let the components of the reaction at A be H_A and V_A as shown in Figure 2.11.

The board is at rest under the action of four forces H_A, V_A, T and W.

Taking moments of all forces about A, we get

$$\Sigma M_A = 0$$

$$\therefore \quad - (T \times \sin 30^\circ) \times 1.6 + W \times 0.8 = 0$$

$$\therefore \quad - 0.8T + 5 \times 0.8 = 0$$

$$\therefore \quad T = \frac{4}{0.8} = 5 \text{kN}$$

Figure 2.11

and, $\Sigma F_x = 0$

$\therefore \qquad H_A - T \cos 30^o = 0$

$\therefore \qquad H_A = T \cos 30^o$

On putting $T = 5$ kN, we have

$$H_A = 5 \times 0.866$$

$$= 4.33 \text{ kN}$$

and, $\Sigma F_y = 0$

$\therefore \qquad V_A + T \sin 30^o - W = 0$

$\therefore \qquad V_A = W - T \sin 30^o$

$$= 5 - 5 \times 0.5$$

$$= 2.5 \text{ kN}$$

Now $\qquad R_A = \sqrt{H_A{}^2 + V_A{}^2}$

$$= \sqrt{4.33^2 + 2.50^2}$$

$$= 5\text{kN}$$

$$\theta = \tan^{-1} \frac{V_A}{H_A}$$

$$= \tan^{-1} \frac{2.50}{4.33}$$

$$= 30^o$$

Hence, reaction R_A has a magnitude of 5kN and is inclined at 30^o to the horizontal.

Note : You may attempt the same problem considering the board to be subjected to 3 forces only viz. R_A, T and W. As three forces keep the body at rest, these must be concurrent, find the point of concurrence and get the values of unknowns R_A and T; and, knowing the directions of R_A, you may also use Lami's Theorem either.

And, consider $\quad \Sigma F_x = 0$ and $\Sigma F_y = 0$

2.4.3 Beam Reactions for Various Types of Loads

The conditions of equilibrium for rigid bodies can be used to find the reactions of a beam. Depending on support conditions, beams are classified as cantilever, simply supported, continuous, propped cantilever and fixed beams. If a beam is fixed at one end and free at the other it is called a cantilever. As at fixed end neither translation nor rotation is possible there will be three unknown reaction components, H_A, V_A and M_A. There are three equations of equilibrium. Hence, these unknowns can be found out. Similarly, the reactions in case of simply supported beam can also be found out. In simply supported beam or in a beam hinged at one end and roller at other end, the number of unknowns is limited to three. Hence these beams are called statically determinate beams. You can determine the reactions using the static conditions of equilibrium. But if the number of unknowns exceed three as in the case of continuous beams, propped cantilever and fixed beams, you cannot determine the unknowns just by making use of statical equations of equilibrium. Therefore, these beams are called statically indeterminate beams. Here, in this unit, you will learn to find the reactions in case of statically determinate beams only.

A beam is a structural member generally carrying transverse loads. These loads can be concentrated, uniformly distributed or with triangular distribution. Let us study each type of loading by solving some problems.

Example 2.6

A cantilever AB, 1.8 m long is fixed at A and carries uniformly distributed load of 20 kN/m over its entire length, and a point load of 30 kN at the free end. Determine the reactions at A (Figure 2.12).

Figure 2.12

Solution

Let the reaction components at A be H_A, V_A and M_A as shown in Figure 2.12. Let us replace the uniformly distributed load of 20 kN/m by a single force of 20 × 1.8 = 36 kN acting at the centre of AB, i.e. at 0.9 m from end A. There is no horizontal force acting on the beam.

$\Sigma F_x = 0$

$\therefore \qquad H_A = 0$

$\Sigma F_y = 0$

$\therefore \qquad V_A - (20 \times 1.8) - 30 = 0$

$\therefore \qquad V_A = 36 + 30$

$\qquad = 66$ kN

Taking moments about A, we get

$\Sigma M_A = 0$

$\therefore \qquad - M_A + (20 \times 1.8)\ (0.9) + 30 \times 1.8 = 0$

$\therefore \qquad M_A = 32.4 + 54$

$\qquad = 86.4$ kN-m

Therefore, the reaction at A consists of a vertical force acting upwards with a magnitude of 66 kN, and an anticlockwise moment of 86.4 kN-m.

Example 2.7

A beam AB is hinged at A and is supported at C. It is loaded as shown in Figure 2.13. Find out the reactions at A and C.

Figure 2.13

Solution

Let the reaction components at A be H_A and V_A as shown in Figure 2.13. The reaction at C will be acting vertically upwards being a roller point. As the beam is at rest under the action of the forces, the conditions of equilibrium can be applied.

Taking moments of all forces about A, we get

$\Sigma M_A = 0$

$$(18 \times 3) \times \frac{3}{2} + (40 \sin 60^{\circ}) \times 5 - R_c \times 7 + 24 \times 9 = 0$$

$$\therefore \quad 81 + 173.2 - 7 R_c + 216 = 0$$

$$\therefore \quad R_C = \frac{81 + 173.2 + 216}{7} = 67.172 \text{ kN}$$

Note : Moments due to H_A, V_A and 40 cos 60° about A are zero as they pass through point A. Clockwise moments are taken as positive.

Now, $\Sigma F_x = 0$

$$\therefore \quad H_A - 40 \cos 60^{\circ} = 0$$

$$\therefore \quad H_A = 40 \times 0.5 = 20 \text{ kN}$$

$\Sigma F_y = 0$

$$\therefore \quad V_A - (18 \times 3) - 40 \sin 60^{\circ} + R_c - 24 = 0$$

$$\therefore \quad V_A = 54 + 34.64 - 67.172 + 24$$

$$= 45.468 \text{ kN}$$

$$R_A = \sqrt{H_A^2 + V_A^2} = \sqrt{20^2 + 45.468^2} = 49.672 \text{ kN}$$

$$\tan \theta = \frac{V_A}{H_A} = \frac{45.468}{20} = 2.2734$$

$$\therefore \quad \theta = 66.257^{\circ} = 66^{\circ}\ 15'\ 24''$$

The reaction at A has a magnitude of 49.672 kN and is inclined at 66° 15′ 24″ with respect to horizontal whereas reaction at C is acting vertically upwards and has a magnitude of 67.172 kN.

SAQ 3

Determine the reactions of the beams loaded as shown in Figures 2.14(a), (b), (c) and (d).

(a)

(b)

(c)

(d)

Figure 2.14

2.5 PLANE TRUSSES

A truss is defined as "a structure which is formed by assembling of straight, rigid bars joined together at their ends, so as to form a framework". In engineering the truss is used in a variety of conditions. A few examples are :

(i) roofs,

(ii) bridges,

(iii) transmission line towers,

(iv) chassis of vehicles etc.

Figure 2.15 shows the outlines of some common types of trusses.

Figure 2.15 : Common Types of Trusses

2.5.1 Truss Analysis

Loads applied to a truss cause axial forces in its members if the joints of the truss are appropriately done; and thus there occur no bending moments in the members. This obviously requires comparatively thinner sections of members – saving the material used. Hence, trusses that way are economical to use. These forces acting at their ends either tend to pull them apart or to compress or crush them. In the first case, the member is said to be "in tension", and in the second "in compression". Computation of these forces, i.e. their magnitudes and sense, for each member of the truss is called its analysis.

The primary analysis of a plane truss is based on the following assumptions :

(i) Every member is straight and is connected to the truss only at its ends, The axes of all the members lie in one plane, called the plane of the truss.

(ii) The self weights of the members are assumed to be very small in comparison to the applied loads and hence neglected (i.e. assuming no bending moments to operate).

(iii) Every joint where two or more members are connected together is made by a single pin passing through them and considered to be frictionless. The pin at the joint can, therefore, exert only forces but no moments on the members connected by it.

(iv) Forces are applied to the truss only at the pins of the joints in the plane of the truss and never on a member at a point in between its two ends.

In a real truss, however, members may be connected at points between their ends, the weights of the members are not negligible, the joints may be welded, rivetted or bolted and hence are far from being frictionless and loads are often applied at points other than the joints. The assumptions above are therefore not always fully valid. However the analysis of a truss based on the above assumptions is sufficiently accurate for the preliminary engineering design.

Above assumptions in truss analysis leads to following observations :

(i) Every member becomes a 2-force body in equilibrium, and

(ii) Every joint becomes the centre of a concurrent force-system in equilibrium.

Let us consider the Free Body Diagram of a member [*AB* of Figure 2.15(b)] which is shown in Figure 2.16(a).

Figure 2.16

As consequence of the assumptions, it can be concluded that :

(i) Since it has negligible weight, no force acts on it corresponding to its weight.

(ii) Since forces are applied to the truss only at the joints, no load acts on it between A and B.

(iii) It is connected to the truss at joints A and B, but since they are hinged, each of the pins to which AB is connected can exert on AB only a force such as P_{AB} and P_{BA} at A and B respectively but no moment at the two points. As shown by the dotted arrows in Figure 2.16(a). They are shown as such because their directions are unknown at this stage.

Thus, since there are only two forces on AB, which keep it in equilibrium, AB becomes a two force body with the result, that P_{AB} and P_{BA} are equal, opposite in direction, and have a common line of action along the axis of the member *AB*. This force-pair which is axial to the member must have senses of its individual components either away from each other as in Figure 2.16(b) or towards each other as in (c). In the former case, the member will be in tension (T) while in the latter case it will be in compression (C).

It must be clearly understood that the forces shown in Figures 2.16(a), (b) and (c) are those exerted *by the joints on the member*. Since they have the same magnitude and direction and differ only in sense, we may identify the pairs by their common magnitude and by whether the member is in tension or compression due to them. Each member is subject to either axial tension or axial compression and there is no bending, shearing or twisting of it.

Let us now consider the free-body diagram (F. B. D) of point B, of Figure 2.16(c). It (i.e. AB) is connected to members 1, 2, 3, and therefore they will exert forces. Assuming that member AB is in tension, i.e. a situation as in figure joint P_B exerting the force $P_{BA} = P_3$ at B on AB. Hence by action and reaction law, we must have the member AB exerting a Force P'_{BA} equal to P_{BA}, at B, P_3 as shown in Figure 2.16(d). Similarly, the other members at B, viz. 1, 2, and 4 will exert forces P_1, P_2 and P_4, away from B if the members are in tension and towards B if they are in compression. In addition to these forces on B, which are internal to the truss, an external force F may be acting on the truss at B. Evidently, the joint B will be in equilibrium under this concurrent force-system. In short, each joint of a truss becomes the centre of a concurrent force-system in equilibrium, which will have as many forces as the number of members meeting there, plus the external loads on the truss acting at the joint.

Statically Determinate and Indeterminate Trusses

The magnitudes of the forces in the members, their type and the reactions developed at the supports are unknown in a problem of truss analysis. If the number of the members is 'm' and of reactions is 'r', we have '$(m + r)$' unknowns. Now for the equilibrium of each joint we have two equations ($\Sigma F_x = 0$ and $\Sigma F_y = 0$). Thus, if the total number of joints in the truss is 'j', we shall have '$2j$' equations based on statics. If we assume that the truss is simply supported at ends, i.e. it has one hinged support and the other roller support, we have three unknown external reactive components, i.e. $r = 3$. The total number of unknowns therefore becomes $(m + 3)$. Evidently, if $2j = (m + 3)$, we have as many equations as the number of unknowns and by solving them it will be possible to determine all the unknowns. For a simple truss, we have $m = 2j - 3$ from which $2j = (m + 3)$, i.e. such a truss satisfies the above condition of the solvability of a truss by statics. A truss in which the force in all the members can be determined by statical equations alone is said to be 'statically determinate'.

If in a certain truss $m > (2j - 3)$, we have more members than those which can be analysed from the number of available statical equations, the truss is then said to be 'statically indeterminate' internally' or 'internally redundant' because it has some bars in excess. On the other hand, if $m < (2j - 3)$, we have too few unknowns, and then physically the number of bars is insufficient to keep the truss rigid and stable. The truss is then said to be 'Deficient'. Only when $m = (2j - 3)$, we have a statically determinate; and merely satisfying the relation $m = (2j - 3)$ is however not a sufficient condition in this behalf.

The '$2j$' equation corresponds to 'j' concurrent force-systems in equilibrium, which forces (corresponding to 'm' members) and also the forces external to the truss (including applied loads and reactions) which are separately in equilibrium. Hence the '$2j$' equations will include the three fundamental equilibrium equations for the truss as a whole also. For the solution of a truss, however, it is common to use the latter three equations first, to evaluate the external reactions. If the reaction components at two ends are more than three they cannot be determined by statical equations alone : we have an externally statically indeterminate truss.

2.5.2 Method of Joints

The diagram of any given truss can be converted into a diagram showing the corresponding concurrent force-systems in equilibrium, one at each of its joints, by erasing the middle portions of the lengths of all the members and marking arrows for the forces on the joints next to them, along the remaining portions of those lengths. The senses of the arrows for any single member may be marked either towards or away from each other. We may show each of these arrows away from the particular joint to start with; this amounts to assuming that all the members are in tension.

The statically determinate truss can be analysed by applying the equations of equilibrium $\Sigma F_x = 0$ and $\Sigma F_y = 0$ for all the forces at each joint, viz. the unknown 'P' values and any loads there. This is known as method of joint for truss analysis. Instead of setting up the equations for all the joints first and then solving them simultaneously, it is better to set them up for one joint, solve them, obtain the values of the unknown member-force there, then proceed to another joint, and continue the process till all the unknowns are obtained. In order that we obtain solvable equations in the beginning and also at every subsequence step, we must find a joint or joints where there are not more than two unknown forces which include forces in members and reactive elements. If we get the value of an unknown member-force with a negative sign, it only means that the member is in compression and not in tension. At the other end of the particular member, therefore, when we put down the equilibrium equations, we either reverse the sense of the force and take its magnitude as positive. It needs to be emphasised here that the arrows next to the joints as marked, show the force exerted by the members on the joints and not vice-versa.

A common case of a member carrying zero force occurs when at a joint we have three members meeting, two of which are in the same straight line, the third at an inclination to them and there is no load there. By resolving all the forces at the joint in a direction normal to the first two we shall find that the force in the third is zero. All such members should be disregarded from the analysis merely on inspection.

Example 2.8

Analyse the truss with geometry, loading and supports as illustrated in Figure 2.17(a).

Figure 2.17(a)

Solution

The members and joints are numbered as shown in Figure 2.17(b); and diagrams in Figure 2.17(c), represent the free body diagrams for each joint. Reactive elements at the hinge A are V_A and H_A, that at the roller support G is H_G.

(b) **(c)**

Figure 2.17

The length of each of the inclined members is 5 m ($l = \sqrt{3^2 + 4^2} = 5$) and hence cos θ = sin ϕ = 0.8, sin θ = cos ϕ = 0.6, θ and ϕ being the inclinations of the diagonals with the vertical and horizontal respectively.

A joint where the number of unknown forces is two or less is located. It is clear that D is the only joint where we have two unknowns S_3 and S_4.

(i) Joint D

$\Sigma F_y = 0$ gives

$$S_3 \cos\theta - 12 = 0$$

$$S_3 = \frac{12}{0.8} = +15 \text{ kN}$$

the positive sign shows that the sense of the arrow S_3 at D is correct, i.e., member 3 is pulling D up and towards the left, i.e. it is in tension =15 kN.

$\Sigma F_x = 0$ now gives

$$S_3 \cos\phi + S_4 = 0,$$

i.e. $(+15)\,0.6 + S_4 = 0.$ $\therefore \; S_4 = -9.0$ kN.

The negative sign shows that the sense of the arrow S_4 at D is to be reversed; member 4 is actually pushing the joint D to the right and it itself is under compression. We may reverse both S_4 arrows to correct the situation, and take $S_4 = +9.0$ kN.

Since S_3 is now known, we notice that at joint C we have only two unknowns S_2 and S_{11} and so we proceed to joint C.

(ii) Joint C

$\Sigma F_x = 0$ gives

$$S_3 \cos\phi - S_2 = 0$$

$$\therefore \quad S_2 = 15 \times 0.6 = 9 \text{ kN},$$

i.e. the member '2' carries a tensile force = 9 kN.

$\Sigma F_y = 0$ gives

$$S_3 \cos\theta + S_{11} = 0.$$

$$\therefore \quad S_{11} = -15 \times 0.8 = -12 \text{ kN}$$

i.e. the member '11' carries a compressive force = 12 kN.

With S_{11} and S_4 known, at joint E we have only two unknowns S_{10} and S_5.

Hence we turn to joint E.

(iii) Joint E

$\Sigma F_y = 0$ gives

$$S_{10} \cos\theta + S_{11} - 12 = 0$$

But $S_{11} = -12$

$$\therefore \quad S_{10} = \frac{+12 + 12}{0.8} = 30 \text{ kN},$$

i.e. member 10 carries 30 kN force in tension.

$\Sigma F_x = 0$ gives

$$S_5 + S_{10} \cos\phi - S_4 = 0.$$

But $S_4 = -9$ and $S_{10} = +30$

$$\therefore \quad S_5 = -9 - 30 \times 0.6 = -27 \text{ kN},$$

i.e. member '5' is in compression under 27 kN force. Now we go to B.

(iv) Joint B

$\Sigma F_y = 0$ gives

$$S_9 = -S_{10} \cos\theta = -30 \times 0.8 = -24 \text{ kN}$$

$\Sigma F_x = 0$ gives

$$S_1 = S_{10} \cos\phi + S_2 = 18 + 9 = +27 \text{ kN}$$

i.e. member 9 carries 24 kN compression and member '1', 27 kN tension.

(v) Joint F

$\Sigma F_y = 0$ gives

$$S_8 \cos\theta = 12 + 24$$

$$\therefore \quad S_8 = 45 \text{ kN tension.}$$

$\Sigma F_x = 0$ gives

$$S_6 + S_8 \cos\phi - S_5 = 0$$

$$S_6 = -45 \times 0.6 - 27 = -54 \text{ kN},$$

i.e. member '8' carries 45 kN tension and member '6', 54 kN compression.

(vi) Joint G

$\Sigma F_y = 0$ gives

$$S_7 = + 12 \text{ kN}$$

and $\Sigma F_x = 0$ gives

$$H_G + S_6 = 0, \text{ i.e. } H_G = + 54 \text{ kN},$$

i.e., member 7 carries 12 kN tension and the reaction H_G is 54 kN to the right as assumed.

(vii) Joint A

$\Sigma F_y = 0$ gives

$$V_A - S_8 \cos \theta - S_7 = 0,$$

$$\therefore \quad V_A = 36 + 12 = + 48 \text{ kN}.$$

$\Sigma F_x = 0$ gives

$$H_A + S_1 + S_8 \cos \phi = 0,$$

$$\therefore \quad H_A + 27 + 45 \times 0.6 = 0.$$

$$\therefore \quad H_A = - 54 \text{ kN}$$

negative sign shows that H_A is to the left, and = 54.00 kN.

The three fundamental equations can be applied to the entire truss now to check the values of V_A, H_A and H_G.

The result of the analysis is best given by a sketch of the truss as given in Figure 2.17(c).

In Figure 2.17(c), the numbers indicate the magnitudes of the forces exerted by the member on the joints and "pairs of approaching arrows" on a member indicate that it is in tension and "pair of receding arrows" indicate compression. This may be indicated alternatively by giving the magnitude of the force a positive sign or T suffix for tension and negative sign or C suffix for compression.

SAQ 4

Analyse the warren truss shown in Figure 2.18.

Figure 2.18

2.5.3 Method of Sections

When analysing a truss by the method of joints, it is necessary that we proceed from joint to joint in a definite sequence which is determined by the progressive availability of joints where the number of unknown forces is two or less. **Now if the force in a particular member only is required the labour of solving all the joints coming earlier in the analysis will have to be put in. The Method of Sections avoids this labour and yet enables one to attain the objective**. It is achieved by cutting the structure such that only the member/members in which force is required to be obtained are cut, and then drawing the free body diagram of the cut portion of truss as explained in the following example.

Example 2.9

A pratt truss having a span 24 m is shown in Figure 2.19. Compute the forces in the members DE, marked 1, 2, and 3 (encircled) by the method of sections. Loads marked are in kN (ignore cuts in members as shown). Values of V and V_1 are not a part of the data.

Figure 2.19

Solution

From the geometry of the truss,

$$\theta = \tan^{-1} \frac{3}{5} = 30°.964$$

and $\therefore\ \phi = 59°.036.$

Also each of the support reactions = half the load (by symmetry).

$$\therefore \quad V_A = V_1 = \frac{70}{2} = 35 \text{ kN} \uparrow ; \text{ and horizontal reaction, } H_A = 0.$$

Having worked out the end reactions, it will be possible to start the analysis by the method of joints either from A or from I. If we start from A, we will have to solve the joints A, P, B, O and C (in that order) first. Then we can solve joint N, which will give S_3. Proceeding next to joint D, we shall get S_1. Then solving joints E and M, we shall get S_2. Obviously this will involve considerable time, effort and calculations. Rather than pursuing this laborious procedure, we adopt the method of sections as follows :

We imagine that; (a) the truss is sawn off along XX into two portions severing members 1, DM and 3 completely and (b) at the same time the force pairs S_1, S_{DM} and S_3 of the appropriate magnitude and sense (as developed in the uncut truss) are replaced so as to act along their cut lengths. [For the sake of understanding, these are all shown as tensile in the figure.] If any of them happens to be compressive, say S_1, it will only mean that S_1 has a negative sign. With this done, the force systems on the two cut portions of the truss-one left of XX and the other right of XX-remain exactly the same as they were before the cut was made and hence both the cut portions will remain in equilibrium. We now consider the equilibrium of the portion left of XX (Figure 2.19) which is subject to all the known external forces on that portion of the truss plus three other forces viz. S_1, S_{DM} and S_3, which are internal to the truss and are unknown. The effect of sectionalising has been that the three internal forces in the truss have been, so to say, exposed. We can now apply the three fundamental equations of equilibrium to this portion, to obtain the above three unknown forces. Since we want S_1, we take moments about M, the point of intersection of S_{DM} and S_3.

$\Sigma M_M = 0$ gives (Figure 2.19).

$$S_1 \times 5 + 35 \times 12 - 10 \times 9 - 10 \times 6 - 10 \times 3 = 0$$

[S_{DM} and S_3 will not enter the equation as they pass through M.]

$S_1 = -48$ kN, i.e. Bar 1 carries 48 kN compression

and, $\Sigma M_D = 0$ gives

$$S_3 \times 5 - 35 \times 9 + 10 \times 6 + 10 \times 3 = 0.$$

[S_{DM}, S_1 and the 10 kN force at N pass through D]

$\therefore$ $S_3 = +45$ kN, i.e. member 3 carries a tensile force = 45 kN.

For obtaining S_2, we take section YY and consider the F. B. D. of the portion right of YY (Figure 2.19) which is in equilibrium under the action of V_1, 10 kN loads at L, K, and J and the bar forces S_{EF}, S_2 and S_{ML}.

Now, $\Sigma F_y = 0$, we get

$$S_2 \cos \theta + 10 + 10 + 10 - V_1 = 0,$$

i.e. $$S_2 = \frac{35 - 30}{\cos \theta} = +5.831 \text{ kN tension.}$$

All the above values can be checked by the method of joints.

From the above example, the following points should be noted:

(i) When a truss is in equilibrium, its every part is in equilibrium under the action of the forces external to that part.

(ii) Since the conditions of equilibrium gives only three equations unknown forces (only if numbering 3 or less) are obtainable from them.

(iii) For the successful application of the method of sections we should select such a cutting section that besides the member in which the force is desired, not more than two others are cut in which the forces are unknown, and sketch the complete F.B.D. of that portion.

SAQ 5

Calculate the forces in the member marked 1, 2 and 3 for the bow-string truss shown in Figure 2.20. Loads are in kN.

Figure 2.20

2.6 SUMMARY

In this unit, you have learnt to classify various types of supports and constraints, rollers, rockers, ball and socket joints, frictional surfaces, short links, cables, pins, hinges and the examples of support connections. These connections offer resistance to the movement of the bodies which are called **reactions**. Following tables give the various types of connections and their reactions.

Table 2.2

Sl. No.	Types of Connections	Reactions
1	Cable	A tension force (pull acting away from the member along the direction of the cable.
2	Link	A force acting along the axis of the link.
3	Roller	A force acting perpendicular to the surface at the point of contact.
4	Rocker	A force acting perpendicular to the surface at the point of contact.
5	Smooth surface	Same as roller.
6	Pin or Hinge	A force having two unknowns : magnitude and direction or two forces with unknown magnitudes in two mutually perpendicular known directions, i.e. H_A and V_A.
7	Member fixed/connected to a collar on smooth rod	A couple (moment) and the force acting perpendicular to the rod.
8	Fixed support	A couple (moment) and two unknowns for components.

The first step in solving equilibrium problems is to draw free-body diagram. Isolate the body from its surrounding. Replace the constraints by reactions. Show all forces acting on the member and apply the equations of static equilibrium.

In the general case of a system of equilibrium, there are six equations of static equilibrium. These ensure that the resultant force and resultant moment both are zero. Hence, we write :

(i) $\Sigma F_x = 0$

(ii) $\Sigma F_y = 0$

(iii) $\Sigma F_z = 0$

(iv) $\Sigma M_x = 0$

(v) $\Sigma M_y = 0$

(vi) $\Sigma M_z = 0$

where *x*, *y* and *z* are three mutually perpendicular axes.

In particular, if the forces are parallel and we take *z* axis parallel to them, then the first, second and last equations are identically satisfied (no force exist along *x* and *y* axes). Therefore, the equations of static equilibrium are reduced to three, viz.

$$\Sigma F_z = 0, \Sigma M_x = 0, \Sigma M_y = 0$$

If the forces are concurrent and if we choose the point of concurrence as the origin then the last three equations will always be satisfied and only three equations will be required to solve the problems. These are

$$\Sigma F_x = 0, \Sigma F_y = 0, \Sigma F_z = 0$$

Similarly, in the case of concurrent forces in a plane there will be only two equations of static equilibrium :

$$\Sigma F_x = 0 \text{ and } \Sigma F_y = 0$$

In case of parallel forces in a plane, these equations will be

$$\Sigma F_z = 0 \text{ and } \Sigma F_y = 0$$

where 0 is the moment centre in the plane containing the parallel forces. 0 may be any point in the plane. The *z* axis will be parallel to the direction of forces.

2.7 ANSWERS TO SAQs

SAQ 1

(a) Free body diagrams for member AB are as follows :

(a)

(b)

(c) **(d)**

(e) **(f)**

Figure for Answer to SAQ 1

(b) Answers may be verified from the preceding text.

SAQ 2

Figure for Answer to SAQ 2

Let the reaction offered by the step on the wheel at the start of motion be R as shown in Figure for Answer to SAQ 2.

The reaction offered by the ground has been taken as zero as the wheel is just on the point of moving over the step.

Since the wheel is in equilibrium, therefore,

$\Sigma M_A = 0$

Taking moments about A,

$$P \sin \alpha \; AC + P \cos \alpha \; OC = W \times AC$$

$$P \sin \alpha + P \cos \alpha \left(\frac{OC}{AC} \right) = 800$$

> Since, $\sin \angle OAC = \frac{OC}{OA} = \frac{0.3}{0.6} = \frac{1}{2}$
>
> $\therefore \; \angle OAC = 30^{o}$
>
> Therefore, $\frac{OC}{OA} = \tan 30^{o} = \frac{1}{\sqrt{3}}$

$$P \sin \alpha + P \cos \alpha \left(\frac{1}{\sqrt{3}} \right) = 800$$

$$\sqrt{3} \; P \sin \alpha + P \cos \alpha = 800 \times \sqrt{3}$$

$$P = \frac{800 \sqrt{3}}{\sqrt{3} \sin \alpha + \cos \alpha}$$

For getting minimum value of force P, $\frac{dP}{d\alpha}$ should be 0.

Hence, $$\frac{d}{d\alpha} = \left(\frac{800 \sqrt{3}}{\sqrt{3} \sin \alpha + \cos \alpha} \right) = 0$$

Then $$\frac{dP}{d\alpha} = - \frac{800 \sqrt{3} \, [\sqrt{3} \cos \alpha - \sin \alpha]}{[\sqrt{3} \sin \alpha + \cos \alpha]^2} = 0$$

or $$\sqrt{3} \cos \alpha - \sin \alpha = 0$$

$$\tan \alpha = \sqrt{3}$$

$$\alpha = \tan^{-1} \sqrt{3} = 60^{o}$$

Now, $$P = \frac{800 \sqrt{3}}{\sqrt{3} \sin \alpha + \cos \alpha}$$

For $\alpha = 60^{o}$, $\quad P = 400 \sqrt{3}$ N

The student may verify the answer by applying Lami's Theorem.

SAQ 3

(a)

Figure for Answer to SAQ 3(a)

Let vertical and horizontal reactions at A be R_{AV} and R_{AH}, respectively. M_A is the total moment coming on the fixed end A of cantilever beam AB. Let us replace the uniformly distributed load of 16 kN/m by a single force of 16 × 1.2 = 19.2 kN acting at a distance of $\left(0.6 + \frac{1.2}{2}\right) = 1.2 \text{ m}$ from A. As the beam is at rest under the action of the forces, the condition of equilibrium can be applied to calculate R_{AV}, R_{AH} and M_A.

$$\Sigma F_x = 0$$

$$\therefore \quad R_{AH} - 28 \cos 45^{\circ} = 0$$

$$\therefore \quad R_{AH} = 19.8 \text{ kN}$$

$$\Sigma F_y = 0$$

$$\therefore \quad R_{AV} - 28 \sin 45^{\circ} - 19.2 = 0$$

$$\therefore \quad R_{AV} = 39 \text{ kN}$$

$$R_A = \sqrt{(R_{AH})^2 + (R_{AV})^2}$$

$$= \sqrt{(19.8)^2 + (39)^2} = 43.74 \text{ kN}$$

$\tan \theta = \frac{R_{AV}}{R_{AH}} = \frac{39}{-19.8} = -19.7$ (–ve sign used to indicate the angle w.r.t –ve x-axis in clockwise direction)

$$\therefore \quad \theta = -63^{\circ}\ 5'\ 0.4'' \text{ or } -296^{\circ}\ 54'\ 59.6''$$

Taking moments about A, we get

$$\Sigma M_A = 0$$

$$\therefore \quad -M_A + (16 \times 1.2)\ (1.2) + 28 \sin 45^{\circ} \times 0.6$$

$$\therefore \quad M_A = 23.04 + 11.88 = 34.92 \text{ kN.m}$$

(b)

Figure for Answer to SAQ 3(b)

Let the reaction components at A be H_A and V_A as shown in figure. The reaction at B, i.e. R_B will be acting vertically upwards. As the beam is in equilibrium, the condition of equilibrium can be applied. The moment of the couple acting at D can be calculated as 20 × 1 = 20 kN-m acting in clockwise direction. The uniformly distributed load of 12 kN/m can be replaced by a single force 12 × 2.5 = 30 kN acting at $7.0 + \frac{2.5}{2} = 8.25$ m from A.

There is no horizontal force acting on the beam. Then,

$\Sigma F_x = 0$

$\therefore \quad H_A = 0$

Taking moments of all forces about A, we get

$\Sigma M_A = 0$

$$32 \times 2 + 20 - R_B \times 7.0 + (12 \times 2.5)\ (8.25) = 0$$

or $\quad 64 + 20 - 7.0\ R_B + 247.50 = 0$

$$R_B = \frac{64 + 20 + 247.5}{7}$$

$$= 47.36 \text{ kN}$$

Note : Moments due to H_A and V_A about A are zero as they pass through point A. Clockwise moments have been taken as positive.

$\Sigma F_y = 0$

$\therefore \quad V_A - 32 + 47.36 - 30 = 0$

$\therefore \quad V_A = 14.64 \text{ kN}$

Therefore, the reaction at A is 14.64 kN, whereas reaction at B is acting vertically upwards and has a magnitude of 47.36 kN.

(c)

Figure for Answer to SAQ 3(c)

Let the vertical and horizontal reactions at A be R_{AV} and R_{AH} respectively. M_A is the total moment coming on the fixed end A of cantilever beam AB. Let us replace the triangular loading by a single force of

$$\frac{1}{2} \times 24 \times 1.2 = 14.4 \text{ kN}$$

acting at a distance of $\frac{1}{3}(1.2) = 0.4$ m from A.

As the beam is at rest, under the action of forces, the condition of equilibrium can be applied to calculate R_{AV}, R_{AH} and M_A.

$\Sigma F_x = 0$

$\therefore \quad R_{AH} = 0$

$\Sigma F_y = 0$

$\therefore \quad R_{AV} - \frac{1}{2} \times 24 \times 1.2 = 0$

$\therefore \quad R_{AV} = 14.4$ kN

Taking moments about A, we get

$\Sigma M_A = 0$

$\therefore \quad - M_A + \frac{1}{2} \times 24 \times 1.2 \times 0.4 = 0$

$\therefore \quad M_A = 5.76$ kN-m.

Therefore, the reaction at A consists of a vertical force acting upwards with a magnitude of 14.4 kN and an anticlockwise moment of 5.76 kN-m.

(d)

Figure for Answer to SAQ 3(d)

There is an internal hinge at C from where the beam may be cut into two sections AC and CB as shown below :

Part 1 **Part 2**

Now, Part 1 and Part 2 can be analyzed separately to get R_{AV}, R_{AH}, M_A and R_B.

Part 1

This part now can be analyzed as cantilever beam with R_C as a concentrated load at the free end. Here R_C = 34 kN (it can be calculated easily) because moment about C is zero – see ahead also i.e., shear force at C = 20 × 1 + 40 – R_B.

Taking moment about A,

$\Sigma M_A = 0$

$\therefore \qquad - M_A + 20 \times 3 \times 1.5 + 34 \times 3 = 0$

$\therefore \qquad M_A = 192 \text{ kN} - \text{m}$

Now, taking moments about C,

$\Sigma M_C = 0$

$\therefore \qquad R_{AV} \times 3 - 192 - 60 \times 1.5 = 0$

$\therefore \qquad R_{AV} = 94 \text{ kN}$

And, $\qquad \Sigma F_x = 0$

$\therefore \qquad R_{AH} = 0$

Therefore, the reaction at A consist of a vertical force acting upwards of the magnitude 94 kN and an anticlockwise moment of 192 kN-m whereas reaction at B is also acting vertically upwards and has a magnitude of 26 kN (i.e., $20 \times 4 + 40 - R_{AV}$ or $R_B \times 5 = 40 \times 3 + 20 \times 1 \times \frac{1}{2}$, i.e. $R_B = 26$ kN).

Part 2

This part can be analysed as simply supported beam.

Taking moment about C,

$$R_B \times 5 = 3 \times 40 + 1 \times 20 \times \frac{1}{2}$$

$$\therefore \qquad R_B = \frac{130}{5} = 26 \text{ kN}.$$

$\Sigma F_y = 0$

$\therefore \qquad R_c - 1 \times 20 - 40 + 26 = 0$

$\therefore \qquad R_c = 34 \text{ kN}.$

SAQ 4

First we draw the diagram of the truss showing all loads and reaction-components on it, and erase the middle portions of the lengths of all members. The concurrent force-system at all the joints can then be clearly exposed by drawing arrows next to each joint along every member there – all arrows away from the joints. Now looking for a joint where the number of unknown forces is two or less, we find that there is no such joint and the method seems to fail at the start itself. Here, therefore, calculation of the reactive components at the supports has to precede the application of the method of joints to the truss.

Figure to Answer to SAQ 4

Considering now, the equilibrium of the truss,

$\Sigma F_x = 0$ gives

$H_A = 8$ kN leftwards.

$\Sigma M_A = 0$ gives

$V_E \times 12 - 8 \times 3.464 - 15 \times 8 - 10 \times 4 = 0$ ($\because$ distance between BD and AE = 4 cos 30 = 3.464)

$\therefore \quad V_E = 15.643$ kN $\uparrow$

$\Sigma M_E = 0$ gives

$V_A \times 12 + 8 \times 3.464 - 15 \times 4 - 10 \times 8 = 0$

$\therefore \quad V_A = 9.357$ kN $\uparrow$

Check : $V_A + V_E = 25$ kN = total downwards load.

Now, we find that at both A and E we have not more than two unknown forces and we can start from either joint.

(i) Joint A

$\Sigma F_y = 0$ gives

$$S_1 \cos 30 + V_A = 0.$$

$$\therefore \quad S_1 = -\frac{9.357}{\cos 30} = -10.805$$

$\therefore \quad$ 10.805 kN (compression)

$\Sigma F_x = 0$ gives

$$S_1 \cos 60 + S_7 - H_A = 0,$$

i.e. $\quad S_7 = 8 + 10.805 \times \frac{1}{2} = +13.402,$

i.e. $\quad$ 13.402 kN (tension)

(ii) Joint B

Since S_1 value is negative, the arrows for it must be reversed. In particular at Joint B, S_1 actually acts upwards and to the right with lines of action of S_8 and S_2 lying at 60^o and 120^o, respectively to it. To sketch the closed triangle of forces at B, we draw S_1 vector = 10.805 units starting from 'SP' [Figure SAQ 4(c)] and two lines one at 60^o to S_1 through its tail and the other also at 60^o through its head and complete the triangle. Obviously the triangle is equilateral, and magnitudes of S_8 and S_2 will both equal to that of S_1.

Starting with the sense of the vector S_1, we mark arrows on S_8 and S_2 so that they chase one another in the sense indicated by that of S_1. From the marked arrows, we see that S_8 is tensile and is 10.805 kN and S_2 is compressive and of the same magnitude. Thus, arrows for S_8 in Figure SAQ 4(b) are alright but those for S_2 must be reversed (or S_2 taken = – 10.805 with the same arrows).

(iii) Joint G

$\Sigma F_y = 0$ gives

$$S_9 \cos 30 + S_8 \cos 30 - 10 = 0.$$

. $$S_9 \cos 30 + 10.805 \times 0.866 - 10 = 0$$

. $$S_9 = + 0.742 \text{ kN},$$

i.e. member 9 carries a force = 0.742 kN (tension).

$\Sigma F_x = 0$ gives

$$S_6 + S_9 \cos 60 - S_8 \cos 60 - S_7 = 0$$

i.e. $$S_6 + 0.742 \times \frac{1}{2} - 10.805 \times \frac{1}{2} - 13.402 = 0.$$

$\therefore$ $$S_6 = + 18.433 = 18.433 \text{ kN} \quad \text{(tension)}$$

(iv) Joint C

$\Sigma F_y = 0$ gives

$$S_{10} \cos 30 + S_9 \cos 30 = 0.$$

$\therefore$ $$S_{10} = - S_9 = - 0.742,$$

i.e. 13.402 kN (tension)

$\Sigma F_x = 0$ gives

$$S_3 + S_{10} \cos 60 - S_9 \cos 60 - (-10.805) = 0$$

$\therefore$ $$S_3 = - 10.805 + 0.742 \times \frac{1}{2} + 0.742 \times \frac{1}{2} = - 10.063 = 10.063 \text{ kN (C)}$$

Having analysed half the truss from the left side it is advisable to analyse the remaining from the right side starting from E, so as to avoid the carrying forward of mistakes, if any, committed in the earlier calculations on the left half.

(v) Joint E

$\Sigma F_y = 0$ gives

$$S_4 \cos 30 + V_E = 0,$$

i.e. $$S_4 = - \frac{15.643}{\cos 30} = - 18.064$$

i.e. 18.064 kN (compression)

$\Sigma F_x = 0$ gives

$$S_5 + S_4 \cos 60 = 0,$$

i.e. $$S_5 + (-18.064) \times \frac{1}{2} = 0,$$

i.e. S_5 is 9.032 kN (tension)

(vi) Joint D

$\Sigma F_y = 0$ gives

$$S_{11} \cos 30 + S_4 \cos 30 = 0$$

$\therefore$ $S_{11} + (-18.064) = 0.$

$\therefore$ $S_{11} = +18.064$

i.e. 18.064 kN (tension)

$\Sigma F_x = 0$ gives

$$S_3 + S_{11} \cos 60 - 8 - (S_4 \cos 60) = 0,$$

i.e. $S_3 + \dfrac{18.064}{2} - 8 - (-18.064) \times \dfrac{1}{2} = 0.$

$\therefore$ $S_3 = -10.064$ kN

$\therefore$ $S_3 = 10.064$ kN (compression)

[tallies with the value of S_3 in (iv) above – check 1].

(vii) Joint F

$\Sigma F_y = 0$ gives

$$S_{10} \cos 30 + S_{11} \cos 30 - 15 = 0$$

$\therefore$ $S_{10} \times 0.866 + 18.064 \times 0.866 - 15 = 0$

$S_{10} = -0.742$ kN

i.e. $S_{10} = 0.742$ kN (compression)

[tallies with the value of S_{10} in (iv) above – check 2.]

$\Sigma F_x = 0$ gives

$$S_6 + S_{10} \cos 60 - S_{11} \cos 60 - S_5 = 0,$$

i.e. $S_6 + (-0.742) \times \dfrac{1}{2} - 18.064 \times \dfrac{1}{2} - 9.032 = 0$

$S_6 = +18.435 = 18.435$ kN (tension)

[tallies with the value of S_6 in (iii) above – check 3.]

The above three checks in the work-out may be noted. Since equilibrium of the entire truss involves three equations which have been included in the joint equations, we shall get only three checks as shown above, using all the $2j$ equations of equilibrium of the j joints.

The answer to the problem is presented in following figure.

Figure for Answer to SAQ 4

SAQ 5

Considering the equilibrium of the whole truss, $\Sigma\, M_B = 0$ gives,

$$V_A \times 24 - 400 \times 20 - 600 \times 16 - 300 \times 12 - 200 \times 8 - 300 \times 4 = 0.$$

$$V_A = 1000 \text{ kN} \uparrow .$$

Taking section XX and considering the Free-body diagram of the portion left of XX as shown in figure, the forces on it are V_A = 1000 kN, loads of 400 kN and 600 kN at H and D respectively (as given) and forces S_1, S_2 and S_3 in bars 1, 2 and 3 respectively, all assumed tensile.

Figure for Answer to SAQ 5

(i) To get S_1, we take $\Sigma\, M_F = 0$, where F is the point of intersection of S_2 and S_3 which although unknown by themselves have known moment about F, viz. zero. Also moment of S_1 about F can be more easily worked out by summing the moments of its x and y components about F, rather than working out its moment-arm. Hence,

$$S_1 \cos\theta \times 5.3 + S_1 \sin\theta \times 4 + 1000 \times 12 - 400 \times 8 - 600 \times 4 = 0$$

But $\quad \tan\theta = \dfrac{0.7}{4};$

$\therefore \quad \theta = 9.9262$

$\therefore \quad \cos\theta = 0.985$

and $\quad \sin\theta = 0.172$

$\therefore \quad S_1 \times 0.985 \times 5.3 + S_1 \times 0.172 \times 4 = -\,64 \text{ N}$

$\therefore \quad S_1\ (5.221 + 0.688) = -\,6400.$

$\therefore \quad S_1 = -\,1083 \text{ kN},$

i.e. member 1 is in compression of magnitude 1083 kN.

(ii) To obtain S_2 we take moments about G, the point of intersection of S_1 and S_3. For computation of moment of S_2 about G, we take it to act at F and sum the moments of its x and y components.

Location of G is obtained by $\tan\theta = \dfrac{EF}{GF}$, also $\tan\theta = \dfrac{0.7}{4}$.

$\therefore \quad GF = \dfrac{6 \times 4}{0.7} = 34.286 \text{ m}.$

$\therefore \quad G_A = 34.286 - 12 = 22.286$

$\Sigma M_G = 0$ gives

$$S_2 \cos\phi(GF) - 1000(22.286) + 400 \times 26.286 + 600 \times (30.286) = 0$$

$$S_2 \cos \phi \text{ (GF)} - 22286 + 10514.4 + 18171.6$$

or $\quad S_2 \cos \phi \text{ (GF)} = -64$ N

But $\quad \tan \phi = \dfrac{4}{5.3} = 0.7547;$

$\therefore \quad \phi = 37.0418^{o}$

$\therefore \quad \cos \phi = 0.798.$

This gives $S_2 \times 0.798 \times 34.286 = -6400.$

$$S_2 = -233.8 \text{ kN}$$

which means that member 2 is in compression whose magnitude = 233.8kN.

(iii) To obtain S_3, we take $\sum M_C = 0$ which gives

$$S_3 \times 5.3 + 400 \times 4 - 1000 \times 8 = 0$$

from which, $\quad S_3 = 1208$ kN,

i.e. member S_3 is in tension and the force developed in it = 1208 kN.

2.5.4 Graphic Statics

Problems of Statics-particularly those of coplanar force systems-can be easily and elegantly solved by the methods of Graphic Statics. If in a particular problem, such a solution can be obtained while the exact one as required by analysis mathematical is either not obtainable or is too time-consuming to arrive at for many engineering problems, methods of graphic statics using a reasonable degree of care is not only entirely acceptable but it also results in giving the engineer a good physical conception of the problem.

Space Diagram, Bow's Notation and Force Polygon

The problem of finding the resultant of a force-system (co-planar) acting on a body is outlined in Figure 2.19(a). This figure is called the *space diagram* for the force-system and must be drawn to scale when the problem is to be solved graphically.

Figure 2.19

There are four forces in the system, viz, P_1, P_2, P_3 and P_4, acting at the points 1, 2, 3, and 4 respectively on a body [Figure 2.19(a)], whose resultant R is to be obtained. For designating the forces, graphic statics adopts a two letter notation for each force, called the *Bow's Notation* which is based on the fact that the line of action of each force divides the plane in which it acts, into two 'spaces' one on each of its sides. According to this notation, we name these spaces in order, say from left to right by the capital letters of the alphabet A, B, C, D etc. for convenience and call each force by the name of the two spaces between which its line of action is situated, taken in that order. Thus, forces P_1, P_2, P_3 and P_4 will be called forces 'AB', 'BC', 'CD', 'DE' respectively. Next draw to scale the force polygon 'abcde' for these forces as in Figure 2.19(b), naming the force-vectors by the corresponding small case letters, i.e. vector **ab** for force 'AB', vector **bc,** for force 'BC' and so on. This correspondence constitutes one of the advantages, afforded by adopting the Bow's notation. Vector **ae** gives the resultant R in all respects except its location on the space diagram.

Select any point 'O' near the force polygon and join it to its vertices viz. a, b, c, d and e. The point 'O' is called the 'Pole' and the lines oa, ob, oc, od and oe radiating from it the 'rays'. This part of the diagram is called the "polar diagram". We draw a line ***IJ***, in space A parallel to the ray 'ao' cutting the force 'AB' at a point J. We next draw through J, a line JK in space B, parallel to the ray 'bo' cutting the force BC in K and follow up by drawing the line KL through K in space C parallel to the ray 'co' cutting the force CD in L and another through L in space D parallel to the ray 'do' cutting the next force DE in M. Finally, we draw a line through M in space E parallel to ray 'eo'. In brief, each of the steps on the construction involves drawing through the previously obtained point on the line of action of a force, a line in the next space parallel to the ray of the same name as that space, so as to cut the next force. The lines IJ, JK, KL, LM, and MN are called the 'strings' and the entire figure IJKLMN, the string polygon or the funicular polygon. We then produce the end-strings IJ and EM to meet at 'S', which is a point on the line of action of the resultant 'R'.

The polar diagram should be looked upon as a device to split each of the given forces or force-vectors into two oblique components-one formed by the vector from the tail, i.e. the starting point of the particular force-vector to the pole and the second from the pole to the head, i.e. the end point of that force-vector, with these result that we have, force P_1, i.e. the force AB represented by **ab**.

$$= \mathbf{ao} + \mathbf{ob} \text{ (see arrows inside } \Delta\, abo).$$

Force P_2, i.e. the force BC represented by **bc**

$$= \mathbf{bo} + \mathbf{oc} \text{ (see arrows inside } \Delta\, bco).$$

Force P_3, i.e. the force CD represented by **cd**

$$= \mathbf{co} + \mathbf{od} \text{ (see arrows inside } \Delta\, cdo)$$

Force P_4, i.e. the force DE represented by **de**

$$= \mathbf{do} + \mathbf{oe} \text{ (See arrows inside } \Delta\, deo).$$

With the help of single pole O it can be concluded that

(i) If the given force-system has 'n' forces, the polar diagram splits them into '$2n$' vectors, which are all concurrent, and

(ii) For any two adjacent force-vectors of the given system, the second component of the first on the polar diagram cancels the first components of the second, the two being equal and opposite, e.g. **ob** of **ab** and **bo** of **bc, oc** of **bc** and **co** of **cd** and so on. The pairs of components represented by each of the intermediate rays thus cancel out each other, with the result that eventually the first component of the first force, viz. **ao** and the second of the last one, viz. **oe,** are left uncancelled, whose resultant as is already known, is 'R' which is represented in magnitude, direction and sense by their vector – sum = **ae**.

In the funicular polygon, each of its vertices can be considered as the point where the force of the given system, acts and where this force is considered to have been replaced by its two oblique components as given by those on the polar diagram, e.g. at J, forces **a′o′** and **o′b′** represented by vectors **ao** and **ob** respectively act in place of P_1, similarly at K, forces

b′o′ and **o′c′** represented by vectors **bo** and **oc** respectively, act in place of P_2 and so on. Thus we now assume that the four forces P_1, P_2, P_3 and P_4 no more act at P_1, P_2, P_3 and P_4 respectively but in their place, we have the system of eight forces, viz. **a′o′** is the equivalent of the original system in all respects including locations. Out of these eight forces, it is easy to see that the force-pairs acting along each of the intermediate strings cancel out being equal, opposite and coincident, e.g. forces **o′b′** and **b′o′** along JK (each force being = | **bo** | in the polar diagram), **o′c′** and **c′o′** along KL (each being = | **co** | in the polar diagram) etc. Thus the system is seen to reduce to a statically equivalent one to only two forces viz., **a′o′** and **o′e′** (represented by vectors **ao** and **oe** respectively on the polar diagram), acting along the end strings, *IJ* and *NM* respectively. The intersection point of these two strings is therefore a point on the line of action of the resultant *R*.

In essence, the funicular polygon construction reduces a given system of forces to a system of just two forces one acting on each of its end strings, which is equivalent in all respects to the original system. If the sense of each of these two forces is reversed, the two together, will represent the equilibrant of the system, and therefore, if we run a string along the profile of the funicular polygon, i.e. along *IJKLMN* in this case, and apply a force **o′a′** equal in magnitude to | **oa** | on the polar diagram, along *JI* from *J* to *I* and another **e′o′** equal in magnitude to | **eo |** on the polar diagram, along *MN* from *M* to *N*, it will hole the P_1, P_2, P_3, P_4 system acting at points p_1, p_2, p_3 and p_4 in equilibrium, with the string *IJKLMN* keeping to its shape. The tension in the string in its A and B portions will balance P_1 at J, and those in B and C portions balancing P_2 at K and so on. For this reason, the line *IJKLMN* is called the string polygon or funicular polygon.

Briefly, therefore, it may be stated that in the case of a force-system, which is known to be in equilibrium, the force-polygon, as well as the funicular polygon must close-this fact is made use of in solving problems involving coplanar force-system in equilibrium.

Table 2.1 : Properties of Force Polygon and Funicular Polygon

Case No.	System Reduces to	Force Polygon	Funnicular Polygon	End Strings of the Funicular Polygon
1	Single resultant force '*R*'	Open vector from '*SP*' to '*EP*' gives '*R*' in mag., direction and sense.	Open has (n + 1) strings	Their intersection gives a point on the line of action of '*R*'.
2	Couple having moment = '*M*'	Closed '*EP*' falls on '*SP*'. No resultant force	Open has (n + 1) strings	Parallel and distinct forces acting on these form the couple whose moment = M = magnitude of one of the force given by the 1st (or last) ray acting on the 1st (or last) end string × the

				perpendicular distance, between the end strings.
3	System is in Equilibrium	Closed '*EP*' falls on '*SP*' indicates zero resultant force	Closed has '*n*' strings	Coincident since the two forces to which the system is reduced and which act on the end strings neutralize each other.

Note : 'SP' and 'EP' denote the starting point and the end point, respectively of the force polygon.

Example 2.10 : Reactions of Simply Supported Beams

A simply supported beam AB (span 12 m) has its end A hinged and is provided with a roller support whose surface is inclined at an angle of 20^{o} to the horizontal, at the other end B. Figure 2.20, shows the beam and the loads- 10 kN at 60^{o} to the horizontal, a u. d. (uniformly distributed load) with intensity 4 kN/m (3 m long), a 20 kN and another 8 kN load.

Figure 2.20

Obtain the reaction R_A and R_B developed at the supports by the graphical method.

Solution

First replace the u. d. l. by its statically equivalent load = $4 \times 3 = 12$ kN placed at its mid-point. This is shown by the dotted vector [Figure 2.20(a)]. In addition to the four forces, viz., the four point loads now, we have R_A, the reaction at A passing through A, with direction not known, and R_B, the reaction at B inclined at 20^{o} to the vertical as shown. Marking the various spaces as shown, we have PQ, QR, RS and ST as the four known loads of 10 kN, 12 kN, 20 kN and 8 kN, respectively. The reactions R_B and R_A will be denoted by TU and UP respectively. (Being a case of equilibrium, there will be as many spaces as there are forces, including reactions – six in this case). Coming to the force polygon, we can easily, draw the part force polygon pqrstu ′where *tu*′ is a line through '*t*' parallel to the reaction R_B. We cannot mark the point u corresponding to '*U*' at this stage – in fact the problem is to locate the point *u*, so as to obtain the reactions R_B and R_A from the vectors **tu** and **up**.

In spite of this incomplete force-polygon, however, we find that we can draw the funicular polygon completely. Regarding it we recall that :

(i) each of its vertices lies on the line of action of a force of the system and that

(ii) it is closed in the case of equilibrium (as here).

In the present problem, we know the locations of the lines of action of all the forces except R_A, in respect of whose line of action, however, we know that it passes through A. Using this fact, we *start drawing the funicular polygon from A* and obtain the funicular polygon A-1-2-3-4-5. The complete funicular polygon whose last string A-5 with vertices lying on the two reactions can now be drawn on the polygon. It is therefore called the "closing line" of the funicular polygon and will be the line along which both of its end-strings must lie clearly in space '*U*'. Hence, the corresponding ray '*ou*' of the polar-diagram will be a line, through the pole O parallel to the closing line and this line for the ray *ou* can accordingly be drawn, [see dotted line in Figure 2.20(b)]. Since the point *u* lies on this ray and the line *tu′***,** their intersection point must be *u*. This at once gives the two reactions represented by the vectors **tu** as R_B and **up** as R_A. By scaling off, we find that $R_A = 26.2$kN inclined at 100^o to the vertical and $R_B = 24$ kN has an inclination as marked. This solution can be checked by the analytical method, and would be found to be reasonably accurate.

Example 2.11

A warren truss made up of equilateral triangles, each side 3 m, is shown in the Figure 2.21. On the top boom there are 2 vertical loads 6 kN and 12 kN and a horizontal load of 4 kN. On the bottom boom, there is a load of 8 kN inclined at 20^0 to the vertical and another 6 kN which is vertical. The support A is hinged while B is a roller support.

Determine the reactions graphically.

Figure 2.21

Solution

The two loads of 8 kN and 6 kN acting on the lower boom, whose line of action are seen to lie in between the two supports are shifted to the upper boom as shown in the Figure 2.21(a) by show the dotted lines. Having named the spaces now, we find that the loads are : PQ, QR, RS, ST, TU and the reactions are UV and VP. We now draw the part-force polygon pqrstu and the line *uv* parallel to known direction of R_B. The problem now reduces to locating the point v on *uv′* so that **uv** will represent R_B and **vp**, R_A.

Figure 2.22

Proceeding exactly as in the previous example; choosing a pole O we complete the polar diagram. Then starting from the hinge A (*this is mandatory)* on the space diagram which is the only known point on R_A, we complete the funicular polygon A-1-2-3-4-5-6 and obtain the closing line A-6. A line through the pole O, parallel to the closing line cuts *uv′* at the point *v* which gives the vectors **uv** and **vp** representing R_B and R_A respectively. By scaling off these, we find that R_A = 18.3 kN inclined at 22^0 to the vertical and R_B = 14.7 kN vertical, an acceptably accurate solution.

2.5.5 Maxwell Diagram or Stress Diagram for Trusses

Drawing of a Maxwell Diagram (also called 'Stress Diagram') for a truss under a given loading is a fast method of obtaining the forces in its members. It is primarily based on the fact that every joint in a truss, is in equilibrium under the action of the system of coplanar concurrent forces meeting there. These forces consist of the applied loads if any at the joint and the axial forces developed in the members or bars which meet there. In fact by following this method and drawing a separate. Closed. The Maxwell diagram for a truss is however one single diagram which combines the force-polygons at all of its joints and in which the vector for any bar-force is drawn just once. Furthermore, as the external forces on the truss (such as loads and support-reactions) also act at the joints, the vectors for them, must appear in this single diagram and because they constitute independent force-system in equilibrium, this fact provides a check on the diagram of graphical construction of the Maxwell diagram.

Consider a Warren truss shown in Figure 2.22. The truss consists of three equal equilateral triangle shaped panels joined by two horizontal bars at the top boom and is hinged at the joint L_0 and supported by a roller at L_3. It carries vertical loads of 4 kN each at the joints U_0, U_1 and U_2 and 12 kN and 6 kN at L_1 and L_2 respectively. There is also a horizontal load of 6 kN at U_2. It is required to obtain the forces in the various members by the Maxwell diagram construction.

Example 2.12

Consider a Warran truss as shown in Figure 2.22. Carrying in loads as shown.

Solution

To start first compute the reactions at supports L_0 and L_3 either graphically or analytically. These will be $V_{L0} = 14.3$ kN $\uparrow$.

$H_{Lo} = 6.0$ kN $\leftarrow$ and $V_{L3} = 15.7$ kN $\uparrow$.

Then the spaces between external loads including reactions are then identified using Bow's notations by letters A, B, C, . . . The internal space between members are then also identified similarly by letters J, K, L, . . . etc. to indicate member force vectors. For example force in member $U_o V_1$ (i.e. internal force $S_{Uo\,V1}$ is denoted by B. K at U_o (vector *bk*) equal at opposite internal force at U_1 i.e. $U_1 U_0$ is also denoted by K B (vector *kb*) (Figure 2.22(c)).

Now the Maxwell diagram proper is drawn, first by drawing to scale the complete closed external force diagram abcdefghta (Figure 2.22(d)). The closed force polygons for various joints are then drawn proceeding from one joint to another. It can be noted that only joints L_o and L_3 represent joints which do not have more than two unknown member forces. Any one of them is selected as starting point. Vector *aJ* is drawn on force polygon parallel to $L_o U_1$ and vector *hJ* parallel to $L_o L_1$ intersecting at *J*. *aJ* and *hJ* represent forces in LoU_1 and LoL_1 respectively and can be measured to scale. Force polygon *i a j h i* represent closed force polygon with arrows indicating the direction of forces acting at L_o.

Next we may move to joint U_o and proceeding in the same way, trace abbk-kj-JA to complete force polygon at U_1 with arrows as marked to get point *k* on the Maxwell diagram. The process in continued till all remaining points e.g. *LM* and *N* are located on Maxwell diagram.