UNIT 4 CENTRE OF GRAVITY AND MOMENT OF INERTIA

Structure

4.1 INTRODUCTION

This unit will help you in understanding the concepts of centre of gravity of thin uniform plates of various shapes with or without openings, and also the centroid of irregular shaped areas. The concept of centroid of plane areas is similar to centre of gravity of thin uniform plates of same shape as the plane areas. Moment of Inertia of plane areas about given axes are mathematically referred to as second-moment of areas. The knowledge of moment of inertia of plane areas is useful in the analysis of bending stresses of beams under flexure. The moment of inertia of rods, discs, or spheres are useful in studying the problems in dynamics of rigid-bodies.

Objectives

After studying this unit, you should be able to determine

- position of centre of gravity of thin uniform rods,
- position of centre of gravity of masses of plates of various shapes with or without openings,
- position of mass centre of a system of coplanar particles,
- position of centroid of irregular shaped areas, and
- compute moment of inertia of various areas and masses, about the given axes.

4.2 CONCEPT OF CENTRE OF GRAVITY

A body can be considered as an agglomeration of a large number of particles each of which is adhering to its adjacent particles. These particles may have different sizes, density or may have uniform size and density. Each of these particles is subjected to gravitational force directed towards the centre of the earth. The magnitude of the gravitational force will depend on its mass. For all earthly bodies whose sizes are very small as compared to its distance upto the centre of the earth, the forces of gravitational attraction on various particles of the body can be assumed to be parallel. Hence the total weight of the body is the summation of all these parallel forces acting vertically down towards the centre of the earth. The line of action of the resultant R of these forces can therefore be found.

Consider a case of a triangular metal plate ABC placed in a vertical plane with its side AB vertical (Figure 4.1). Divide the area of triangle in large number of imaginary vertical strips parallel to BA; for each of the strips the gravitational force can be computed and resultant R of all these forces will be along a vertical line (1-G) as shown in Figure 4.1(a), where G is a point at which entire weight of the plate is concentrated. When the plate is oriented at right angle to its previous position, line BC is placed vertically as shown in Figure 4.1(b). New vertical strips can be formed and resultant R, which is obviously of same magnitude R as in the case of first orientation (a), will be acting along vertical line (2-G). The location of G will be the intersection of lines of action of resultants obtained in cases (a) and (b). The mass centre is a unique point G in the plate ABC through which the resultant weight of all the strip-masses will always pass irrespective of the orientation of the plate. Figure 4.1(c) shows the third orientation of triangle ABC where one of the medians BG3 in the triangle is kept vertical. It will be proved later on that the point of intersection of all the medians represents the mass centre of the triangular plate.

(a) **(b)** **(c)**

Figure 4.1 : Position of Resultant R in Different Orientation of Plate ABC

Centre of gravity or Mass-centre is a point in the body where the entire mass or weight is assumed to be concentrated and, for convenience, a single resultant gravity load R can be used as a replacement for distributed gravity loads at various locations of its particles.

4.2.1 System of Two Equal Masses

Consider a system consisting of two masses m_A and m_B with their centres at locations A and B which are rigidly connected by a thin rod of negligible weight as shown in Figure 4.2

The resultant force R due to combined effect of the weight of the two masses is equal to $(m_A g + m_B g)$, and the location G, on the thin rod, of this resultant of a single equivalent force $(m_A + m_B) \times g$ is the centre of gravity of the two masses.

When the two masses are of equal magnitude, $m_A = m_B = m$, the resultant R of magnitude 2 mg will act at mid-point of AB as shown in Figure 4.2.

Figure 4.2

Hence, the two equal masses m at A and B can be looked upon as a single equivalent mass 2 m at the centre of gravity G, where $GA = GB$. The case of two unequal masses will be dealt with later in this unit.

4.2.2 Centre of Gravity of a Thin Uniform Rod

Let $AB = L$ be the length of a uniform rod or a uniform strip with its mid point G. The mass per unit length of this rod or strip is same at all points. For any elemental mass (δm) at point P located at a distance x from the mid-point G as shown in Figure 4.3, there is an equal mass (δm) at P' at the same distance $x' = x$ on the other side of centre line passing through point G of the bar. The centre of gravity of these two equal masses at P and P' is at their midpoint G. This is true for all other pair of masses : one set in the portion GB of the bar and the corresponding masses in the counterpart GA. Thus G, the midpoint of AB, happens to be the centre of gravity of the uniform rod. However, if the bar is of non-uniform weight, the position of G will shift from the central point.

Figure 4.3

4.2.3 Centre of Gravity of a Uniform Rectangular Plate

Let the dimensions of uniform rectangular plate ABCD be $L \times B$, where $AB = L$ and $BC = B$ as shown in Figure 4.4. It is already seen from the previous section that the centre of gravity or mass-centre of any uniform strip of length L parallel to AB is at its midpoint G. Since the plate ABCD has a uniform mass all over its area, the centre of gravity of the first strip from AB or DC is indicated by G_1, while those of second strip, third strip and so on, can be indicated by G_2, G_3 etc. depending upon the number of even strips chosen for the rectangular plate. Further, if the widths of the strips chosen are equal, the masses associated with

G_1, G_2 and G_3 etc. will be of same magnitude, say m. In fact, you can realize that all these masses (m) are spread uniformly on line $M_1\, M_2$, where M_1 and M_2 are midpoint of AB and DC. Obviously the C. G. of all these masses will be at G_0 which is the midpoint of $M_1\, M_2$.

Figure 4.4

4.2.4Mass-centre of a System of Two Unequal Masses

Consider two spherical bodies of masses m_1 and m_2 with centres at points A and B respectively, where distance $AB = L$. Let us assume that these masses are connected by a massless thin rod so that distance AB is not changed. Let AB be a horizontal line along X direction.

Let the weight W_1 of mass m_1 act vertically along $A\, A_1$ and weight W_2 of mass m_2 along $B\, B_1$ as shown in Figure 4.5

Figure 4.5

Let G be the centre of gravity of the two masses at a distance $\bar{x}_1$ from A.

Now the resultant weight, $R = (m_1 + m_2) \times g$ acts vertically downward at G.

Using the theorem of moments, we have

$$\begin{bmatrix} \text{Moment of Resultant } (R) \\ \text{about a point } (A_1) \end{bmatrix} = \begin{bmatrix} \text{Algebraic sum of moment of fall of} \\ \text{the masses in the system about } (A_1) \end{bmatrix}$$

$$R \times \bar{x}_1 = m_2\, g \times L$$

where $R = (m_1 + m_2) \times g = [\Sigma\,(m)] \times g$

$\therefore$ $(m_1 + m_2) \times \bar{x}_1 = m_2\, L$

$\therefore$ $\bar{x}_1 = \dfrac{m_2\, L}{\Sigma\,(m)}$;

where $\Sigma\,(m) = m_1 + m_2$

Similarly, by considering equilibrium of moments about point B_1,

$$\bar{x}_2 = \frac{m_1\, L}{\Sigma\,(m)}$$

Case I

As a special case, when $m_1 = m_2 = m$

$$R = m + m = 2\,m;$$

$$\bar{x}_1 = \frac{mL}{2m} = \frac{L}{2}$$

Thus, midpoint of line AB is the location of C.G. for two equal masses, as expected.

Case II

When AB = 6 m (say), and masses are given as follows :

m_1 = 2 kg at point A_1

m_2 = 4 kg = 2 m_1 at point B_1

$\Sigma\, m = m_1 + 2\,m_1 = 3\,m_1$ and $L = 6\,\text{m}$

$$\bar{x}_1 = \frac{m_2\, L}{\Sigma\,(m)} = \frac{2\,L}{3}$$

and $\bar{x}_2 = \dfrac{m_1\, L}{\Sigma\,(m)} = \dfrac{L}{3}$

It is clear that in general,

$$\frac{\bar{x}_1}{\bar{x}_2} = \frac{AG}{GB} = \frac{m_2}{m_1}$$

This implies that distances of the C.G. from the two particle masses are inversely proportional to their masses.

or $m_1\, \bar{x}_1 = m_2\, \bar{x}_2$

$$m_1\, g\, \bar{x}_1 = m_2\, g\, \bar{x}_2$$

Therefore, the moment of mass m_1 about G just balances the moment of mass m_2 about G. Thus G, the centre of gravity of two masses, happens to be fulcrum of an imaginary levels AB of negligible weight having two masses (m_1) and (m_2) placed at x_1 and x_2 from the fulcrum F as shown in Figure 4.6.

The resultant mass (m_1+ m_2) is concentrated at location G so that total resultant weight R = (m_1+ m_2) g acts downward at G and if a fulcrum or support F is provided to the lever at G, the vertical reaction V offered by this support will balance the resultant weight R to keep the system of masses in equilibrium.

Figure 4.6

4.2.5 Centre of Gravity of a System of Collinear Masses

It is well understood by now, that centre of gravity of several masses is a point through which resultant weight R of the masses passes.

Consider the case of two unequal masses with centres A and B as given in previous section. If the line AB is kept vertical, the two weights ($m_A g$) and ($m_B g$) will be collinear since they act along same vertical direction AB. Hence, the resultant, $R = (m_A + m_B) \times g$, will also be along the line AB passing through the centre of gravity G as shown in Figure 4.7(a). The distance $\bar{x}_a$ (i.e. $\bar{x}_1$) has already been worked out in the previous section.

This is also true for a system of masses m_A, m_B, m_C, m_D etc., all having their centres A, B, C, D lying on a rigid massless rod along line AD. Hence, all these collinear masses can be placed along a single vertical line AD as shown in Figure 4.7(b).

(a) **(b)**

Figure 4.7

Since the resultant force R of all these weights must also act along the line AD, the centre of gravity of the collinear masses is always on the line joining their centres. This logical thinking is even otherwise appealing to our commonsense. The same type of reasoning can be applied to determine the location of centre of gravity of a triangular plate of uniform mass per unit area.

4.2.6Centre of Gravity of a Uniform Triangular Plate

The centre of gravity of a uniform triangular plate (i.e. with uniform thickness) lies at its centroid, which is the point of intersection of its medians. This can be proved on the basis of following logic.

Consider the triangular plate ABC as shown in Figure 4.8

Figure 4.8

Let M_1 and M_2 be the midpoint of sides BC and AB of the triangle. Consider a typical small strip $B_1\, C_1$ parallel to side BC. The midpoint M_1' of this strip lies on the median AM_1. Since the strip has uniform mass per unit of length, mass centre of this strip is at M_1'.

This result is based on again the old logic that for every elemental mass δm at B'_1 there is an equal mass δm at C_1' which are equidistant from M_1'. Similarly, mass-centre of any other strip parallel to BC will lie on this median AM_1. Further, since the mass-centre for all such strips (parallel to BC) in the triangle lie on the median AM_1, it can be concluded that the mass-centre of all these masses must lie on the median AM_1.

Similarly, considering the strips, like $A_2\, B_2$ parallel to AB, it can be proved that mass-centre of the triangular plate should lie on the median CM_2. Hence, mass-centre of the triangular plate must be at the intersection of the medians.

4.2.7Mass-centre of a System of Coplanar Particles

Consider a system of coplanar particles A_1, A_2, A_3 etc., having masses $m_1, m_2, m_3, \ldots$ (or weights w_1, w_2, w_3 etc.) which have their co-ordinates (x_1, y_1), (x_2, y_2), (x_3, y_3) and so on, as shown in Figure 4.9.

For simplicity of understanding, assume that all the particles lie in a horizontal plane XOY and weights w_1, w_2, w_3 of these particles A_1, A_2, A_3 are acting vertically downward along z-direction.

Let the resultant W of all these weights act at G whose co-ordinates are $(\bar{x}, \bar{y})$.

Then, $W = (w_1 + w_2 + w_3 + \ldots) = \sum_i^n w_i$, where n is the number of coplanar particles.

Figure 4.9

Hence, applying the Varignon's theorem which states that "moment of the resultant force at G is equal to the algebraic sum of the moment of all the individual forces at A_1, A_2, A_3 etc. about the horizontal axis OY".

$$\therefore \qquad W\,\bar{x} = w_1 x_1 + w_2 x_2 + w_3 x_3 + \ldots$$

$$= \sum_i w_i\, x_i$$

$$\bar{x} = \frac{\sum_i (w_i\, x_i)}{W} = \frac{\sum_i (w_i\, x_i)}{\sum_i w_i}$$

Similarly, considering the theorem about axis OX.

$$W\,\bar{y} = w_1\, y_1 + w_2\, y_2 + w_3\, y_3 + \ldots$$

$$\therefore \qquad \bar{y} = \frac{\sum_i w_i\, y_i}{\sum_i w_i}$$

where, $\sum_i$ implies the sum of all terms where i varies from 1 onwards for all masses.

The concept of centre of gravity or mass-centre of thin plates can be extended to that of centroid of plane areas. Thus, replacing the masses by corresponding areas, we have centroid of a rectangular area ABCD at its geometric centre or the point of intersection of diagonals. The centroid of a triangular area is at its point of intersection of the medians. Note that centroid of a plane area of a given shape and centre of gravity of thin plate of the same shape are at the same location provided the plate has uniform-mass per unit area.

4.2.8Centroid of Irregular Shaped Area

Consider an irregular area, A, as shown in Figure 4.10 (having a small thickness throughout) which can be divided into a large number of elemental areas, such as, δA_i.

We can write A as a summation of elemental areas

$$\therefore \qquad A = \delta A_1 + \delta A_2 + \ldots + \delta A_i + \ldots$$

$$= \sum_i \delta A_i \text{ , where, } i = 1, 2, 3, \ldots, n \text{ (say).}$$

Figure 4.10 : An Irregularly Shaped Area

Considering the centroid of the area A at $C\ (\bar{x}, \bar{y})$ the principle of moments can be applied on the same lines as that for the centre of gravity of weights of various elemental masses, δm_i.

$\therefore \quad A\bar{x} = \sum_i (A_i\, x_i)$ . . . (taking moments of areas about y-axis)

or $\quad \bar{x} = \dfrac{\sum_i A_i\, x_i}{\sum_i A_i}$

Similarly, $\quad \bar{y} = \dfrac{\sum_i A_i\, y_i}{\sum_i A_i}$ . . . (taking moments of areas about x-axis)

Considering a thin plate of uniform mass m per unit area, i.e., $\delta m_i = \delta A_i \times m$

Total mass $\quad M = \sum_i m_i = m = \sum_i \delta A_i = mA.$

$\therefore \quad (m\, g\, A)\,.\,\bar{x} = \sum_i m_i\, g\, x_i$ i.e., taking moments about y-axis.

Hence, centre of gravity of the uniform plate is given by

$$\bar{x} = \frac{\Sigma\, m_i\, x_i}{\Sigma\, m_i} = \frac{\Sigma\, A_i\, x_i}{\Sigma\, A_i} \text{ (as derived above) } mA = \Sigma\, m_i$$

Similarly, we can find the corresponding expression for $\bar{y}$.

Hence, numerically the position of centroid of a given area is the same as centre of gravity of thin uniform plate of the same area.

4.2.9First Moment of Area

The terms $A_i\, x_i$ and $A_i\, y_i$ are termed as the first moment of area A_i about y and x axes respectively, considering XOY as a horizontal plane. It is to be noted that x_i and y_i refer to the co-ordinates of centre of small area, A_i.

Similarly, the terms $(m_i\, g)\, x_i$ is the first moment of the weight of mass m_i about y axis, where x_i is the x co-ordinate of centre of the small mass, m_i.

The numerator $(\Sigma\, A_i\, x_i)$ or $(\Sigma\, m_i\, x_i)$ in the expression for $\bar{x}$ referred to in the previous section is obtained from the concept of summation of **First Moment of area (A_i) or mass (m_i)**.

Example 4.1

Determine the centroid of the area OBCD as shown in Figure 4.11.

Figure 4.11

Solution

The trapezium area is divided into two simple areas :

(i) A_1 = Rectangle OLDC = 3 × 6 = 18 m^2 with centroid at G_1 (1.5, 3)

(ii) A_2 = Triangular area $\text{DLB} = \frac{6 \times 3}{2} = 9\,\text{m}^2$ with its centroid at G_2 (4, 2)

The computation of co-ordinates $(\bar{x}, \bar{y})$ for G and other aspects are given in Table 4.1.

Table 4.1

Sl. No.	Area (m^2) A_i	x_i	y_i	$A_i\, x_i$	$A_i\, y_i$
1	$A_1 = 18$	1.5	3	27	54
2	$A_2 = 9$	4	2	36	18
	$\Sigma\,(A_i) = 27$			$\Sigma\, A_i\, x_i = 63$	$\Sigma\, A_i\, y_i = 72$

Then, $\bar{x} = \frac{63}{27} = 2.33$ m

$\bar{y} = \frac{72}{27} = 2.67$ m

Hence, the co-ordinates of centroid G are $\bar{x} = 2.33$ m and $\bar{y} = 2.67$ m

Example 4.2

Determine the centroid of an area shown in Figure 4.12(a).

Figure 4.12(a)

Solution

This problem can be attempted in two ways as shown in Figure 4.12(b) and Figure 4.12(c).

The total area A is divided in to the three parts A_1, A_2 and A_3 in Figure 4.12(b) while the same area A can be divided in two parts A_1' and A_2' (shown shaded) in Figure 4.12(c).

(b) **(c)**

Figure 4.12

Let G be the centroid of area A as shown in Figure 4.12(a) with $(\bar{x}, \bar{y})$ co-ordinates.

Table 4.2

Refer Figure 4.12(b)						**Refer Figure 4.12(c)**		
Sl. No	A_i **(m²)**	x_i **(m)**	$A_i x_i$ **(m³)**	y_i **(m)**	$A_i y_i$ **(m³)**	A_i' **(m²)**	x_i' **(m)**	$A_i' x_i'$ **(m³)**
1	$A_1 = 18$ (OLM_2B)	1.5	27	3	54	$A_1' = 36$ $(= 6 \times 6)$	3	108
2	$A_2 = 9$ ($ALKM_1$)	4.5	40.5	1.5	13.5	$A_2' = (-4.5)$	5	-22.5
3	$A_3 = 4.5$ (M_1M_2K)	4	18.0	4	18	$\Sigma A_i' = 31.5$ m²		$\Sigma = 85.5$
$\Sigma A_i = 31.5$ m²			$\Sigma = 85.5$		$\Sigma = 85.5$	$\therefore \bar{x} = \bar{y} = \dfrac{85.5}{31.5} = 2.714$ m		

$$\therefore \qquad \bar{x} = \frac{85.5}{31.5}; \qquad \bar{y} = \frac{85.5}{31.5}$$

$$\therefore \qquad \bar{x} = 2.714 \text{ m}, \qquad \bar{y} = 2.714 \text{ m}$$

It is to be noted from Table 4.2 that whichever way we consider the division of regular area, either all positive or some of them negative, we should finally get a unique value of

(i) ΣA_i,

(ii) $\Sigma A_i x_i$, and

(iii) $\Sigma A_i y_i$,

For a given area A, the co-ordinates, $\bar{x}$ and $\bar{y}$ will be a unique set of values.

Note : If the problem of C.G. (centre of gravity) of a uniform plate is considered, where out of a square plate OACB a triangular part

$M_1\,CM_2$ is cut out, the position of centre of gravity of such a plate is the same as the position of centroid of area in Figure 4.12(a).

Example 4.3

A square plate of uniform thickness and density is bent along $M_1\,M_2$ till corner C coincides with centre C′ as shown in Figures 4.13(a) and (b). Determine the centre of gravity of the area thus formed.

(a) **(b)**

Figure 4.13

Solution

Let w be the uniform weight of the plate per unit area. The entire plate after it is bent can be considered to be made up of three parts.

(i) W_1 = Weight corresponding to a square plate OACB

= (36 w) at location (3, 3)

(ii) W_2 = weight corresponding to overlapped portion M_1 C′ M_2

= (4.5 w) at location (4, 4)

(iii) $(-W_3)$ = Portion (M_1 (M_2)) which is removed

= (− 4.5 w) at location (5, 5)

$\therefore$ Resultant $W = \Sigma\, W_i = (w_1 + w_2 - w_3) = 36$ w.

Table 4.3

Sl. No.	w_i	x_i	$w_i\,x_i$	y_i	$w_i\,y_i$
1	w_1 = 36 w	3	108 w	3	108 w
2	w_2 = 4.5 w	4	18 w	4	18 w
3	w_3 = − 4.5 w	5	− 22.5 w	5	− 22.5 w
	$\Sigma\, w_i$ = 36 w		$\Sigma\, w_i\,x_i$ = 103.5 w		$\Sigma\, w_i\,x_i$ = 310.5 w

$$\therefore \quad \bar{x} = \frac{103.5}{36} = 2.88 \text{ m} \quad \text{and} \quad \bar{y} = 2.88 \text{ m}$$

This example can also be solved by other alternative way by considering the total plate in parts as follows :

(i) W_1' = weight of rectangular plate ($BM_2\,M_3\,O$)

18 w with its mass-centre at (1.5, 3)

(ii) W_2' = weight of square plate ($C'\ M_1\ AM_3$)

9 w with its mass-centre at (4.5, 1.5)

(iii) W_3' = weight of two triangular plates ($M_1\ M_2\ C'$)

9 w with its mass-centre at (4, 4)

Note that resultant weight $W = \Sigma\, w_i = (18 + 9 + 9)\, w = 36\, w$ as before.

Example 4.4

Determine the centroid of the shaded area shown in Figure 4.14.

Figure 4.14

Solution

Net area of shaded portion of Figure 4.14

= The area (A_1) of full circle of radius r − The area (A_2) of cut out circle of radius $\frac{r}{2}$.

$$= \pi r^2 - \frac{\pi r^2}{4} = \frac{3\pi r^2}{4}$$

Area A_2 is to be regarded as negative area.

Consider moment of areas about G, the required centroid.

$\therefore\ \Sigma\, A_i x_i = 0$ since the lever-arm of the resultant area A about G is zero,

$\therefore\ A_1\, x_1 + (-\,A_2)\, x_2 = A \times (0) = 0$ (i.e., taking moments of areas about G)

$\therefore\ \pi r^2\, (-\,\bar{x}) + \left(-\frac{\pi r^2}{4}\right) \times \left\{-\left(\frac{r}{2} + \bar{x}\right)\right\} = 0$ (distances to the left of G are taken as – ve, or say anticlockwise moments are taken as –ve).

$$\therefore \qquad -\frac{3\pi r^2}{4}\,\bar{x} + \frac{\pi r^3}{4 \times 2} = 0$$

$$\therefore \qquad \bar{x} = +\frac{r}{6}$$

Positive sign of $\bar{x}$ indicates that with respect to the origin O of reference axes x and y, $\bar{x}$ = OG is along positive direction of x axis. Since the centres

O and A of the two areas (A_1) and ($-A_2$) are taken along x axis; G lies on AO.

SAQ 1

Determine the centroid of a plate with uniform mass per unit area having a shape given in Figure 4.15.

Figure 4.15

SAQ 2

Determine the centroid of the area as shown in Figure 4.16, where D is the contact point of the circle to the edge CF.

Figure 4.16

SAQ 3

A thin uniform triangular plate OAB where $\angle$ AOB = 90° is bent along M_1 M_2 where M_1 and M_2 are midpoints of AB and AO, respectively till apex A is made to coincide with point O. Here OA = 12 cm and OB = 6 cm.

Determine the location of C.G. of such a plate with respect to axes OA (x axis) and OB (y axis).

4.3 MOMENT OF INERTIA OF AREA

4.3.1Definition

Consider an area of a surface as consisting of a large number of small elements of area dA each. The area integral of all such elements can be written mathematically as under :

$$\int_A (dA) = A$$

Referring to Figure 4.17, the Area Moment of Inertia of elemental area dA about x-axis, in its plane is defined as :

$$I_x \text{ (element)} = dA \times y^2$$

$$I_y \text{ (element)} = dA \times x^2$$

Since, the axis x lies in the plane of element, these are also called as **axial moment of inertia of the element *dA***.

Figure 4.17

Polar moment of inertia of dA about z-axis perpendicular to plane of A is defined as :

$$I_{z\,(\text{Element})} = dA\,(x^2 + y^2)$$

Also, we defined the product of Inertia of element dA with respect to axes x and y,

$$I_{xy\,(\text{Element})} = dA\,(x \times y)$$

Moment of Inertia of Area A

Moment of Inertia (M. I.) of full area A about centroidal axis CX, where C is the centroid of the area A is given by surface integral as follows :

$$I_{x(A)} = \int_A dA \times y^2$$

Similarly, about axis CY, $I_{y(A)} = \sum dA \times x^2$

4.3.2Perpendicular Axis Theorem

Polar Moment of Inertia of A about z axis passing through C. Referring Figure 4.17, where axis ZC is perpendicular to the plane of area A, we have

$$I_{z(A)} = \sum dA\, r^2 = \sum dA\,(x^2 + y^2)$$

i.e., $$I_{z(A)} = I_{y(A)} + I_{x(A)}$$

Moment of inertia of the area A about any axis X_1X_1 shown in Figure 4.17 is given by

$$I_{(X_1X_1)} = \text{(square of distance from } X_1\, X_1 \text{ axis)}$$

$$= \sum dA\,(y + y_1)^2$$

where y_1 is the perpendicular distance between X_1X_1 and CX. Thus for a given axis X_1X_1, y_1 is constant.

Similarly, $$I_{X_2X_2} = \sum dA\,(y + y_2)^2$$

4.3.3Parallel Axis Theorem

With reference to Figure 4.17, it is noted that first moment of elemental area dA about centroidal axis is given by ($dA \times y$). By the definition of centroid C of the area, it is further noted that

$$\sum dA \times y = 0$$

This means that horizontal plate of area A gets balanced about axis CX. When Moment of Inertia of areas are computed about any random axis X_1X_1, then

$$I_{(X_1X_1)} = \sum dA\,(y + y_1)^2$$

$$= \sum dA\,(y^2 + 2yy_1 + y_1^2)$$

$$= \sum dA \times y^2 + 2y_1 \sum dA \times y + y_1^2 \sum dA$$

$$= I_{cx} + 0 + A\,(y_1)^2$$

Similarly, $$I_{X_2X_2} = I_{cx} + A(y_2)^2.$$

This equation is termed as parallel axis theorem, whereby it is observed that out of all axes parallel to centroidal axis, CX, the Moment of Inertia about the centroidal axis is minimum, for a given direction of the axis.

Similarly, Referring Figure 4.17,

$$I_{Y_1Y_1} = I_{cy} + A(x_1)^2$$

where x_1 is the perpendicular distance between the axes y_1 y_1 and CY.

$$I_{Z_1Z_1} = G_{CZ} + A(x_1^2 + y_1^2)$$

where the perpendicular distance between axes z_1z_1 passing through point K (x_1, y_1) and CZ is

$$r_1 = \sqrt{x_1^2 + y_1^2}$$

Example 4.5

Determine the axial moment of inertia of a rectangular area of base b and height d about centroidal axis GX and the base B_1B_2.

Solution

Referring Figure 4.18, where centroidal axis GX divides the area at mid-depth, i.e. $\frac{d}{2}$.

Figure 4.18

For a thin strip shown shaded, of width b and thickness (very small) dy, all points on it are at a constant distance y from GX,

$$\therefore \qquad dA = b\, dy$$

Considering y as positively upward from centroidal axis GX, for elements below GX, y will be treated as negative.

$$\therefore \qquad \int A\, dy = 0$$

$$I_{GX} = \sum dAy^2 = \sum (b\, dy) \times y^2$$

$$\therefore \qquad b \int_{(-d/2)}^{(+d/2)} y^2\, dy = b \left[\frac{y^3}{3} \right]_{(-d/2)}^{(+d/2)}$$

$$= \frac{bd^3}{12}$$

Similarly, referring to y axis through centroid G,

$$I_{GY} = \frac{db^3}{12}$$

Moment of inertia about base B_1 B_2 can be computed either directly or by using parallel axis theorem.

Direct approach is as follow. Referring Figure 4.19,

$$I_{B_1B_2} = \int dA \times (y')^2$$

$$I_{B_1B_2} = \int_{(y'=0)}^{(y'=d)} b\,(dy') \times (y')^2$$

$$= \frac{b}{3}[y'^3]_0^d$$

$$= \frac{bd^3}{3}$$

Figure 4.19

Alternatively, using theorem of parallel axis, we have

$$I_{B_1B_2} = I_{GX} + A(y_1)^2$$

where, $y_1 = \frac{d}{2} =$ perpendicular distance between GX and B_1 B_2.

$$= \frac{bd^3}{12} + bd\left(\frac{d}{2}\right)^2$$

$$= \frac{bd^3}{3}$$

Example 4.6

Determine the Moment of Inertia of a triangular area ABC having base *b* and height *d* about its base BC. Hence or otherwise determine the Moment of Inertia about the centroidal axis parallel to the base.

Solution

Figure 4.20 shows the triangular area ABC with its centroid G, where perpendicular distance from G to BC (= *b*) is $\left(\frac{d}{3}\right)$.

Figure 4.20

Consider a thin strip LM, of thickness *dy*, at distance *y* from the base BC. Considering similar triangles, ALM and ABC, we have :

$$\frac{LM}{(d-y)} = \frac{BC}{d} = \frac{b}{d}$$

$$LM = \frac{b}{d}\,(d-y)$$

$$= b\left(1-\frac{y}{d}\right)$$

Now Elemental Area (shaded) $dA = b\left(1-\frac{y}{d}\right)dy$

$$I_{BC} = \int_{y=0}^{y=d} b\left(1-\frac{y}{d}\right)dy\,(y^2)$$

$$= b\int_0^d \left(y^2 - \frac{y^3}{d}\right)dy$$

$$= b\left(\frac{y^3}{3} - \frac{y^4}{4d}\right)_0^d$$

$$= bd^3\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{bd^3}{12}$$

Using theorem of parallel axes, we get :

$$I_{GX} = I_{BC} - A\left(\frac{d}{3}\right)^2$$

where, GX is the axis through G and parallel to BC.

$$= \frac{bd^3}{12} - \frac{bd}{2}\left(\frac{d^2}{9}\right)$$

$$= \frac{bd^3}{6}\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{bd^3}{36}$$

Example 4.7

Determine the Moment of Inertia of an *I*-shaped area about its centroidal axis as shown in Figure 4.21(a) (dimensions are given in mm).

(a) **(b)**

Figure 4.21

Solution

(a) The area of *I* section can be divided into three parts namely A_1, A_2 and A_3.

$$A = A_1 + A_2 + A_3 = (400 \times 150 \times 2 + 300 \times 200) = 18 \times 10^4 \text{ mm}^2$$

The centroid G is at mid-depth as shown in Figure 4.21(a).

Now, $$I_{A(x)} = I_{A_1(x)} + I_{A_2(x)} + I_{A_3(x)}$$

where, $$I_{A_3(x)} = I_{A_1(x)} = \frac{1}{12} \times 400 \times 150^3 + 400 \times 150 \times (225)^2$$

(where, $150 + \frac{150}{2} = 225$).

or, $$I_{A_3(x)} = I_{A_1(x)} = 10^6 \, [112.5 + 3037.5] = 3150 \times 10^6 \text{ mm}^4$$

$$I_{A_2(x)} = \frac{1}{12} \times 200 \times 300^3 = 450 \times 10^6 \text{ mm}^4$$

$\therefore$ $$I_{A(x)} = [(2 \times 3150) + 450] \times 10^6$$

$$= 6750 \times 10^6 \text{ mm}^4$$

(b) **Alternatively,** the M. I. of *I*-section about *x* axis can be obtained by subtracting the M. I. of area $[2 \times (A_2')]$ from the M. I. of area (A_1') as shown in Figure 4.21(b).

Here, $A_1' = 400 \times 600 \text{ mm}^2$ and $A_2' = 100 \times 300 \text{ mm}^2$

$\therefore$ $$A = A_1' - 2A_2'$$

$$= 24 \times 10^4 - 6 \times 10^4 = 18 \times 10^4 \text{ mm}^2$$

Then, $$I_{A(X)} = I_{(A_2')} - [I_{(A_2')} \times 2]$$

$$= \frac{400 \times 600^3}{12} - \frac{2 \times 100 \times 300^3}{12}$$

$$= \frac{1}{12} \times 10^6 \, [86400 - 5400] = 6750 \times 10^6 \text{ mm}^4$$

SAQ 4

Determine the area moment of inertia of a *T*-section about its centroidal axis as shown in Figure 4.22. Determine also the radius of gyration about its centroidal axis CX.

Figure 4.22

SAQ 5

Compute the M. I. of a hollow section shown in Figure 4.23 about its centroidal axis OX.

Figure 4.23

SAQ 6

Determine the centroid of the shaded area shown in Figure 4.24.

Figure 4.24

4.4 SUMMARY

- Centroid C of an area A is a point in the plane of the area where entire area can be assumed to be concentrated.
- Centre of gravity G of a thin plate of same surface area A referred above is a point where the entire mass or weight of the plate is assumed to be concentrated. In fact, it is possible to support this plate in a horizontal plane by a tip of the vertically placed single pin at point G.
- The geometric centre of a rectangular area or a circular area also represents its centroid. Centroid lies on an axis of symmetry of the area. When there are two axes of symmetry, the centroid is the point of intersection of the two axes.
- For the triangular area A (or triangular plate of same area A with mass M and of small uniform thickness), the centroid of A (or centre of gravity of the plate of mass M) is at the point of intersection of the medians of the triangle.
- Generally, any given area can be divided into parts A_1 to A_n for each of area A_1 can be computed along with the location of its centroid (x_i, y_i), the values of $(\bar{x}, \bar{y})$ of centroid can be computed as

$$A\,\bar{x} = \sum_{i=1}^{n} A_i\; x_i$$

and

$$A\,\bar{y} = \sum_{i=1}^{n} A_i\; y_i$$

- For a uniform plate of mass M, the entire mass can be divided into individual masses M_i for which centre of gravity is known as (x_i, y_i), the values of $\bar{x}, \bar{y}$ for centre of gravity is given by

$$M \ \bar{x} = \sum_{i=1}^{n} M_i \ x_i$$

and $$M \ \bar{y} = \sum_{i=1}^{n} M_i \ y_i$$

- Area moment of inertia about x axis $= \int_A dA \, (y^2) = I_{XX\,(A)}$
- Area moment of inertia about y axis $= \int_A dA \, (x^2) = I_{YY\,(A)}$
- Parallel Axis Theorem : Area moment of inertia about $X_1 X_1$ axis at distance y_1 from XX $= I_{xx} + A\,(y_1)^2$
- Perpendicular Axis Theorem : If z axis is perpendicular to the plane of area A, then $I_{zz} = I_{xx} + I_{yy}$

4.5 ANSWERS TO SAQs

SAQ 1

Total area is divided into three parts A_1, A_2 and A_3 as shown in figure below.

Figure for Answer to SAQ 1

Area mark A_i	Area (cm^2)	$\bar{x}_i$ (cm)	$\bar{y}_i$ (cm)	$A_i\,\bar{x}_i$ (cm^3)	$A_i\,\bar{y}_i$ (cm^3)
A_1	36	1.5	6	54	216
A_2	36	6	13.5	216	486
A_3	36	7	2	252	72
Total	108			522	774

$\therefore$ $$\bar{x} = \frac{522}{108} = 4.83 \text{ cm}$$

$$\bar{y} = \frac{774}{108} = 7.17 \text{ cm}.$$

SAQ 2

Referring to Figure for Answer to SAQ 2, total area is divided into three parts A_1, A_2 and A_3.

Figure for Answer to SAQ 2

Area mark A_i	**Area (cm^2)**	$\bar{x}_i$ **(cm)**	$\bar{y}_i$ **(cm)**	$A_i\,\bar{x}_i$ **(cm^3)**	$A_i\,\bar{y}_i$ **(cm^3)**
A_1	20	5	1	100	20
A_2	20	1	7	20	140
A_3	50.26	6	8	301.59	402.12
Total	90.26			421.59	562.12

$$\therefore \qquad \bar{x} = \frac{421}{90.26} = 4.67 \text{ cm}$$

$$\bar{y} = \frac{562.12}{90.26} = 6.23 \text{ cm}\,.$$

SAQ 3

Refer Figure for Answer to SAQ 3. Since portion AM_2M_1 takes up a new position $A'M_2M_1$, weight of plate $A'M_2M_1$ add to weight of plate OM_2M_1, then portion OM_2M_1B is considered as areas A_1 and $2A_2$.

Figure for Answer to SAQ 3

A_i	Area	$\bar{x}_i$	$\bar{y}_i$	$A_i\, \bar{x}_i$	$A_i\, \bar{y}_i$
A_1	20	5	1	100	20
$2A_2$	20	1	7	20	140
A_3	50.26	6	8	301.59	402.12
Total	90.26			421.59	562.12

$\therefore \qquad \bar{x} = 3 \text{ and } \bar{y} = 2$

SAQ 4

Let G be the centroid at distance y_T from top face as shown in figure.

A **(cm^2)**	y_T **(cm)**	Ay_t
1000	10	10^4
800	40	3.2×10^4
$\sum A_i = 1800$		4.2×10^4

Figure for Answer to SAQ 4

$$\bar{y}_T = \frac{420}{18} = 23.3 \text{ cm}$$

$$\bar{y}_B = 26.7 \text{ cm}$$

$$I_{(A)} = I_{A_1} + I_{A_2}$$

$$I \text{ about } CX = \frac{50 \times 20^3}{12} = 1000 \times (13.3)^2 + \frac{20 \times 40^3}{12} + 800 \times (6.7)^2$$

$$= 33.33 \times 10^3 + 176.89 \times 10^3 + 106.66 \times 10^3 + 800 \times (6.7)^2$$

$$\therefore \qquad I_{(cx)} = 377 \times 10^3 \text{ cm}^4 = A(r_{cx})^2$$

So, $$r_{cx} = \left(\frac{377 \times 10^3}{1800} \right)^{\frac{1}{2}} = 14.47 \text{ cm}$$

SAQ 5

Referring to Figure for Answer to SAQ 5.

Figure for Answer to SAQ 5

$$I_{OX\left\{\frac{\text{Hollow}}{\text{Section}}\right\}} = I_{OX\left\{\frac{\text{Full}}{\text{Section}}\right\}} - I_{OX\{VOID\}}$$

$$= \frac{20 \times 20^3}{12} - \frac{\pi (7.5)^4}{4}$$

$$= 10848 \text{ cm}^4$$

SAQ 6

The shaded area A can be considered as algebraic sum of three areas A_1, A_2 and A_3 as shown in Figure for Answers to SAQ 6.

$$A = A_1 + A_2 - A_3$$

Figure for Answer to SAQ 6

where $A_1 = 200 \times 200 = 40{,}000 \text{ mm}^2$ with centroid at (100, 100)

$A_2 = \dfrac{\pi \times 100^2}{2} = 15{,}700 \text{ mm}^2$ with centroid at (2424, 100) and

$A_3 = \left(\dfrac{-\pi \times 60^2}{2}\right) = -5655 \text{ mm}^2$ with centroid at (100, 35.44).

Sl. No.	Area A_i (mm^2)	x_i (mm)	$A_i\, x_i$ (mm^3)	$\bar{y}_i$ (mm)	$A_i\, y_i$ (mm^3)
1	$A_1 = 40{,}000$	100	400×10^4	100	400×10^4
2	$A_2 = 15{,}700$	242.4	380×10^4	100	157×10^4
3	$A_3 = -5655$	100	-56×10^4	25.44	-14×10^4

$\therefore \Sigma\, A_i = 50{,}045 \text{ mm}^2 \qquad \Sigma\, A_i\, x_i = 724 \times 10^4 \qquad \Sigma\, A_i\, y_i = 543 \times 10^4$

$$\therefore \bar{x} = \frac{724}{5.0045} \qquad \therefore \bar{y} = \frac{543}{5.0045}$$

$$= 144.67 \text{ mm} \qquad = 108.50 \text{ mm}$$

Example 4.8

Determine Moment of Inertia of circular area of radius a = 10 cm about its centroid axis OX as shown in Figure 4.23.

Figure 4.23

Solution

Consider a thin strip of width B_y at distance y from axis OX where

$$y = a \sin\theta \quad \text{and} \quad B_y = 2(a\cos\theta)$$

differentiating this equation :

$$dy = a \cos\theta \, d\theta$$

$$dA = B_y \, dy = (2a\cos\theta) \, a\cos\theta \, d\theta$$

$$I_x = \int_A dA \times y^2$$

All elements have to be considered for values of θ ranging from – 90° to + 90°.

$$I_x = \int_{-90^o}^{+90^o} dA \times y^2 = 2\int_{0^o}^{90o} \frac{dA}{2} \times y^2$$

$$= 2\int_{0^o}^{90^o} a^2 \cos^2\theta \, d\theta \times a^2 \sin^2\theta$$

Note that, $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

and $\cos 2\theta = 1 - 2\sin^2\theta$ or $-2\sin^2\theta = 1 - \cos^2\theta$

$\therefore$ $2\cos^2\theta = 1 + \cos 2\theta$

$\therefore$ $$I_x = \frac{a^4}{2}\int_{(-\pi/2)}^{(+\pi/2)} (2\sin\theta\cos\theta)^2 \, d\theta$$

We know $$\frac{a^4}{4}\int 2[\sin 2\theta]^2 \, d\theta = \frac{a^4}{4}\int (1 - \cos 4\theta) \, d\theta$$

And, $\cos 4\theta = 1 - 2(\sin 2\theta)^2$, $\therefore$ $2(\sin 2\theta)^2 = 1 - \cos . 4\theta$

$\therefore$ $$I_x = \frac{a^4}{4}\left(\theta - \frac{\sin 4\theta}{4}\right)_{-\pi/2}^{+\pi/2}$$

$$= \frac{\pi a^4}{4}$$

$$= \frac{\pi D^4}{64}$$

Where, D = diameter of circle = 2 a in the present problem.

$$\therefore \qquad I_x = \frac{\pi \times 10^4}{4} = 7857 \text{ cm}^4 .$$

if $\qquad 2a = 10$ cm.

It is to be noted that circular cross-section is axis-symmetric i.e. it is symmetric about both x and y axes or any other radial direction and also has same nature of shape about all centroidal axes.

$$I_y = I_x = \frac{\pi a^4}{4} = \pi a^2 \left(\frac{a^2}{4} \right) = A \left(\frac{a^2}{4} \right)$$

where A = Area of the circle of radius a.

Considering perpendicular axis theorem,

$$I_z = I_x + I_y = A \left(\frac{a^2}{2} \right)$$

SAQ 6

Determine the moment of inertia of the semicircle about axis AT which is tangential to the circle as shown in Figure 4.26.

Figure 4.25