UNIT 1 COPLANAR AND NON-COPLANAR FORCES

Structure

1.1 INTRODUCTION

This unit seeks to introduce you the different systems of forces. The prerequisite for this is the concept of a force and the various forms of forces occurring in nature. In addition to this, you should have the basic knowledge of algebra. The study of this unit will enable you to understand the effect of forces on different types of structures.

Objectives

After studying this unit, you should be able to

- identify the different systems of forces,
- add concurrent forces vectorially,
- resolve forces into components,
- add forces by components,
- find the moment of a force, and
- find the resultant of non-concurrent forces.

1.2 SYSTEM OF FORCES

Newton's First Law of Motion helps to define a force as an external agency which tends to change the state of rest or of uniform motion of a body. Force tends to produce motion in a body, changes the motion of a body or checks the motion of a body. In simple words, the action of one body on any other body can be called a force. These actions may be of various forms: Pull or push on a body, gravitational force known as weight of a body, force exerted by an elastic spring, force exerted by a locomotive on the train, resistance offered by the track.

To specify a force, one need to know its magnitude, direction and the point of application. Its magnitude is expressed as Newton in SI unit. Graphically, a force is represented by drawing a line to scale, showing its magnitude and an arrowhead indicating its direction. Such a straight line is called vector.

A combination of several forces acting on a body is called a system of forces or a force system.

Example 1.1

Consider a sphere of mass m suspended by means of a string resting against a smooth wall, as shown in Figure 1.1(a). What are the forces acting on it?

Solution

Let us identify the forces acting on the sphere. These are as follows :

(i) Weight of the sphere, $W = mg$, acting vertically downwards from the centre of gravity of the sphere.

(ii) Tension, T, in the string.

(iii) Reaction, R, offered by the wall.

Thus, the sphere is subjected to a system of three forces as shown in Figure 1.1(b).

(a) **(b)**

Figure 1.1

Example 1.2

Consider a dam retaining water as shown in Figure 1.2(a). What are the forces acting on it?

Solution

The forces acting on the dam are as follows :

(i) Weight of the masonry acting vertically downwards through the centroid of the cross-section, and

(ii) The horizontal water pressure, p, which goes on increasing as the depth of water increases.

Thus, the structure shown is subjected to a system of forces as indicated in Figure 1.2(b)

(a) **(b)**

Figure 1.2

Force System

The system of forces can be classified according to the arrangement of the lines of action of the forces of the system.

The forces (i.e., system of forces) may be classified as

(i) coplanar or non-coplanar,

(ii) concurrent or non-concurrent, and

(iii) parallel or non-parallel.

All these systems of forces have been explained in subsequent paragraphs.

Coplanar Forces

Forces acting in the same plane are called **coplanar forces.** In Figure 1.3(a), forces $\bar{a}$ and $\bar{b}$ are acting in vertical plane *ABCD*. They are called coplanar forces. Forces $\bar{c}$ and $\bar{d}$ are also called coplanar forces as they are acting in one plane. But forces $\bar{a}$ and $\bar{c}$ are not coplanar forces as they are acting in two different plane. If all forces acting on a body meet at a point, they are called **concurrent forces.** Forces $\bar{a}$, $\bar{b}$ and $\bar{c}$ shown in Figure 1.3(b) are concurrent forces as they are meeting at point ***O*,** whereas forces $\bar{d}$, $\bar{e}$ and $\bar{f}$ are called as **non-concurrent forces** because all the three forces are not meeting at a point.

Coplanar forces can also be classified as parallel forces and non-parallel forces. If the line of action of forces are parallel then the forces are called **parallel forces.** If the forces point the same direction they are called parallel forces, and if they point in opposite directions, they are called **unlike parallel** forces, and the forces $\bar{p}$ and $\bar{q}$ are **like parallel** forces, whereas $\bar{q}$ and $\bar{r}$ or $\bar{p}$ and $\bar{r}$ are unlike parallel forces as shown in Figure 1.3(c). Forces $\bar{s}$, $\bar{t}$ and $\bar{u}$ are **non-parallel forces**. Concurrent forces are non-parallel forces. But non-parallel forces may be concurrent or non-concurrent.

(a) (b)

(c)

Figure 1.3

Non-Coplanar Forces

If the line of action of various forces do not lie in the same plane then the forces are called **non-coplanar forces.** These forces may be concurrent or parallel as shown in Figures 1.4(a) and 1.4(b), respectively. Forces $\bar{l}$, $\bar{m}$, $\bar{n}$ and $\bar{o}$ are non-coplanar concurrent forces, and forces $\bar{x}$, $\bar{y}$ and $\bar{z}$ are non-coplanar parallel forces.

(a) (b)

Figure 1.4

SAQ 1

Identify the system of forces in Figures 1.5(a), (b) (c) and (d) and classify them.

(a) **(b)**

(c) **(d)**

Figure 1.5

1.3 COPLANAR FORCES

If a system of coplanar forces is acting on a body, its total effect is usually expressed in terms of its resultant. Force is a vector quantity. The resultant of the system of forces can be found out by using vector algebra, e.g., if the resultant of two forces is to be found out then the law of parallelogram of forces is used for the purpose.

1.3.1Law of Parallelogram of Forces

If the two coplanar forces meet at a point, their resultant may be found by the law of parallelogram of forces, which states that, "If two forces acting at a point are such that they can be represented in magnitude and direction by the two adjacent sides of parallelogram, the diagonal of the parallelogram passing through their point of intersection gives the resultant in magnitude and direction".

Consider two forces *P* and *Q* acting at a point *O* in the body as shown in Figure 1.6(a). Their combined effect can be found out by constructing a parallelogram using vector *P* and vector *Q* as two adjacent sides of the parallelogram as shown in Figure 1.6(b). The diagonal passing through *O* represents their resultant in magnitude and direction. You can prove by the geometry of the figure that the magnitude *R* of the resultant and the angle it makes with *P* are given by :

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}\text{ , and}$$

$$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha} \quad \text{or} \quad \tan\beta = \frac{P\sin\alpha}{Q + P\cos\alpha}$$

where, α is the angle between $\overline{P}$ and $\overline{Q}$

θ is the angle between $\overline{R}$ and $\overline{P}$

β is the angle between $\overline{R}$ and $\overline{Q}$.

The above two forces can also be combined by using the law of triangle of forces which states that if the second force is drawn from the end of the first force then the line joining the starting point of first force to the end of the second force represents their resultant [Figure 1.6(c)].

(a) **(b)**

Figure 1.6

From the triangle of forces, by using trigonometric relations, you can find that,

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

and $$\sin\theta = \frac{Q}{R}\sin\alpha, \quad \sin\beta = \frac{P}{R}\sin\alpha$$

Figure 1.6(c)

1.3.2 Law of Polygon of Forces

If more than two forces are acting on a body, then their resultant can be found by repeated applications of the parallelogram law or the triangle law for one pair of forces at a time. You may start with any two forces and find their resultant first and then add vectorially to this resultant the remaining forces taking one at a time. In the final form a polygon would be completed.

Therefore, if more than two coplanar forces meet at a point, their resultant may be found by the law of polygon of forces, which states that, "If number of forces acting at a point are such that they can be represented in magnitude and direction by the sides of an open polygon taken in order, then their resultant is represented in magnitude and direction by the closing side of the polygon but taken in the opposite order".

Example 1.3

Consider five forces each of 80 N acting at *O* in a body. Draw the polygon of forces and show the resultant of all the forces.

(a) **(b)**

Figure 1.7

Solution

Let us construct a polygon such that the forces $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D}$ and $\overline{E}$ represent the sides of a polygon taken in order (obviously, each side will be equal in magnitude), each force being drawn from the end of earlier force then their resultant is represented by the line joining the starting point of the first force $\overline{A}$ to the end of the last force $\overline{E}$ (Figure 1.7).

SAQ 2

Determine the resultant in magnitude and direction of two forces shown in Figures 1.8(a) and (b) using the parallelogram law and the triangle law.

(a) **(b)**

Figure 1.8

SAQ 3

Four forces are acting at *O* as shown in Figure 1.9. Find the resultant in magnitude and direction by using

(i) polygon law, and

(ii) method of resolution of forces.

Figure 1.9

1.3.3 Resolution and Composition

In many engineering problems, it is desirable to resolve a force into rectangular components. This process of splitting the force into components is called the **resolution** of a force, whereas the process of finding the resultant of any number of forces is called the **composition** of forces. The resolution of forces helps in determining the resultant of a number of forces acting on a body as it **reduces vectorial addition to algebraic addition**.

A force $\overline{F}$ making an angle θ with respect to *x*-axis, as shown in Figure 1.10, can be resolved into two components $\overline{F}_x$ and $\overline{F}_y$ acting along *x* and *y* axes, respectively. If $\bar{i}$ and $\bar{j}$ are the unit vectors acting along *x* and *y* axes, respectively then the force *F* can be expressed as,

$$\overline{F} = F_x \bar{i} + F_y \bar{j}$$

where, F_x and F_y are the magnitude of the components along *x* and *y* axes. Referring to Figure 1.10, F_x and F_y are determined as,

$$F_x = F \cos \theta$$

$$F_y = F \sin \theta, \text{ and } \tan \theta = \frac{F_y}{F_x}$$

(a) **(b)**

Figure 1.10

Note : θ is measured in anticlockwise direction with respect to positive *x*-axis.

The magnitude of the force can also be expressed as

$$F = \sqrt{(F_x)^2 + (F_y)^2}$$

Example 1.4

A force of 120 N is exerted on a hook in the ceiling as shown in Figure 1.11. Determine the horizontal and vertical components of the force.

Figure 1.11

Solution

As θ is to be measured in anticlockwise direction from positive x-axis, then we take $\theta = 300^o$.

$$F_x = F \cos \theta = 120 \cos 300^o = +60 \text{ N.}$$

$$F_y = F \sin \theta = 120 \sin 300^o = -103.92 \text{ N.}$$

The vector components of force $\overline{F}$ are :

$$\overline{F_x} = (+60 \text{ N})\, i \quad \text{and} \quad \overline{F_y} = (-103.92 \text{ N})\, j$$

Therefore, $\overline{F}$ can be expressed as

$$\overline{F} = +60\ \bar{i} - 103.92\ \bar{j}$$

Example 1.5

A force of 80 N is acting on a bolt as shown in Figure 1.12. Find the horizontal and vertical components of the force.

Figure 1.12

Solution

By principle of transmissibility of a force, the force can be considered acting at any point on the line of action of the force.

$\therefore \quad \theta = 60^o + 180^o = 240^o$

with respect to positive x-axis measured in anticlockwise direction (i.e. dotted line making with +ve x-line).

$\therefore \quad F_x = F \cos\theta = 80 \cos 240^o = -40$ N

$F_y = F \sin\theta = 80 \sin 240^o = -69.28$ N

$$\overline{F} = (-40\,\text{N})\,\bar{i} + (-69.28\,\text{N})\,\bar{j}$$

$$= -40\,\bar{i} - 69.28\,\bar{j}$$

(–ve signs indicating –ve x and y directions).

1.4 RESULTANT OF CONCURRENT FORCES

The resultant of a concurrent force system can be defined as the simplest single force which can replace the original system without changing its external effect on a rigid body. For the non-concurrent force system, the resultant will not necessarily be a single force but may be a force system comprising a force or a couple or a force and a couple together. The types of force systems along with their possible resultants are given in Table 1.1

Table 1.1

Types of Force System	Possible Resultant
Concurrent	Force
Coplanar, non-concurrent	Force or a couple
Parallel, non-coplanar, non-concurrent	Force or a couple
Non-parallel, non-coplanar, non-concurrent	Force or a couple or a force and a couple

1.4.1 Resultant of Coplanar Concurrent Forces

The technique of resolution of a force can be used to determine the resultant of coplanar concurrent forces. If 'n' concurrent forces F_1, F_2, F_3, . . . ,F_n are acting at a point in a body then each force can be resolved into two mutually perpendicular directions. Thus, we get '2 n' components in all. Each set of 2 n components acts in one direction only. Therefore, we can algebraically add all these components to get the components of the resultant,

$$\boldsymbol{R_x = \Sigma F_{ix}}$$

$$\boldsymbol{= F_{1x} + F_{2x} + F_{3x} + \ldots + F_{nx}}$$

$$\boldsymbol{R_y = \Sigma F_{iy}}$$

$$\boldsymbol{= F_{1y} + F_{2y} + F_{3y} + \ldots + F_{ny}}$$

Finally, combining these components $\boldsymbol{R_x}$ and $\boldsymbol{R_y}$ vectorially, we get the resultant $\overline{\boldsymbol{R}}$.

Thus, $$\overline{\boldsymbol{R}} = \boldsymbol{R_x}\,\bar{\boldsymbol{i}} + \overline{\boldsymbol{R}}_{\boldsymbol{y}}\,\bar{\boldsymbol{j}}$$

and $$|R| = \sqrt{(R_x)^2 + (R_y)^2}$$

$$\theta = \tan^{-1} \frac{R_y}{R_x}$$

where, θ is the angle of inclination of the resultant $\overline{R}$ with respect to positive x-axis.

Example 1.6

Four forces act on a body as shown in Figure 1.13. Determine the resultant of the system of forces.

Figure 1.13

Solution

Resolving all forces along x-axis, we get :

$$R_x = \Sigma \boldsymbol{F}_x$$

$$= F_1 \cos \theta_1 + F_2 \cos \theta_2 + F_3 \cos \theta_3 + F_4 \cos \theta_4$$

$$= 40 \cos 30^o + 50 \cos 315^o + 30 \cos 180^o + 20 \cos 240^o$$

Note : The angle made by 50 N force is measured in anticlockwise direction from positive x-axis after making the force act away from O by principle of transmissibility of the force.

$$R_x = 40 \cos 30^o + 50 \cos 45^o - 30 \cos 0^o - 20 \cos 60^o$$

$\theta \leq 90^o$ may be chosen in appropriate quadrant with proper signs as indicated above.

$$R_x = 34.64 + 35.36 - 30.00 - 10.00$$

$$R_x = 30 \text{ N} \qquad \ldots (1.1)$$

Similarly, resolving all forces along y-axis, we get,

$$R_y = \Sigma \boldsymbol{F}_y$$

$$= F_1 \sin \theta_1 + F_2 \sin \theta_2 + F_3 \sin \theta_3 + F_4 \sin \theta_4$$

$$R_y = 40 \sin 30^o + 50 \sin 315^o + 30 \sin 180^o + 20 \sin 240^o$$

$$R_y = 20.00 - 35.36 + 0.00 - 17.32$$

$$R_y = -32.68 \text{ N} \qquad \ldots (1.2)$$

Thus, the resultant in vector form may be expressed by

$$\overline{\boldsymbol{R}} = (30 \text{ N}) \bar{i} + (-32.68 \text{ N}) \bar{j}$$

The magnitude of the resultant is given by

$$R = \sqrt{(R_x)^2 + (R_y)^2}$$

$$= \sqrt{(30)^2 + (-32.68)^2}$$

$$= 44.36 \text{ N}$$

The direction θ can be worked out from

$$\theta = \tan^{-1} \frac{R_y}{R_x}$$

$$= \tan^{-1}\left(\frac{-32.68}{30}\right) = -47^\circ\, 26'\, 54''$$

$$= 312^\circ\, 33'\, 6''$$

The resultant has a magnitude of 44.36 N and is acting in IVth quadrant making an angle of $312°33'6''$ in anticlockwise direction from positive x-axis.

1.4.2 Resultant of Non-coplanar Concurrent Forces

In case of non-coplanar forces system also, the technique of resolution of forces can be used to determine the resultant. If three non-coplanar forces F_1, F_2 and F_3 are acting at a point O in a body, the resultant R_{12} of the two forces F_1 and F_2 can be determined by law of parallelogram of forces. The force R_{12} can next be combined with F_3 by means of the parallelogram, giving the resultant of three forces F_1, F_2 and F_3 as R. If there are more forces in the system, the same process can be continued until all the forces have been covered. Here, note that the resultant of non-coplanar force system must pass through the point of concurrence.

The resultant of concurrent force system can also be determined as the vector sum of the forces of the system. The vector sum of the forces can be obtained very easily if each force is resolved into rectangular components. Thus, the vector sum of a non-coplanar system of concurrent forces F_1, F_2 and F_3.

$$\overline{\boldsymbol{R}} = \overline{F}_1 + \overline{F}_2 + \overline{F}_3$$

which can be written in rectangular component form as

$$R_x\, i = R_y\, j\; R_z\, k = \overline{F}_{1x}\, \bar{i} + \overline{F}_{1y}\, \bar{j} + \overline{F}_{1z}\, \bar{k} + \overline{F}_{2x}\, \bar{i} + \overline{F}_{2y}\, \bar{j} + \overline{F}_{2z}\, \bar{k}$$

$$+ \overline{F}_{3x}\, \bar{i} + \overline{F}_{3y}\, \bar{j} + \overline{F}_{3z}\, \bar{k}$$

$$= (F_{1x} + F_{2x} + F_{3x})\, \bar{i} + (F_{1y} + F_{2y} + F_{3y})\, \bar{j} + (F_{1z} + F_{2z} + F_{3z})\, \bar{k}$$

$$= (\Sigma F_x)\, \bar{i} + (\Sigma F_y)\, \bar{j} + (\Sigma F_z)\, \bar{k}$$

Therefore, $R_x = F_{1x} + F_{2x} + F_{3x} = (\Sigma F_x)$

$$R_y = F_{1y} + F_{2y} + F_{3y} = (\Sigma F_y)$$

$$R_z = F_{1z} + F_{2z} + F_{3z} = (\Sigma F_z)$$

Finally, combining these components R_x, R_y and R_z vectorially, we get the resultant $\overline{R}$.

Thus, $\overline{R} = \overline{R}_x\, \bar{i} + \overline{R}_y\, \bar{j} + \overline{R}_z\, \bar{k}$

and $$|R| = \sqrt{(R_x)^2 + (R_y)^2 + (R_z)^2}$$

or $$= \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}$$

$$\cos \theta_x = \frac{\Sigma F_x}{R}, \cos \theta_y = \frac{\Sigma F_y}{R}, \cos \theta_z = \frac{\Sigma F_z}{R}$$

where θ_x, θ_y and θ_z are the angles which the resultant R makes with x, y and z axes respectively.

1.5 MOMENT OF A FORCE

If a force F is acting on a body resting at O and the line of action of force does not pass through G, the centre of gravity of the body, it will not give the body a straight line motion, called the translatory motion, but will try to rotate the body about O as shown in Figure 1.14. The measure of this property of a force by virtue of which it tends to rotate the body on which it acts is called the **moment of a force**. The rotation of the body may be happening either about a point or a line.

Figure 1.14

Referring to Figure 1.15, if F is the force (in N) acting on the body along AB and x is the perpendicular distance (in m) of any point, O (to which the body may be pinned) from AB, then,

Moment of the force F about $O = M = F \times x = F \times OC$

Figure 1.15

Here, point O is known as moment center or fulcrum and distance x is termed as moment arm. We can state in words, therefore :

Moment of Force = Force × Perpendicular distance between the line of action of the force and the point or the axis about which moment is required

Moment is a vector quantity and the vector direction is along the axis about which the moment is taken. In terms of vector algebra, we can define "the moment M of a force $\overline{F}$ with respect to point O as the cross product of the perpendicular distance of point O from the line of action of the force $\overline{F}$ ".

If the moment of the force about a point is zero, it means either the force itself is zero or the perpendicular distance between the line of action of the force and the point about which moment is to be calculated is zero, i.e. the force passes through that point.

Varignon's Theorem

It states that the moment of a force about any point is equal to the sum of the moments of its components about the same point. This principle is also known as principle of moments.

Varignon's theorem need not be restricted to the case of only two components but applies equally well to a system of forces and its resultant. For this it can be slightly modified as, "the algebraic sum of the moments of a given system of forces about a point is equal to the moment of their resultant about the same point". This principle of moment may be extended to any force system.

1.5.1 Moment of Coplanar Forces

Let F_1, F_2 and F_3 be the three coplanar forces acting on a body and let θ_1, θ_2 and θ_3 be the angles which these forces make with positive x-axis as shown in Figure 1.16(a).

(a) **(b)**

Figure 1.16

Now, the magnitude and direction of resultant R can be found out very easily by resolving all the forces horizontally and vertically. Let the resultant R makes an angle θ with positive x-axis as shown in Figure 1.16(b). Now by computation of moment of forces, the position of resultant force R can be ascertained.

To determine the point of application of the resultant, let it cut the horizontal axis XOX at A at a perpendicular distance d from O as shown in Figure 1.16(b). For point O in Figure 1.16(a), let the algebraic sum of the moments of the given forces about O be given by ΣM_0 (anticlockwise).

Then, $$\Sigma M_0 = F_1 d_1 + F_2 d_2 + F_3 d_3$$

Now, by applying Varignon's theorem, the position of resultant R will be such that the moment of R about point O, ($R \times d$) is equal to ΣM_0, and the direction of the moment due to R about moment centre O must be the same as of ΣM_0 due to given system of forces.

Thus, $$R \times d = \Sigma M_0$$

The distance d is computed from the above relation and R, whose magnitude and direction have already been determined earlier, is now completely located.

1.5.2Moment of a Force about a Point and an Axis

The moment of a force can be determined with respect to (about) a point and also with respect to a line or any axis.

The moment of a force F with respect to a point A is defined as a vector with a magnitude equal to the product of the perpendicular distance from A to F and the magnitude of the force and with a direction perpendicular to the plane containing A and F. The sense of the moment vector is given by the direction a right-hand screw would advance if turned about A in the direction indicated by F as shown in Figure 1.17.

Figure 1.17

The moment of a force about a line or axis perpendicular to a plane containing the force is defined as a vector with a magnitude equal to the product of the magnitude of the force and the perpendicular distance from the line to the force and with a direction along the line. Thus, it is the same as the moment of the force about the point of intersection of the plane and the moment axis.

Since the moment of a force about an axis is a measure of its tendency to turn or rotate a body about the axis, the force parallel to an axis has no moment with respect to the axis, because it has no tendency to rotate the body about the axis.

The moment of a force about various points and axes is illustrated in Figure 1.18. The moment of the horizontal force F about point A has a magnitude of Fd_1 in the direction shown by M_A (Figure 1.18(a)). Similarly, M_B, the moment about point B, has a magnitude of Fd_2 and is perpendicular to the plane determined by B and the force F. The moment of force F about the line AB is the same as M_A (as shown in Figure 1.18(a)) or M_{AB} (as shown in Figure 1.18(b)). The moment of force F about line BC can be obtained by resolving force F into components F_1 and F_2. Since F_1 is parallel to line BC, it has no moment about BC. The resultant F_2 is in a plane perpendicular to BC and its moment is $F_2\, d_3$ in the direction shown. Similarly, the moment about line BD is $F_1\, d_3$ as indicated. Here, you can note that M_{AB}, M_{BC} and M_{BD} are the orthogonal components of M_B.

(a) **(b)**

Figure 1.18

Example 1.7

The side of a square ABCD is 1.60 m long. Four forces equal to 6, 5, 4 and 8 N act along the line CB, BA, DA and DB, respectively. Find the moment of these forces about *O*, the point of intersection of the diagonals of the square (Figure 1.19).

Figure 1.19

Solution

Taking moments about *O*,

Resultant moment $M_0 = -\,6x_2 - 5x_3 + 4x_1 + 8 \times 0$

where x_1, x_2 and x_3 are the perpendicular distances of the forces of 4, 6 and 5 N, respectively from *O*, and force of 8 N has zero moment about *O* as its line of action passes through this point.

Here, $$x_1 = x_2 = x_3 = \frac{1.6}{2} = 0.8\text{m.}$$

$$\therefore \quad M_0 = -\,6(0.8) - 5(0.8) + 4(0.8) + 0$$

$$= -\,4.8 - 4.0 + 3.2$$

$$= -\,5.6 \text{ N-m or } 5.6 \text{ N-m (clockwise)}$$

SAQ 4

The side of a regular hexagon ABCDEF is 0.6m. Forces 1, 2, 3, 4, 5 and 6 N are acting along the sides AB, CB, DC, DE, EF and FA, respectively. Find the algebraic sum of the moments about A (Figure 1.20).

Figure 1.20

1.5.3Couples and their Properties

A couple is a force system consisting of two equal, coplanar, parallel forces acting in opposite directions. Since a couple constitutes two equal and parallel forces, their resultant is zero and hence a couple has no tendency to produce translatory motion but produces rotation in the body on which it acts.

Figure 1.21 shows two equal and opposite forces, each equal to *P* and acting at *A* and *B* along parallel lines, thus constituting a couple. The perpendicular distance AB is called the arm of the couple and is denoted by *p*.

Figure 1.21

Moment of a Couple

The moment of a couple about any point in the plane containing the forces is constant and is measured by the product of any one of the forces and the perpendicular distance between the lines of action of the forces, i.e. $M = P \times p$.

Properties of Couples

The properties which distinguish one couple from every other couple are called its characteristics. A couple whether positive or negative, has the following properties/characteristics.

(i) The algebraic sum of the forces constituting a couple is zero.

(ii) The algebraic sum of the moments of the forces forming a couple is the same about any point in their plane

(iii) The couple can be balanced only by another couple of the same moment but of the opposite sense.

(iv) The net effect of a number of coplanar couple is equivalent to the algebraic sum of the effects of each of the couples.

A couple is frequently indicated by a clockwise or counter-clockwise arrow when coplanar force systems are involved instead of showing two separate forces.

Replacement of a Force by a Force and Couple

Consider a force F acting at a point O_1. Imagine now two equal and opposite forces F parallel to the given force acting at O_2 as shown in Figure 1.22, as an addition to the system.

Figure 1.22

These two additional forces do not alter the system in its effect on a body. The new system is equivalent to a force F acting at O_2, plus a couple of moment $M = F.d$.

Replacement of a Couple by two Forces

Consider a couple of moment M, where the axis of the couple is through O perpendicular to the plane of paper as shown in Figure 1.23.

Figure 1.23

This couple is equivalent to any two parallel forces of magnitude F acting at a distance d apart such that $F.d = M$ and the direction of the forces chosen so as to give the correct direction of M.

1.6 RESULTANT OF NON-CONCURRENT FORCES

As stated earlier, the resultant of a system of forces is the simplest force system which can replace the original forces without altering their external effect on a rigid body. The equilibrium of a body is the condition wherein the resultant of all the forces is zero. The properties of force, moment and couple discussed in the preceding sections will now be used to determine the resultant of non-concurrent force systems.

1.6.1 Resultant of Coplanar Non-concurrent Forces

The resultant of a system of coplanar non-concurrent forces can be obtained by adding two forces at a time and then combining their sums. The three forces F_1, F_2 and F_3, shown in Figure 1.24(a), can be combined by first adding any two forces such as F_2 and F_3. They can be moved along their lines of action to their point of concurrency A by the principle of transmissibility.

(a) **(b)**

Figure 1.24

Their vector sum R_1 is formed by the law of parallelogram of force. The force R_1 may then be combined with F_1 by the parallelogram law at their point of concurrency B to obtain the resultant R of the three given forces. Here, the order of combination of the forces is immaterial as may be verified by combining them in a different sequence. Now, the force R may be applied at any point on its established line of action.

Algebraically, the same result may be obtained by forming (i.e. resolving the forces into) the rectangular components of the forces in any two convenient perpendicular directions. In Figure 1.24(b), the x and y components of R are seen to be the algebraic sums of the respective components of the three forces. Thus, in general, the rectangular components of the resultant R of a coplanar system of forces may be expressed as

$$R_x = \Sigma F_x \quad \text{and} \quad R_y = \Sigma F_y$$

where, $$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

The angle made by R with x-axis is given by :

$$\theta = \tan^{-1} \frac{\Sigma F_y}{\Sigma F_x}$$

The location of the line of action of R may be computed with the help of Varignon's theorem. The moment of R, Figure 1.24(a), about some point must equal to the sum of the moments of its two components F_1 and R_1 about the same point. The moment of R_1, however, must equal to the sum of the moments of its components F_2 and F_3 about the same point. It follows that the moment of R about any point equals the sum of the moments of F_1, F_2 and F_3 about this same point. Application of this principle of moments about the point O, shown in Figure 1.25, gives the equation

$$Rd = F_1\, d_1 - F_2\, d_2 + F_3\, d_3$$

Figure 1.25

For this system of forces where the clockwise direction has been taken as positive, the distance d is computed from this relation, and R, whose magnitude and direction have already been determined earlier, may now be completely located. In general, then, the moment arm d of the resultant R is given by

$$R.d = \sum M_0$$

where $\sum M_0$ stands for the algebraic sum of the moment of the forces of the system about any point O.

Figure 1.26

For a system of parallel forces, the magnitude of the resultant is the algebraic sum of the several forces, and the position of its line of action may be obtained from the principle of moments.

Now, consider a force system such as shown in Figure 1.26, where the polygon of forces may be closing and consequently there will be no resultant force R. Direct combination by the law of parallelogram shows that for this case, the resultant is a couple of magnitude $F_3\,d$. The value of the couple is equal to the moment sum about any point. Thus, it is seen that the resultant of a non-concurrent coplanar system of forces may be either a force or a couple.

1.7 SUMMARY

In this unit, you have studied the definition of a force and its nature. Different types of force systems are explained, e.g. coplanar, concurrent, parallel, non-coplanar and non-concurrent force systems etc. Composition of forces by laws of triangle of forces, parallelogram of forces and polygon of forces are discussed. You have also studied how the forces can be resolved along an orthogonal axes system. The translational and rotational tendencies of forces were also studied. With this knowledge base you will be able to identify the force system acting on a body and find its resultant.

1.8 ANSWERS TO SAQs

SAQ 1

(a) Consider forces acting at A.

The gravity force acting on the mass of the body will cause stretching of the tie member and shortening of the jib member. Therefore, there are three forces acting at A :

(i) Weight of the body acting vertically downwards,

(ii) Tension in the tie member, and

(iii) Compression in the jib member.

These three forces pass through the common point A. This is the system of concurrent forces.

(b) Consider forces acting on the beam.

If the string at P is cut, point P will move downward. Thus, the string is offering a force acting upwards to keep P in position shown. Similarly, the string at Q is also offering an upward force. There are five forces acting on the beam

(i) Weight W_1 acting vertically downward on the beam,

(ii) Weight W_2 acting vertically downward,

(iii) Tension in string at P acting vertically upward,

(iv) Tension in string at Q acting vertically upward, and

(v) Weight of the beam acting vertically downward.

The lines of action of these forces are parallel to each other. This is the system of parallel forces.

(c) Consider forces acting at *C*.

There are four forces acting at *C*.

(i) Weight W acting vertically downward,

(ii) Tension in the tie,

(iii) Compression in left leg, and

(iv) Compression in right leg.

The system is of concurrent forces as all the forces pass through the common point *C*.

(d) If forces acting on the roof truss are considered, the lines of all forces do not pass through any common point. Neither the lines of action of all forces are parallel. This is a system of non-concurrent non-parallel forces.

SAQ 2

(a) Parallelogram Law

Here P = 60N and Q = 80N

Represent the forces P and Q in magnitude and direction by drawing lines OA and OB to scale and parallel to the line of action of forces P and Q, respectively.

Figure for Answer to SAQ 2(a) : Parallelogram Law

Complete the parallelogram of forces. Join OC which is the diagonal of the parallelogram passing through the point of concurrence '*O*' of P and Q. The diagonal OC, therefore, represents their resultant in magnitude and direction. Measure the length OC and get the magnitude of the resultant. Measure angle COA and get the direction of the resultant with respect to force P.

Using trigonometrical relations, we get

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

where $P = 60$ N and $Q = 80$ N

and α = angle between $\overline{P}$ and $\overline{Q}$

$= 180 - 30 - 45$

$= 105^{\circ}$

$\therefore \quad R = \sqrt{60^2 + 80^2 + 2 \times 60 \times 80 \cos 105^{\circ}}$

$= 86.691 \text{ N}$

$$\tan\theta = \frac{Q \sin\alpha}{P + Q\cos\alpha}$$

$$= \frac{80 \sin 105^\circ}{60 + 80 \cos 105^\circ}$$

$= 1.9665$

$\therefore \quad \theta = 63.046^\circ = 63^\circ\ 2'\ 46''$ . . . w.r.t. force P.

Also $\quad \tan\beta = \dfrac{P\sin\alpha}{Q + P\cos\alpha} = \dfrac{60 \sin 105^\circ}{80 + 60 \cos 105^\circ}$

$= 0.8989$

$\therefore \quad \beta = 41.954^\circ = 41^\circ\ 57'\ 14''$ . . . w.r.t force Q.

Triangle Law

Here $p = 60$ N and $Q = 80$ N

Represent the force P in magnitude and direction by drawing line OA to scale and parallel to the line of action of force P. From point A, represent force Q in magnitude and direction by drawing line AB to scale and parallel to the line of action of force Q. Then line OB joining the starting point to the end point B represents their resultant in magnitude and direction. Length OB gives the magnitude and angle BOA determines and direction of the resultant with respect to force P.

The resultant can also be found by drawing Q as the first vector and P as the second as shown in the adjoining diagram.

Using trigonometric relations, we get

$$R = \sqrt{60^2 + 80^2 + 2 \times 60 \times 80 \cos 105^\circ}$$

$= 86.691 \text{ N}$

$$\sin\theta = \frac{Q}{R}\sin\alpha$$

$$= \frac{80}{86.691}\sin 105^\circ$$

$= 0.8914$

$\therefore \quad \theta = 63.046^\circ = 63^\circ\ 2'\ 59''$

Also $\quad \sin\beta = \dfrac{P}{R}\sin\alpha$

$$= \frac{60}{86.691}\sin 105^\circ$$

$= 0.66853$

$\therefore \quad \beta = 41.9537 = 41^\circ\ 57'\ 14''$

(Note : $\theta + \beta = \alpha = 105^\circ$)

Figure for Answer to SAQ 2(a) : Triangle Law

(b) Parallelogram Law

Here P =100 N and Q =120 N

Represent forces P and Q in magnitude and direction by drawing lines OA and OB to scale and parallel to the line of action of forces P and Q, respectively.

Complete the parallelogram OABC and join OC. As per the law of parallelogram of forces, OC represents their resultant in magnitude and direction. Measure the length to get the magnitude and angle COA to get the resultant with respect to P.

Using trigonometrical relations, we get

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha}$$

where, P =100 N and Q = 120 N

α = angle between $\overline{P}$ and $\overline{Q}$

$= 20^o + 60^o = 80^o$

$$\therefore \quad R = \sqrt{100^2 + 120^2 + 2 \times 100 \times 120 \cos 80^o}$$

$= 169.019$ N

$$\tan \theta = \frac{Q \sin \alpha}{P + Q \cos \alpha}$$

$$= \frac{120 \sin 80^o}{100 + 120 \cos 80^o}$$

$= 0.97798$

$\therefore \quad \theta = 44.362^o = 44^o\ 21'\ 43''$ w.r.t. force P.

Also $$\tan \beta = \frac{P \sin \alpha}{Q + P \cos \alpha}$$

$$= \frac{100 \sin 80^o}{120 + 100 \cos 80^o}$$

$= 0.7169$

$\therefore \quad \beta = 35.638^o = 35^o\ 35'\ 16''$ w.r.t. force Q.

Figure for Answer to SAQ 2(b) : Parallelogram Law

Triangle Law

Here P = 100 N and Q = 120 N

Represent the force P in magnitude and direction by drawing line OA to scale and parallel to the line of action of force P. From point A, represent force Q in magnitude and direction by drawing line AB to scale and parallel to the line of action of force Q. Then line OB joining the starting point to the end point B represents their resultant in magnitude and direction as per triangle law of forces. Length OB gives the magnitude and angle BOA determines the direction of the resultant with respect to P.

The resultant can also be found by drawing Q as the first vector and P as the second vector as shown in the above Figure for Answer to SAQ 2(b).

Figure for Answer to SAQ 2(b) : Triangle Law

Using trigonometric relations, we get

$$R = \sqrt{100^2 + 120^2 + 2 \times 100 \times 120 \cos 80^\circ}$$

$$= 169.019 \text{ N}$$

$$\sin \theta = \frac{Q}{R} \sin \alpha \quad \text{(sine Rule)}$$

$$= \frac{120}{169.019} \sin 80^\circ$$

$$= 0.6992$$

$\therefore$ $\theta = 44.362^\circ = 44^\circ\ 21'\ 43''$ w.r.t force P.

Also $\sin \beta = \dfrac{P}{R} \sin \alpha$

$$= \frac{100}{169.019} \sin 80^o$$

$$= 0.5827$$

$\therefore \qquad \beta = 35.638^o = 35^o\ 38'\ 16''$ w.r.t. force Q.

(**Note :** $\theta + \beta = \alpha = 80^o$)

SAQ 3

(a) We construct a polygon such that the forces 60 N, 80 N, 40 N and 50 N represent the sides of a polygon taken in order, each force being drawn from the end of earlier force as shown in Figure for Answer to SAQ 3(b). Here, the scale to construct the polygon has been taken as 1 cm = 30 N.

(a) **(b)**

Figure for Answer to SAQ 3

Now, the resultant is represented by the line joining the starting point of the first force i.e. *O* to the end of the last force i.e. *D*. This line OD measures 3.35 cm.

Magnitude of resultant = (Linear measurement of OD) × (Scale of drawing)

= 3.35cm × 30 N/cm = 100.50N.

To determine the direction of resultant, draw a co-ordinate system at point *D*. We find that line OD makes 80^o angle with positive x axis measured in clockwise direction whereas θ is measured in anticlockwise direction from positive x axis.

So, the direction of resultant $\theta = 280^o$ (measured in anticlockwise from positive x axis).

(b) Resolving all the forces along x-axis, we get

$$= 60 \cos 45^o + 80 \cos 90^o + 40 \cos 150^o + 50 \cos 240^o$$

$$= 42.42 + 0 - 34.64 - 25.0$$

$$= - 17.22$$

Similarly, resolving all forces along y-axis, we get

$$= 60 \sin 45^o + 80 \sin 90^o + 40 \sin 150^o + 50 \sin 240^o$$

$= 42.42 + 80 + 20 - 43.30$

$= 99.12$ N

Thus, the resultant in vector form may be expressed as :

$\overline{R} = (-\ 17.22 \text{ N})\ i + (99.12 \text{ N})\ j$

The magnitude of the resultant is given by

$$R = \sqrt{(R_x)^2 + (R_y)^2}$$

$$= \sqrt{(-17.22)^2 + (99.12)^2}$$

$R = 100.60$ N

The direction θ can be worked out as

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

$$= \tan^{-1}\left(\frac{99.12}{-17.22}\right) = -\ 80^{\text{o}}\ 8'\ 40'' \ \text{(clockwise)}$$

$$\theta = 279^{\text{o}}\ 51'\ 20'' \qquad \text{(anticlockwise)}$$

The resultant has a magnitude of 100.60 N and is acting in IVth quadrant making an angle of $290^{\text{o}}\ 51'\ 20''$ in anticlockwise direction from positive x-axis.

SAQ 4

In the hexagon ABCDEF shown in Figure for Answer to SAQ 4, the forces 1, 2, 3, 4, 5, and 6 N are acting along the sides AB, CB, DC, DE, EF, and FA respectively.

Figure for Answer to SAQ 4

Let $x_1, x_2,\ x_3$, and x_4 be the respective perpendicular distance of the forces of 2, 3, 4 and 5 N along the sides CB, DC, DE, and EF respectively.

Taking moment about A,

$$M_A = 1 \times 0 - 2x_1 - 3x_2 + 4x_3 + 5x_4 + 6 \times 0$$

As the forces of 1 N and 6 N acting along AB and FA pass through A, their moment about this point is zero.

Also $x_1 = x_4 = \text{AB} \sin 60^\circ$

$$= \frac{0.6\sqrt{3}}{2}$$

$$= 0.3\sqrt{3} = 0.52 \text{ m}$$

Considering the Δ ABC and using cosine formula

$$x_2 = x_3 = \sqrt{AB^2 + BC^2 - 2AB \times BC \cos 120^\circ}$$

$$= \sqrt{0.6^2 + 0.6^2 - 2 \times 0.6 \times 0.6 \times (-0.5)}$$

$$= 1.04 \text{ m}$$

$$\therefore \quad M_A = -(2 \times 0.52) - (3 \times 1.04) + (4 \times 1.04) + (5 \times 0.52)$$

$$= -1.04 - 3.12 + 4.16 + 2.6$$

$$= 2.6 \text{ N-m} \quad \text{(anticlockwise)}$$

EXTRA

SAQ 4

Resolving all the forces along x axis, we get,

$R_x = \Sigma F_x$

$$= F_1 \cos\theta_1 + F2\cos\theta_2 + F3\cos\theta_3$$

$$= 60\cos 0^0 + 90\cos 210^0 + 80\cos 315^0$$

=60-77.94+56.57 =38.63N

Similarly, resolving all forces along y axis, we get,

$$R_y = \sum F_y$$

$$= F_1 \sin\theta_1 + F_2 \sin\theta_2 + F_3\theta_3$$

$$= 60\sin 0^0 + 90\sin 210^0 + 80\sin 315^0$$

=0-45-56.57=-101.57N

Thus, the resultant in vector form may be expressed as

$\bar{R}$ =(38.63N)i +(-101.57N)j

The magnitude of the resultant is given by

$$R = \sqrt{(R_x)^2 + (R_y)^2}$$

$$= \sqrt{(38.63)^2 + (-101.57)^2}$$

R= 108.67N

The direction θ can be worked out as

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

$$= \tan^{-1}\left(\frac{-101.57}{38.63}\right) = -69^0 10' 36'' \text{ (clockwise)}$$

$$\theta = 290^0 49' 24'' \text{ (anticlockwise)}$$

The resultant has a magnitude of 108.67 N and is acting in Ivth quadrant making an angle of 290^0 49’ 24”

In anticlockwise direction from positive x axis.

APPLIED MECHANICS

Mechanics is a physical science dealing with the study of forces and motion of bodies when acted upon by external forces. Applied Mechanics, which forms the subject matter of this course, is that branch of mechanics which deals with application of the principles of mechanics to the solution of practical engineering problems. It provides the opportunity to develop logical thinking, analytical capability, reasoning and judgement, which are essential for the solution of great variety of problems for all branches of engineering and technology.

This course, comprises six units, covers Statics (Units 1, 2, 3 and 4) and Dynamics (Units 5 and 6).

Statics is a branch of Applied Mechanics concerned with the study of bodies at rest or in equilibrium under the action of applied external forces. In Unit 1, you will learn how to determine the resultant effect of various force system acting on rigid bodies. Unit 2 will enable you to draw the "free-body diagrams" which are useful in analyzing and solving engineering problems. Apart from description of various trusses and their analysis, you will also learn the static conditions of equilibrium in this unit. Unit 3 introduces the concept of friction and its role in engineering situations. It also includes the description of different machines, which works on the concepts covered. Unit 4 discusses the geometric properties of bodies like centre of gravity and moment of inertia, which are required in the analysis of problems.

Dynamics is a branch of Applied Mechanics concerned with the study of bodies in motion. Unit 5 deals with rectilinear motion, projectiles and relative motion of two bodies. Unit 6 explains the laws of motion. Apart from describing Newton's laws of motion, it also discusses motion on circular path and simple harmonic motion. Towards the end, the unit includes the concepts of work, power and energy.

In each unit, you will find a number of illustrative examples and SAQs (Self Assessment Questions) for better understanding of the concepts. Study the text and illustrative examples carefully. Attempt SAQs on your own and verify your answers with those given at the end of the unit. This will develop your confidence in analysing and solving the practical problems.

At the end, we wish you all the best for your all educational endeavours.