UNIT 6 LAWS OF MOTION

Structure

6.1 INTRODUCTION

The kinematics of motion structure, in Unit 5, did not consider the effect of force on motion. In this unit, we will study the relationship between the motion and the force causing it. Such a study is termed as kinetics. These relationships are governed by what are known as laws of dynamics. Newton's three laws of motion constitute important part of laws of dynamics.

Important Terms

Some significant terms associated with kinetics can be listed as follows :

Mass

It is representative of the matter contained in a body. Units are kilograms, tonnes, pounds etc.

Force

It is defined as the cause which produces or tends to produce a change in state of rest or of uniform motion of a body, commonly expressed as Newton (N), kilo Newton (kN), kgf etc.

Weight

It is the force produced in the body due to gravitational attraction of earth. It acts towards the centre of earth. Since it is a force, its units are same as that of force, e.g. N, kN, kgf etc.

Inertia

It is inherent property of a body which offer resistance to change in its state of rest or of uniform motion.

Momentum

It is quantitative measure of motion possessed by a body when it is moving in a straight line. It can be expressed as a product of its mass and its velocity, e.g. Momentum = $m \times v$.

Absolute Unit of Force

In C.G.S. units, the absolute unit of force is dyne. One dyne is defined as the force which acting on a mass of 1 gram produces an acceleration of 1 cm/sec^2 with the direction of its action.

In MKS system, the absolute unit of force is Newton which is the force which produces an acceleration of 1 m/sec^2 in a mass of 1 kg. Thus

$$1 \text{ N} = 10^5 \text{ dynes}$$

Gravitational Unit of Force

The force produced on a mass due to attraction of earth (gravity) is the gravitational force. Thus, if 1 Newton signifies the force produced by a mass of 1 kg with an acceleration of 1 m/sec^2 then the same body moving with an acceleration due to gravity of 9.81 m/sec^2 will produce a weight (Force) of 9.81 Newtons. This is also called a weight of 1 kgf which is gravitational of engineers' unit of force, i.e. 1 kgf = 9.81N.

Gravitational unit of force is equal to "g" times the absolute unit where "g" is acceleration due to gravity ($W = m \times g$ or $m = W/g$).

Objectives

After studying this unit, you should be able to

- conceptualise the Newton's laws of motion,
- compute simple problem on laws of motion,
- describe the motion on a circular path,
- explain simple harmonic motion,
- explain the relationship of work to kinetic and potential energy,
- state the range of application of principle of conservation of momentum and conservation of energy,
- understand the concept of power, and
- estimate the motion of bodies after their impact.

6.2 NEWTON'S LAWS OF MOTION

Newton proposed three basic laws of motion which are commonly known as Newton's Laws of motion. These can be enunciated as follows

6.2.1 First Law of Motion

A body continues in its state of rest or of uniform motion in a straight line unless it is acted upon by some external force to change its initial state.

6.2.2 Second Law of Motion

The rate of change of momentum is directly proportional to the applied force and takes place in the direction of application of force in a straight line.

Let a mass of body m moving with an initial velocity u ($x = 0$ in the state of rest) is acted upon by a force p. This force will cause its velocity to change from u to final velocity v in time t. Then, initial momentum of the body is mu and the final momentum of the body after time t is mv.

Then the change in momentum in time t would be $mv - mu = m(v - u)$ or, rate of change of momentum would be $\frac{m(v-u)}{t} = ma$.

where a = acceleration = rate of change of velocity $= \frac{v-u}{t}$.

As per Newton's second Law of motion, rate of change of momentum will be directly proportional to applied force or $P \propto m \, . \, a = kma$, where k is constant of proportionality. If the force is defined such as to produce unit acceleration in a body of unit mass, then

$$1 = k \times 1 \times 1 \quad \text{or} \quad k = 1$$

i.e

$$p = ma \qquad \ldots (6.1)$$

6.2.3 Third Law of Motion

"To every action, there is an equal and opposite reaction." This implies that force is an action which one body applies on another. Thus, if a force is exerted by body A on another body B, an equal and opposite force will be applied by body B on body A. Thus, when a person applies some effort on a stone to lift it up, the stone exerts an equal downward force on the person.

6.2.4 Law of Conservation of Momentum

From the above basic laws of motion as stated by Newton, we can conclude that "Total momentum of any group of objects always remains the same if no external force acts on them".

From Newton's second law, we know that to change the momentum of a system of bodies, external force is required which is proportional to the rate of change of momentum. Hence, total momentum will remain same in the absence of an external force. Any action amongst the system of bodies will cancel out as any action of one body in the system will produce equal and opposite reaction on it by Newton's third law.

6.2.5 Impulse and Impulse Force

Impulse is defined as mass multiplied by change in velocity of the body. If a body of mass m is subjected to a force P which is acting on it for a time period t, causing its velocity to change from u to v.

Then impulse $\quad I = m \times (v - u)$ . . . (6.2(a))

or $\quad \frac{I}{t} = m \times \frac{(v-u)}{t} = m \times a \quad \left\{ \because a = \frac{v-u}{t} \right\}$ (by Newton's Second Law)

or $\quad I = P \times t$ . . . (6.2(b))

Hence, impulse can also be defined as forces × time interval or impulsive force. Its units are Ns (Newton's Seconds) or kgf s. From Eq. (6.2(a)), the impulse of a force *P* can also be called as change of momentum produced in the body on its application.

When a force is applied on a body gradually, it is said that the force system is in steady state. It is sustained on the body for a long period. However, if time duration of force is small, the force is said to be suddenly applied force producing an impact. The concept of impulsive of a force is very useful in analysing problems of suddenly applied force or impacts. The examples of impact could be firing of a bullet or collision of two bodies.

Simple Example of Recoil of Gun

Principle of impulse and the conservation of momentum can be of great use in considering problems associated with a recoil of a gun and likewise. Before firing, the gun is loaded with the bullet and both are in a state of rest. The initial velocity of bullet as well as that of the gun is zero. Hence, the initial total momentum of the system is zero. Upon firing, the bullet moves in one direction while the gun in the opposite direction, e.g. it recoils. By principle of conservation of momentum the momentum of the bullet will be equal and opposite to that of the gun, because the total final momentum of the system shall also be zero.

Example 6.1

A gun has a mass of 30 tonnes. It fires a bullet whose mass is 450 kg with a velocity of 300 m/s.

(i) Calculate the initial velocity of gun recoil.

(ii) If a resistive force of 600 kN is applied on gun on an average, calculate the distance travelled by the gun during recoil?

(iii) Also compute the time period of recoil.

Solution

Mass of gun (M) = 30,000 kg; Mass of bullet (m) = 450 kg, its velocity (v) = 300 m/s.

Let V is recoil velocity of gun.

(i) Then by principle of conservation of momentum

$$MV + mv = 0$$

or $\quad V = -\frac{mv}{M} = \frac{450 \times 300}{30000} = -4.5 \text{ m/s}$ . . . (a)

(i.e. gun velocity is opposite to that of bullet)

(ii) Impulse $I = P \times t = M(v - u)$

(Initial velocity of gun = $u = -4.5$ m/s and Final velocity of gun = $v = 0$)

or $\quad 600000 \times t = 3 \times 10^4 \times 4.5$

or $$t = \frac{3 \times 10^4 \times 4.5}{6 \times 10^5} = 0.225 \text{ seconds} \quad \ldots (b)$$

e.g., time period of recoil = 0.225 seconds.

a = acceleration / retardation $\frac{v - u}{t} = \frac{0 - 4.5}{0.225} = -20 \text{ m/sec}^2$

(iii) We know $v^2 = u^2 + 2\,as$ (where s is distance travelled), then

$$s = \frac{v^2 - u^2}{2a} = \frac{0 - (4.5)^2}{2 \times (-20)} = \frac{-20.25}{-40} = 0.506 \text{ m}$$

Example 6.2

A pile of mass 1000 kg is driven 30 cm into ground by a pile driver of mass 250 kg falling from a height of 2.0 m. Find the average resistance of the ground to penetration of pile, assuming $g = 10$ m/sec^2.

Solution

Let the velocity of pile driver after falling 2.0 m is v.

Then $$V = \sqrt{2 \times gh} = \sqrt{2 \times 10 \times 2} = 6.325 \text{ m/sec} \quad \ldots (a)$$

After the pile driver strikes the pile, the common velocity of system is V while its mass is $(M + m) = 1000 + 250 = 1250$ kg, by principle of conservation of momentum.

Momentum before impact = momentum after impact.

Then $$m \times v + m \times 0 = (M + m) \times V$$

or $$250 \times 6.325 = 1250 \times V$$

or $$V = \frac{250 \times 6.325}{1250} = 1.265 \text{ m/s.} \quad \ldots (b)$$

Since the pile is driven 0.3 m into ground before coming to rest (e.g. final velocity = 0).

$$v^2 = u^2 + 2\,as$$

$$0^2 - 1.265^2 = 2 \times a \times 0.3$$

or $$a = -\frac{(1.265)^2}{2 \times 0.3} = -2.67 \text{ m/sec}^2 \text{ (retardation)}$$

By Newton's Second law of motion, $P = M \times a$

Then $$P = (M + m) \times a = 1250 \times 2.67 = 3337.50 \text{ N}$$

The resistance of ground R will be the retarding force so created plus the weight of the pile driven system

Hence, $$R = 1250 \times 10 + 3337.50 = 15837.50 \text{ N}$$

Example of Motion of a Lift

A lift or elevator normally moves vertically carrying its own weight and of the passengers/cargo it carries. Without much less of accuracy it can assumed to be moving under uniform acceleration.

It has two types of motion

(i) Moving upwards (against gravity) (Figure 6.1(a))

(ii) Moving downwards (along gravity) (Figure 6.1(b))

Let us assume that

m_s = self mass of the lift assembly and $m_s g$ its weight

m_L = mass of cargo carried by it and $m_L g$ its weight

m = total mass moving with mg, i.e. the weight carried.

Then $m = m_s + m_L$ and

$$W = (m_s + m_L)\, g \qquad \ldots (6.3(a))$$

α = uniform acceleration of lift

g = acceleration due to gravity

T = tension in the cable supporting the lift.

(a) **(b)**

Figure 6.1 : Motion of a Lift

For upward movement against gravity

Net upward force on the lift

$$= T - W \qquad \ldots (i)$$

$$\text{Inertia} = m \times \alpha \qquad \ldots (ii)$$

Satisfying equation of dynamic equilibrium, we get

$$T - mg = m\,\alpha \quad \text{or} \quad T = m\,(g + \alpha) \qquad \ldots (6.3(b))$$

And for downward movement along gravity, it would be

$$W - T = m\,\alpha \quad \text{or} \quad T = m\,(g - \alpha) \qquad \ldots (6.3(c))$$

Example 6.3

A cage of self mass 1000 kg is carrying a shift load of six passengers of average mass 65 kg. If the cage is moving with

(a) a uniform acceleration of 2 m /sec^2 upwards

(b) a uniform acceleration of 2 m/sec^2 downwards or

(c) a uniform velocity of 5 m/s downwards, find

(i) the tension in wire supporting the cage

(ii) the reaction of cage on the crew and of the crew on the cage

Assume g = 10 m/sec^2.

Solution

Mass of cage = 1000 kg

Mass of crew $6 \times 65 = 390$ kg

(a) Upward $\alpha = 2$ m/sec^2

Total mass = 1000 + 390 = 1390 kg (acting vertically downwards)

(i) Equation of dynamic equilibrium

$T - (1390)\,g = 1390\,\alpha$ . . . (i)

or $T = 1390\,(10 + 2) = 16680$ N . . . (ii)

(ii) If R is reaction of cage on crew, then dynamic equilibrium gives

$R - mg = m\alpha$

or $R = 390\,(10 + 2) = 4680$ N . . . (iii)

(iii) By Newton's third law of motion, the force exerted by crew on cage will be same as force exerted by cage on crew.

(b) Downward $\alpha = 2$ m/sec^2.

(i) $W - T = M\alpha$

or $1390 \times 10 - T = 1390 \times 2$

or $T = 1390\,(10 - 2) = 11120$ N . . . (iv)

(ii) $R = 390 \times (10 - 2) = 3120$ N . . . (v)

(c) Downward velocity $v = 5$ m/s, $\alpha = 0$

(i) $1390 \times 10 - T = 0$

or $T = 13900$ N . . . (vi)

(ii) $R = 390 \times 10$

or $R = 3900$ N . . . (vii)

Example 6.4

An elevator of total mass 500 kg starts from position of rest and moves upwards at a constant acceleration. It gains a velocity of 2 m/sec in a travel distance of 3 m. While stopping a uniform retardation it comes to rest in 2 seconds from a velocity of 2 m/sec.

Assuming $g = 10$ m/sec^2. Calculate the pull in cable during upward movement and pressure transmitted by a person of mass 70 kg to the floor during stopping.

Solution

While accelerating, the initial velocity of elevator $u = 0$; while final velocity $v = 2$ m/sec, and distance travelled = 3 m.

Let uniform acceleration during this period be α.

Then from equation $v^2 = u^2 + 2\,\alpha s$, we get

$$2^2 = 0 + 2\,\alpha \times 3$$

or $$\alpha = \frac{4}{6} = 0.667 \text{ m/sec}^2$$ . . . (i)

From equation of dynamic equilibrium of elevator

$$T - 500 \times 10 = 500 \times 0.667$$

or $$T = 5000 + 333.3 = 5333.3 \text{ N} \quad \ldots \text{(ii)}$$

While stopping during retardation

$$u = 2 \text{ m/sec}, \quad v = 0 \quad \text{and} \quad t = 3 \text{ seconds}$$

From equation $v = u + \alpha t$, we get

$$0 = 2 + \alpha \times 2 \quad \text{or} \quad \alpha = -\frac{2}{2} = -1 \text{ m/sec}^2 \quad \ldots \text{(iii)}$$

From equation of dynamic equilibrium of person

$$W - R = -\alpha \times \text{m}$$

or $$R = 70 \times 10 - 1 \times 70 = 630 \text{ N} \quad \ldots \text{(iv)}$$

SAQ 1

A cannon of mass 20000 kg fires a shell of mass 100 kg with a muzzle velocity of 800 m/s. Calculate the recoil velocity of cannon, average uniform force required to stop in a recoil distance of 400 mm and time period required for recoil.

SAQ 2

A shell of mass 1 kg is fired from a cannon of mass 1000 kg with a velocity of 300 m/s in a horizontal direction. The cannon is resting on a smooth horizontal surface against a spring buffer of spring constant 15 N/mm compression. Calculate the spring displacement due to firing.

6.3 MOTION ON A CIRCULAR PATH

6.3.1 Angular Motion, Relationship with Linear Motion

When a body is rotating about an axis, its motion will be along a circular path. Its position at any time can be described by its distance from axis of rotation (radial distance) and angle θ it subtends at centre with respect to the radial line drawn at the position $t = 0$ (Figure 6.2).

(a) Angular Displacement

In a rotational motion, angular displacement θ or the movement of body along its circular path will vary with time. In general, θ can be expressed as a function of time equation

$$\theta = f(t) \qquad \ldots (6.4(a))$$

(b) Angular Velocity

Like rectilinear velocity which was defined as the rate of change of rectilinear displacement with time, angular velocity can also be defined as rate of change of angular displacement with time.

(a) **(b)**

Figure 6.2

Let the body moves from θ to θ + δθ position while moving from time t to $t + \delta t$. Then the body moves an angular distance of δθ in time interval δt or

angular velocity denoted by ω (omega) $= \dfrac{\delta\theta}{\delta t}$ . . .

(i)

If the body is moving with a uniform angular velocity, same angular distance δθ will be travelled in a similar time interval δt, however small or large δt could be.

However, when the angular velocity is changing with time, the average angular velocity over the time period δt would be $\omega_{av} = \dfrac{\delta\theta}{\delta t}$.

Hence if time interval δt is taken very small

$$\omega = \underset{\delta t \to 0}{\text{Lt}} \frac{\delta\theta}{\delta t} = \frac{d\theta}{dt} \qquad \ldots (ii)$$

This is termed instantaneous angular velocity at time t. If the body's angular rotation is measured in terms of revolutions per minute (= N rpm),

Then the angular velocity would be

$$\omega = \frac{2\pi N}{60} \text{ rad/sec.} \qquad \ldots (iii)$$

(c) Tangential Velocity

If r is the radius of circular path and v is the rectilinear velocity of the body at time t then

$$v_t = \underset{\delta t \to 0}{\text{Lt}} \frac{\delta s}{\delta t} = \underset{\delta t \to 0}{\text{Lt}} r \frac{\delta \theta}{\delta t} = r \frac{d\theta}{dt} \quad \ldots$$

(iv)

or $v_t = \omega r$

where s is the distance travelled by body along the circumference of the circular path and δs is the circumferential distance travelled by the body in time δt.

(d) Acceleration

Let the angular velocity ω changes with time t. Then angular acceleration

α (alpha) is defined as rate of change of angular velocity. If the change in angular velocity during time interval δt is $\delta \omega$.

Then $$\alpha = \frac{\delta \omega}{\delta t} \quad \ldots$$

(vi)

Hence instantaneous angular acceleration would be

$$\alpha = \underset{\delta t \to 0}{\text{Lt}} \frac{\delta \omega}{\delta t} = \frac{d\omega}{dt} \quad \ldots \text{(vii)}$$

Let the tangential velocity be v_t at any given time t (Figure 6.2(b)). Then the change in velocity in tangential direction would be dV in forward direction of θ while it would be $Vd\theta$ in radial direction (inward towards centre of rotation).

Since acceleration of a particle would be its ratio of change of velocity with time, the components of acceleration would be

$$\alpha_t \text{ (tangential)} = \frac{dV_t}{dt} = \frac{rd}{dt}\left(\frac{d\theta}{dt}\right) = r \frac{d^2\theta}{dt^2} = r\,\alpha \quad \ldots \text{(viii)}$$

and $$\alpha_r \text{ (radial)} = -V_t \frac{d\theta}{dt} = -V_t\,\omega = -\frac{v_t^2}{r} = -\omega^2 r \quad \ldots \text{(ix)}$$

Negative sign indicates that direction of radial (or normal) acceleration would always be inwards. This normal acceleration directed towards the centre of rotations is also called centripetal acceleration.

6.3.2 Equations of Angular Motion

Let ω_0 be initial angular velocity, then

$$\alpha = \frac{\omega - \omega_0}{t} \text{ or } \alpha t = \omega - \omega_0 \text{ or } \omega = \omega_0 + \alpha t \quad \ldots \text{(6.5(a))}$$

Average angular velocity over time period $t = \frac{\omega + \omega_0}{2}$

then $$\theta = \frac{(\omega_0 + \omega)}{2} \times t$$

or $$t = \frac{2\theta}{\omega_0 + \omega} \quad \ldots$$

(i)

$$\theta = \left(\frac{\omega_0 + \omega_0 + \alpha t}{2}\right) t = \omega_0 t + \frac{1}{2} \alpha t^2$$

or $$\theta = \omega_0 t + \frac{1}{2} \alpha t^2 \qquad \ldots (6.5(b))$$

Substituting Eq. (i) in Eq. (6.5(a))

$$\omega = \omega_0 + \alpha \left(\frac{2\theta}{\omega_0 + \omega}\right) \quad \left(\because\ t = \frac{2\theta}{\omega_0 + \omega}\right)$$

or $$\omega (\omega_0 + \omega) = \omega_0 (\omega_0 + \omega) + 2\ \alpha\theta$$

or $$\omega^2 - \omega_0^2 = 2\ \alpha\theta$$

or $$\omega^2 = \omega_0^2 + 2\ \alpha\theta \qquad \ldots (6.5(c))$$

Eqs. 6.5(a), (b) and (c) are similar to corresponding equations of linear motion.

Example 6.5

A wheel of 1.0 m diameter is mounted on a shaft between two bearings. The wheel is subjected to constant moment of 100 Nm at the rim for 10 minutes to attain a speed of 120 rpm.

Determine

(i) The number of revolutions made during this period.

(ii) The angular acceleration, the tangential acceleration and centripetal acceleration.

(iii) The work done in rotating the wheel during this time.

Solution

Initial angular velocity of wheel = ω_0 = 0.

Final angular velocity, $\omega = \dfrac{2\pi \times N}{60} = \dfrac{2\pi \times 120}{60} = 4\pi$ rad/ sec $\qquad \ldots$ (a)

Radius of wheel =1.0/2 = 0.5 m

Time taken t = 10 minutes = 600 sec.

(i) Then angular rotation θ = average angular velocity × time

$$= \left(\frac{0 + 4\pi}{2}\right) \times 600 = 1200\ \pi \text{ radians} \qquad \ldots (b)$$

Number of revolution made = 1200π / 2π = 600 $\qquad \ldots$ (c)

(ii) We have $\omega = \omega_0 + \alpha t$

$$4\pi = 0 + \alpha \times 600$$

or $$\alpha = \frac{4\pi}{600} = \frac{\pi}{150} \text{ rad/ sec}^2 \qquad \ldots (d)$$

Tangential acceleration, $\alpha_t = \alpha\ r = \dfrac{\pi}{150} \times 0.5 = \dfrac{\pi}{300}$ m/sec^2.

Normal acceleration, $\alpha_n = -\omega^2 r = 0.5 \times (4\pi)^2 = -8\ \pi^2$ m/sec^2.

(iii) Work done = Torque × Angle turned

$$100 \times 1200\ \pi = 120000\ \pi \text{ Joule.}$$

6.3.3 Centripetal and Centrifugal Forces

As shown in Figure 6.2, a body has tendency to move tangentially if it is moving on a circular path. In order to resist this tendency of moving away from the centre, an external force will be required to applied acting radially towards the centre (centripetal). This will ensure that the body continue to move along the circular path. The inertia forces which are equal in magnitude and opposite in direction of centripetal forces are called centrifugal forces.

During its rotation, the body has an acceleration which has two components (i) tangential acceleration $\alpha_t \left(= r \dfrac{d\omega}{dt}\right)$, and (ii) normal acceleration ($\alpha_n = \omega^2 r$). If m is the mass of body then magnitude of centripetal force Fc would be equal to $m \times \alpha_n$.

or $$F_{cp} = -\, m\,\omega^2 r = -\frac{m v_t^2}{r} \text{ (indicating inward force)} \qquad \text{. . .(6.6 (a))}$$

The inertia forces, also known as centrifugal forces, would be

$$F_{cf} = -\, F_{cp} = \frac{m v^2}{r} \text{ (radially outward)} \qquad \text{. . . (6.6(b))}$$

The concepts of centripetal and centrifugal forces are explained by studying the motion of a vehicle on a level curved path.

Let a vehicle of weight W is moving with a linear velocity v on a curved path of radius r (Figure 6.3).

Figure 6.3

Due to rotational movement of vehicle along the circular path of radius r, it will experience a radially outward acting centrifugal force W_{cf}, where $W_{cf} = \dfrac{W}{g}\,\dfrac{v^2}{r}$.

There will be a tendency of the vehicle to skid outwards resulting on producing frictional forces in tyres at point of contact with pathway. From Figure 6.3, the following forces acting on the vehicle can be listed

(i) Centrifugal forces, $F_{cf} = \dfrac{W}{g}\,\dfrac{v^2}{r}$

(ii) Weight of vehicle, W

(iii) Normal Reactions, R_A and R_B

(iv) Frictional force, $F_A = \mu R_A$ and $F_B = \mu R_B$

To study the stability of vehicle in negotiating the curved path, following cases are to be considered.

(a) Skidding of Vehicle

For dynamic equilibrium of vehicle in vertical direction

$\Sigma v = 0$

$$W = R_A + R_B \qquad \ldots \text{(i)}$$

horizontally $\Sigma H = 0$

$$\frac{W}{g}\frac{v^2}{r} = \mu R_A + \mu R_B = \mu(R_A + R_B) = \mu W \qquad \ldots \text{(ii)}$$

i.e.
$$\mu = \frac{v^2}{gr} \text{ or } v_{\max} = \sqrt{\mu \cdot gr} \qquad \ldots (6.7)$$

Hence, the maximum velocity with which the vehicle can negotiate the curve without skidding will be $\sqrt{\mu . gr}$. For safety against skidding, V shall be less than $V_{\max}$.

(b) Vehicle Over Turning

Dynamic equilibrium of vehicle also require $\Sigma M = 0$. Taking moments about A,

$$R_B \times d = W \times \frac{d}{2} + \frac{W}{g}\frac{v^2}{r} \times h$$

or
$$R_B = \frac{W}{2}\left[1 + \frac{2v^2 h}{grd}\right] \qquad \ldots (6.8(a))$$

$$R_A = W - R_B = \frac{W}{2}\left\{1 - \frac{2v^2 h}{grd}\right\} \qquad \ldots (6.8(b))$$

Eq. (6.8(b)) signified that the term $\frac{2v^2 h}{grd}$ can become more than 1 for larger value of v, making reaction R_A negative and vehicle will overturn. The maximum speed with which the vehicle may move without over turning will be

$$V_{\max} = \sqrt{\frac{grd}{2h}} \qquad \ldots (6.8(c))$$

For the vehicle to overturn and skid simultaneously Eq. (6.7) and Eq. (6.8(b)) are satisfied simultaneously, e.g.

$$v^2 = \mu gr = \frac{grd}{2h}$$

or
$$h = \frac{d}{2\mu} \qquad \ldots (6.9)$$

6.3.4 Banking of Roads and Railway Tracks

The vehicle negotiating the curve exerts a frictional force between the tyres and the roads or wheels and rails causing wear and tear. Also, it exerts a centrifugal force at centroid of vehicle to cause its overturning. In order to improve the safety and stability of vehicle and to reduce the wear and tear of road/rail, it is advisable to develop an external centripetal force This can be achieved by providing inclined track surface sloping towards its centre of curvature. This ensures inclined reactive forces with components in centripetal direction as against centrifugal forces produced by angular speed of vehicle. This raising of external portion of track with respect to its inner portion is termed banking or super elevation. The design of banking or super elevation of a curved track can be achieved considering following parameters

(a) Angle of Super Elevation, "β"

Let the angle of super elevation of track be β as shown in Figure 6.4. In ideal condition, β shall be so selected that horizontal component of normal reactions, R_A and R_B, should be equal and opposite to centrifugal force, $\frac{W}{g} \frac{v^2}{r}$, and their vertical component should be equal to vehicle weight, W.

Hence $$(R_A + R_B) \cos \beta = W \qquad \text{. . . (a)}$$

and $$(R_A + R_B) \sin \beta = \frac{W}{g} \frac{v^2}{r} \qquad \text{. . . (b)}$$

From above equations, we get

$$\frac{(R_A + R_B) \sin \beta}{(R_A + R_B) \cos \beta} = \frac{W}{g} \frac{v^2}{r} \times \frac{1}{W} \qquad \text{. . . (6.10(a))}$$

or $$\tan \beta = \left[\frac{v^2}{gr} \right] \qquad \text{. . . (6.10(b))}$$

Hence, for negotiating a curve of radius r with velocity v, the ideal angle of super elevation shall be β as given by Eq. (6.10(b)). Conversely, on a track with super elevation of β, the corresponding velocity of vehicle could be

$$\sqrt{gr \tan \beta} \qquad \text{. . . (6.10(c))}$$

Figure 6.4

(b) Maximum Velocity

To avoid the skidding of vehicle on a super elevated track, the equations of motion can be derived as follows. For dynamic equilibrium, resolve all forces acting on vehicle along the inclined surface and normal to the inclined surface.

$$W \sin \beta + F_A + F_B = \frac{W}{g} \frac{v^2}{r} \cos\beta \qquad \text{(along surface)} \qquad \text{. . . (c)}$$

$$R_A + R_B = \frac{W}{g}\frac{v^2}{r}\sin\beta + W\cos\beta \qquad \text{(normal to surface)}$$

We know that $F_A + F_B = \mu(R_A + R_B) = \mu\left(\frac{W}{g}\frac{v^2}{r}\sin\beta + W\cos\beta\right)$ . . . (d)

Hence $W\left\{\sin\beta + \mu\cos\beta + \frac{\mu v^2}{gr}\sin\beta\right\} = \frac{\mu W v^2}{gr}\cos\beta$ (from Eq. (c)).

or $$\tan\beta + \mu\left(1 + \frac{v^2}{gr}\tan\beta\right) = \frac{\mu v^2}{gr} \qquad \text{. . . (f)}$$

If ϕ is angle of friction between tyre and track, then $\tan\phi = \mu$. The Eq. (f) can be written as

$$\tan\beta + \tan\phi + \frac{v^2}{gr}\tan\beta\tan\phi = \frac{v^2}{gr}\tan\phi$$

or $$\frac{v^2}{gr} = \frac{\tan\beta + \tan\varphi}{1 - \tan\beta\tan\phi} = \tan(\beta + \phi) \qquad \text{. . . (6.11)}$$

Hence to avoid skidding

$$V_{\max} = \sqrt{gr\tan(\beta + \phi)} \qquad \text{. . . (6.11(b))}$$

(c) **Maximum Speed to avoid Overturning on a Super Elevated Track**

As observed in analysing the maximum vehicle speed on a curved plane track, it can be said that to avoid overturning the normal reaction R_A shall always be positive. It could be zero in critical case.

Let us identify the forces acting on a vehicle on a super elevated track as shown in Figure 6.5.

Figure 6.5

In critical case, $N_A = 0$, the total reaction at β_1 will be the combination of tangential frictional force, F_B, and normal reaction, N_B, inclined at an angle of ϕ with normal. For centroid of vehicle G to be in equilibrium the reaction R_B, weight, W, and centrifugal force, $\frac{W}{g}\frac{v^2}{r}$, meeting at G shall be in equilibrium as represented by triangle of force abc in Figure 6.5(b). Hence, we get

$$\tan(\beta + \phi) = \frac{W}{g}\frac{v^2}{r} \times \frac{1}{W} = \frac{v^2}{gr} \qquad \text{. . . (6.12(a))}$$

or $$v = \sqrt{gr \tan(\beta + \phi)} \qquad \ldots (6.12(b))$$

For reaction R_B to pass through point G in Figure 6.5(a), we have

$$\tan\phi = \frac{d}{2} \times \frac{1}{h} = \frac{d}{2h}$$

Substituting this value of tan ϕ in Eq. 6.12(a), we get

$$v = \sqrt{\left\{ gr \frac{\tan\beta + \tan\phi}{1 - \tan\beta \tan\phi} \right\}}$$

or $$v = \sqrt{\left\{ gr \frac{\tan\beta + \frac{d}{2h}}{1 - \frac{d}{2h}\tan\beta} \right\}} \qquad \ldots (6.12(c))$$

Example 6.7

A truck with total weight of 200 kN (with two axles and four wheels) is travelling with a speed of 72 kmph on a plane flat curved road of radius 200 m. The wheel distance of an axle is 1.6 m. The centroid of truck is at a height of 1.5 m above ground along the axis of truck. Assuming the truck load is equally distributed on two axles and g = 10 m/sec^2, obtain the

(i) vertical pressure on each wheel.

(ii) maximum speed at which the vehicle can negotiate the curve without over turning.

(iii) maximum coefficient of friction between tyre and road surface to avoid skidding.

(iv) angle of super elevation required so that the truck can travel on this curve without frictional support.

Solution

Load of truck on each axle = $\frac{200}{2}$ = 100 kN

Speed of truck = 72 km/h = $\frac{72 \times 1000}{60 \times 60}$ = 20 m/s . . .
(i)

Centrifugal force acting on truck = $F_{cf} = \frac{W}{g}\frac{v^2}{2} = \frac{200}{10} \times \frac{20^2}{200} = 40$ kN . . . (ii)

(i) Let the reactions on the outer wheels be R_0 and on inner wheels be N_I (Figure 6.6). Taking moments about i,

$$40 \times 1.5 + 200 \times 0.8 = R_0' \times 1.6$$

Figure 6.6

or $\quad R'_0 = 137.5$ kN

and $\quad R_i = 200 - 137.5 = 62.5$ kN

Thus, pressure on each inner wheel $= \dfrac{R'_0}{2} = 31.25$ kN . . . (iii)

and on each outer wheel $= \dfrac{R_0}{2} = 68.75$ kN . . .
(iv)

(ii) At just the stage of overturning, let the truck speed be V_c then

$$F_{cf} = \frac{200000}{10} \times \frac{v_c^2}{10} \times 1000 .$$

The restoring moment due to truck load must be equal to or less than overturning movement due to centrifugal force about 0.

or $$1000 \times \frac{v_c^2}{10} \times 1.5 = 20000 \times 0.8$$

or $$v_c{}^2 = \frac{200,000}{1000} \times \frac{8 \times 10}{1.5} = 1066.7$$

or $$v_c = 32.664 \text{ m/s}$$

$$= \frac{32.664 \times 60 \times 60}{1000} = 117.6 \text{ kmph} \quad \text{. . . (v)}$$

(iii) For avoiding skidding, the minimum coefficient of friction, $\mu = \dfrac{v^2}{gr}$

(as per Eq. (6.7)).

or $$\mu = \frac{20 \times 20}{10 \times 200} = 0.2$$ . . .
(vi)

(iv) In Eq. (6.11(b)) to avoid assistance of friction $\phi = 0$ hence

$$\tan \beta = \frac{v^2}{gr}$$

or $$\tan \beta = \frac{20^2}{10 \times 200} = 0.2$$

or $$\beta = 11° \ 18.5'$$

SAQ 3

A horizontal bar AB 1.6 m long is rotating about the vertical axis through end A, in an interval of 10 seconds it accelerates uniformly from 1200 to 1800 rpm. Compute the initial and final linear velocities. Also compute the tangential and normal components of acceleration of the mid-point of bar after 8 seconds from start.

SAQ 4

A locomotive weighing 600 kN is negotiating a curve of 300 meter radius at a speed of 100 kmph. Calculate the radial pressure and radial acceleration between wheel and rails. If the wheel base is 1.2 m, find the vertical height at which the outer rail is to be raised above the inner wheel so that the radial pressure is zero at this speed? Also calculate the vertical pressures upon each rail (i) when the track is flat, and (ii) when super elevation is provided as above. Assume $g = 10$ m/sec^2. Engine has four wheels on two axles, and centroid of locomotive is 1.8 m above the rails.

6.4 SIMPLE HARMONIC MOTION

Many types of motion of a particle of a body are such that it acquires the same position and moves in the same direction with the same velocity after a certain fixed time interval, e.g. motion of a piston of the engine, vibration of spring or oscillations of a pendulum. Such motions are termed as periodic motion, and the above referred time interval is called period. Simple harmonic motion is a special case of this general periodic motion, when the body moves in a straight line, its acceleration is always in a direction towards a central fixed point and at any instant is proportional to the distance of the body at that instant from the referred fixed point. Hence the basic characteristics of a simple harmonic motion can be defined as

(i) the fixed point of reference is situated on the path of motion,

(ii) at any instant, the acceleration of the body is directed towards or away from this reference point along the path of motion, and

(iii) the magnitude of acceleration at any instant is proportional to the distance of the body from the fixed point at that instant.

Let us consider the motion of a body as shown in Figure 6.7. The path of motion is XX' rectilinear and O is the fixed reference point.

Figure 6.7

Body A is moving in a simple harmonic motion with extreme positions *P* and p^1 at a distance *r* from fixed point *O*. When the body is moving from *O* towards *P* and is at position A at any instant, it will have a retardation, i.e. negative acceleration towards *O* which is directly proportional to the distance *OA* (= *x*). With increase in *x*, retardation goes on increasing till it reaches *P* ($x = r$) where velocity becomes zero and retardation maximum. As velocity is zero at *P*, the body is at rest. The distance *P*(*r*) is also known as amplitude.

Now the body starts moving towards *O* with decreasing acceleration to attain a value of zero at fixed point *O* ($x = 0$). The velocity is maximum at this point.

6.4.1 Important Terms

Time Period

When the body start from fixed position *O*, traverse the path OPOP′O such that the body is once again at *O* moving in the same direction OP and with same acceleration (*o*) with same velocity, it is said to have completed one oscillation. Time taken by the particle in completing one oscillation is called time period and is denoted by *T*.

Referring Figure 6.8, the motion of body A along rectilinear path XOX′can be treated as projection of point B moving on a circular path of radius

Figure 6.8

and uniform angular velocity ω rad/sec. In one complete oscillation, the distance covered will be 2π rad. Then

$$T = \frac{2\pi}{\omega} \qquad \text{. . . (6.13(a))}$$

Amplitude

The distance of extreme position of body from fixed point (= OP) is called amplitude. It may be noted that fixed point O representing the mean position of the body during a simple harmonic motion.

Frequency

The number of oscillations (also called vibrations or cycles) per second is called frequency. Thus if *f* is the frequency then

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \qquad \text{. . . (6.13(b))}$$

Phase

The position or direction of motion of the body at any time is denoted by the phase while phase difference is referred to the amount by which the phase of a body differ from fixed reference or another body.

6.4.2 Velocity and Acceleration

Referring to Figure 6.8, the angular displacement of the body from position of rest (= 0) after a time internal, t, would be $\omega t = \theta$. The projectile of this on x axis will be OA (= x).

where $x = r \sin \theta = r \sin \omega t$. . . . (6.14(a))

Let point B moves from B_t to $B_{t+\delta t}$ in time t at an angular distance of

$\theta + \delta\theta = \omega\, (t + \delta t)$.

Then project $dx = x + dx - x$

$$= r \sin \omega\, (t + \delta t) - r \sin \omega t$$

$$= r\, [(\sin\ \omega t\ \cos\ \omega \delta t\ + \cos\ \omega t\ \sin\ \omega \delta t\,) - \sin\ \omega t\,)]$$

if δt is chosen as a small interval of time then $\sin\ \omega \delta t = \omega \delta t$ and $\cos\ \omega \delta t = 1$. This makes $dx = r\ [\sin\ \omega t + \omega \delta t\ \cos\ \omega t\ - \sin\ \omega t\,) = r \omega\ \delta t\ \cos\ \omega t$

Hence, velocity of projection, $v_x = \dfrac{\delta x}{\delta t} = \omega r\ \cos \omega t$ . . . (6.14(b))

$$= \omega_1\ AB' = (r^2 - x^2)\, \omega \qquad \text{. . . (6.14(c))}$$

working similarly.

Acceleration projection $\alpha_x = \dfrac{\delta v_x}{\delta t} = -\, \omega^2\ r \sin \omega t$ . . . (6.15(a))

Negative sign indicates it is retardation when x is positive,

hence $\alpha_x = -\, \omega^2\ r\ \sin\ \omega t = -\, \omega^2\ x$ . . . (6.15(b))

or $$\frac{\alpha_x}{x} = -\, \omega^2 = (2\pi\, f\,)^2\ (as\ \omega = 2\pi\, f\,)$$

or $$f = \frac{1}{2\pi} \sqrt{\frac{\alpha_x}{x}} \qquad \text{. . . (6.15(c))}$$

and time period $$T = \frac{1}{f} = 2\pi \sqrt{\frac{x}{\alpha_x}} \qquad \text{. . . (6.15(d))}$$

6.4.3 Graphical Representation of SHM

The algebraic expressions developed for displacement (x), velocity, v_x, and acceleration, α_x, as developed above can be graphically represented as follows (Figure 6.9).

$$x = r\ \sin\ \omega t$$

$$x_{\max} = r\ \text{at}\ \omega t = \frac{\pi}{2}\ \text{ and }\ = -\, r\ \text{at}\ \omega t = \frac{3\pi}{2}$$

$$x_{\min} = o\ \text{at}\ \omega t = 0,\ \pi\ \text{ and }\ 2\pi$$

$$V_x = \omega r\ \cos\ \omega t$$

$$v_{x\max} = \omega r \text{ at } \omega t = 0 \text{ and } 2\pi$$

$$= -\omega r \text{ at } \omega t = \pi$$

$$v_{x\min} = 0 \text{ at } \omega t = \frac{\pi}{2} \text{ and } \frac{3\pi}{2}$$

$$\alpha_x = -\omega^2 r \sin \omega t$$

$$\alpha_{x\max} = -\omega^2 r \text{ at } \omega t = \frac{\pi}{2} \text{ and } = +\omega^2 r \text{ at } \omega t = \frac{3\pi}{2}$$

$$\alpha_x = 0 \text{ at } \omega t = 0,\ \pi \text{ and } 2\pi$$

then $$T = \frac{2\pi}{\omega}$$

Figure 6.9

Example 6.7

The piston of an engine with stroke of 1.0 m is assumed to be executing a simple harmonic motion. The crank rotates at 600 rpm. Find velocity and acceleration of the piston when it is at a distance of 0.3 m from its centre.

Solution

Angular velocity of crank $= \omega = \dfrac{2\pi \times 600}{60} = 20\pi$

Radius of crank circle $= \dfrac{\text{Stroke}}{2} = 0.5$ m

Then velocity at a distance of 0.3 m from centre would be

$$v = \omega \sqrt{(r^2 - x^2)} = 20\pi \sqrt{0.5^2 - 0.3^2} = 8\pi \text{ m/s} \qquad \text{. . . (a)}$$

$$\alpha_x = -\omega^2 x = -(20\pi)^2 \times 0.3 = -120\pi^2 \text{ m/s}^2 \qquad \text{. . . (b)}$$

SAQ 5

A pump plunger weighing 300 N is driven by a crank of uniform velocity of 150 rpm. The crank length is 300 mm. Find

(a) velocity and acceleration of plunger when it is at mid point.

(b) maximum velocity and maximum acceleration of plunger and the max accelerating force.

(c) the periodic time.

Assume $g = 10$ m/sec^2.

6.5 WORK, POWER AND ENERGY

6.5.1 Work

According to Newton's second law of motion, a force when applied on a body displaces or tends to displace the body from its position of rest or of uniform motion. However, this force can be considered to have performed some work only and only when it moves the body. If the body does not get displaced no work is assumed to be performed however large the magnitude of the force might be. In other words, the work is performed only when the force accomplishes some displacement of body in the direction of application of the force.

Work performed is denoted by the product of the magnitude of force (= P) and the displacement in the direction of the force (d) (i.e., $W = P \times d$).

Let a force of magnitude, P, is acting on a body is resting on a frictionless surface (Figure 6.10) to cause its displacement (x) from position A to A_1 and let the force be inclined at an angle α with the direction of displacement force P is resolved in to components $p \cos \alpha$ along the direction of movement and $p \sin \alpha$ normal to it.

Figure 6.10

Then work done by force P in moving the body will be sum of the work done by its components along their respective directions. Thus

$$W = P \,.\, d = P \cos \alpha \times x + P \sin x \,.\, 0 = P\, x \cos \alpha \qquad \ldots (6.16)$$

It may be noted that only the components of the force system in the direction of displacement does perform the work.

Since $W = P.\, d$, the unit of work done will be unit of force unit of distance.

In SI units, the unit of force is N while that of distance is meters (m) hence unit of work would be Nm. It is defined as the amount of work done by the force of magnitude 1 N in moving a body by a distance of 1m in its direction. This is also called Joule. Thus

$$1 \text{ Joule} = 1 \text{ Nm}$$

Gradually Applied Load and Sustained Load

Difference should be made between a sustained load and a gradually applied loads in computing the work done by it. Let us take the example of a beam AB (Figure 6.11(a)) supported at ends AB and assume a load is applied gradually at point 1 to produce a displacement of δ_{11} at point 1. When the load is starting to be applied at 1 the displacement at point 1 is zero. As the magnitude is gradually increased from zero to full value P_1 is displacement of 1 gradually becomes δ_{11}, while that at another point 2 will be δ_{21}. The average load applied during this process is $P_{av} \frac{0 + P_1}{2} = \frac{P_1}{2}$ (Figure 6.11(b)) and work done by this force will be $\frac{P_1}{2} \delta_{11}$ while at point 2 it will be zero (i.e. O $\times \delta_{21}$). Now, let us gradually apply a load at point 2 developing from zero to P_2. The displacements produced would be δ_{12} at 1 and δ_{22} at point 2. If the load P_1 at 1 is continued to be sustained on beam. The work done will be $P_1 \times \delta_{21}$ at 1 and $\frac{p_2}{2} \times \delta_{22}$ at 2 producing a total work $W = p_1 \delta_{21} + \frac{P_2}{2} \delta_{22}$.

This concept can also be explained with the help of graphic representation of work as shown in Figures 6.11(a) and (b). When the load is suddenly applied, the average load is not $\frac{p}{2}$ but full value P and the work done will be $P\delta$ (Figure 6.11(c)).

(a) **(b)**

(c)

Figure 6.11

6.6.2 Power

The rate of doing work by a force is defined as power. If a force of magnitude P causes a displacement in a body d in t units of time. The work done would be W, where $W = P \times d$ and Power would be

$$\frac{W}{t} = \frac{P \times d}{t} = P \times v \qquad \text{. . . (6.17)}$$

Hence, Power can also be defined as force × velocity (since $\frac{d}{t} = v$). The power is measured in S1 units as watts where one watt is the work done by 1 N force is causing a displacement of 1m in one sec (1 N m/s). In practical cases, this unit of power (watt) is too small hence the unit employed is kilo watt where 1 kilo watt

(1 kW = 1000 watts). In Engineering systems, normally we use MKS system, i.e. kgf, meter and sec units. Hence the unit of power would be the work done per second by a force of magnitude 1 kgf in causing a displacement of 1 m. The engineering unit of power generally would be horse power (hp) where
1 hp = 75 m kg f/s or 4500 m kg f/min.

In many engineering problems, we use mechanical appliance like engine or motors etc. Hence we need to define power more explicitly. Two terms, e.g. indicated power (IP) and brake power (BP) are frequently used.

Indicated power is IP or IHP (if power unit used is horse power) is the power developed in the cylinder of the engine or input power of a motor. This represents the rate at which work can be performed at piston or at which power is externally supplied to a motor.

Brake power, i.e. BP or BHP (if unit is horse power) is the power supplied by the engine at pulley or shaft of the engine. It is termed brake power as it is measured by means of a brake drum.

It can be easily observed that BP will always be less than IP as some work will always be lost due to frictional forces in the mechanical system. Hence the mechanical efficiency of a machine can be defined as

$$\text{Efficiency, } \eta = \frac{BP}{IP} \times 100 \text{ percent}$$

which will always be less than 100%.

The effort lost due to these frictional forces can be said to be work lost in friction or friction power (FP) and is representative of the efficiency of the mechanical system. Efforts are made to minimize the frictional losses of power to improve the efficiency of the system.

6.6.3 Energy

When an actual effort is applied in causing a displacement in a mechanical system, work is considered to be performed while the rate of doing this work is termed as power. The mechanical system under consideration may be capable of doing more work than actually performed in a particular case. The capacity of work performance, i.e. the maximum work which can be performed by the system is defined as the energy of the system.

There can be several forms of energy viz. mechanical, electrical, thermal, chemical or nuclear etc. The form of energy can be transformed from one form to another. It can be observed that energy can neither be created nor destroyed. It can only change its form. This concept is known as law of conservation of energy. In formal form, this law can be stated as follows.

The total amount of energy of the universe (or any system) is constant. It can neither be created nor destroyed although it changes its form.

In mechanics, the form of energy with which we are most concerned is mechanical energy. It can exist in a body either by virtue of its position or configuration (termed Potential energy (PE) or by virtue of its motion (termed kinetic energy) KE).

To explain the concept of potential energy and kinetic energy, let us consider a body of mass m raised through a height "h" above ground level, the work done on the body will be force × distance = $mg \times h$. This will be stored in the body in the form of potential energy. Thus the Potential Energy of the body due to its position at h above the ground is

$$PE = mgh \qquad \text{. . . (6.17)}$$

Now, let the body to fall to ground under its own weight. At time $t = 0$, the initial velocity of body $u = 0$ and is moving with a constant acceleration α, where $\alpha = g$ i.e. acceleration due to gravity. Let it attain a velocity, v, when it touches the ground, i.e. it travels through a distance h, then

$$v^2 = u^2 + 2gh \text{ or } v = \sqrt{2gh} \qquad \text{. . . (i)}$$

or $$gh = \frac{v^2}{2} \qquad \text{. . . (ii)}$$

The work done by the body during this process will be force × distance

i.e. $$= mgh \qquad \text{. . . (iii)}$$

Substituting gh from Eq. (ii) into Eq. (i), we get

Work done by body virtue of its motion, i.e. the kinetic energy is KE

where $$KE = \frac{1}{2}mv^2 \qquad \text{. . .}$$

(iv)

By principle of conservation of energy

$$KE = PE \text{ or } mgh = \frac{1}{2}mv^2$$

Next let us consider any position of the body (say c) referred above as shown in Figure 6.12 and consider the energy

At A, $$y = 0, v = 0, PE = mgh, KE = 0$$

$$TE = mgh + 0 = mgh \qquad \text{. . .}$$

(i)

At C, $$y = y, v^2 = 0 + 2\,gy, PE = mg\,(h - y) \qquad \text{. . . (a)}$$

$$KE = \frac{1}{2}m \times v^2 = \frac{m}{2} \times 2gy = mgy \qquad \text{. . . (b)}$$

$$TE = PE + KE = mg\,(h - y) + mgy = mgh \qquad \text{. . . (ii)}$$

At B, $$y = h, v^2 = 2\,gh, PE = mg\,(h - h) = 0$$

$$KE = \frac{1}{2}mv^2 = \frac{m}{2} \times 2gh = mgh \qquad \text{. . . (iii)}$$

Figure 6.12

It can be clearly seen that total energy of the body at any position A, B or C is same (Eqs. (i), (ii) or (iii) = *mgh*). Verifying the principle of conservation of energy which states that total energy remains constant while the form of energy changes from potential energy at A to part potential energy and part kinetic energy at C and ultimately to only kinetic energy at B.

Example 6.6

To stretch a spring by 30 mm certain force is applied to it. Calculate the work done, if spring constant i.e. force required to stretch the spring by 1 mm is 60 N.

Solution

Force required to cause 1 mm displacement is 60 N. Maximum force required for 30 mm = 60 × 30 = 1800 N. Force is gradually applied, i.e. it grows from 0 to 1800 N.

$$\text{Average force, } P_{av} = \frac{0 + 1800}{2} = 900 \text{ N}$$

Hence work done = 900 × 30 = 27,000 Nmm = 27 Nm = 27 J.

Example 6.7

A train with total weight of 1000 tonne is resting on an inclined track of 1 in 100 with tractive resistance of 5 N per kN. The train is pulled downwards by a locomotive with a constant pull of 5 tonnes. Assuming g = 10 m/sec^2, calculate the power developed by the locomotive, after it has travelled a distance of 1 km (Figure 6.13).

Figure 6.13

Distance travelled = s = 1000 m

Initial velocity = 0, final velocity = v (say) : acceleration = a

Then equation of dynamic equilibrium is

$$P + T - F = \frac{W}{g} a \qquad \left\{ \tan\alpha = \sin\alpha = \frac{1}{100} \text{ when } \alpha \text{ is small} \right\}$$

or $$50000 + 100000 \frac{1}{100} - 5 \times 1000{,}0 = \frac{10000000}{10} \times a$$

or $$a = \frac{\left\{ 50000 + \frac{100000}{100} - 50000 \right\}}{100000} = 0.1$$

also $$v^2 = u^2 + 2ax = 0 + 2 \times 0.1 \times 1000 = 200$$

or $v = 14.14$ m/s

Power developed = Work done/sec = Force $\times$ Speed

= 50000 $\times$ 14.14 watts

or = 707000 watts

or = 707 k watts.

Example 6.8

During the impact test on metal specimen, in the strength of material laboratory, the hammer head weighing 100 N is arranged to swing in a circular path and is released at point 1.6 m higher than the specimen fixture. At impact, after breaking the specimen hammer rises to 0.6 m height above specimen fixture. Calculate the speed of hammer and kinetic energy just before the impact and the energy spent in breaking the specimen, assume g =10 m/sec^2.

Solution

Potential energy of hammer head above specimen = 100 $\times$ 1.6

= 160 Nm (or J) . . . (i)

Potential energy of hammer remaining after breaking the specimen = 100 $\times$ 0.6 = 60 Nm (or J) . . . (ii)

Energy observed during breaking = 160 – 60 = 100 Nm or (J) . . . (iii)

The potential energy at the level of specimen (lowest point on the one of swing) = 0. Since total energy of the body must remain same, kinetic energy of body at this point = 160 Nm (from Eq. (ii)), i.e. maximum potential energy. Initial velocity of hammer u = 0, then from $v^2 = u^2 + 2\,gh$

or $v^2 = 0 + 2 \times 10 \times 1.6 = 32$ or $v = \sqrt{32} = 5.66$ m/s

Kinetic energy at this point = $KE = \frac{1}{2} mv^2 = \frac{1}{2} \times \frac{100}{10} \times 32 =$ 160 Nm same as (i).

SAQ 6

In railway yard, spring bumper with stiffness, K, is provided at the end of a side track of track resistance is N per kN. The bumper spring has the total compressibility of 0.5 m. The stiffness of the spring is to be so designed that the wagon weighing 50 kN travelling at a speed of 4 m/s dawn a slope of 50, at a distance of 40 m from the junction A (Figure 6.14) and from A m over on a horizontal track for 100 m before striking the bumper, find the spring stiffness K and the roll back distance of the wagon from the point of maximum compression. Assume g = 10 m/sec^2.

Figure 6.14

SAQ 7

A wagon weighing 50 kN is moving with a velocity of 36 km/hr on a level track with negligible track resistance. At the end of track A, main bumper shield with spring constant of 2 kN/mm is provided. Two auxiliary bumper shields each of spring constant 1 kN/mm are provided 200 mm before main shield. Determine the maximum compression in main bumper. Also how much share of energy is transmitted to auxiliary springs?
Assume $g = 10\ \text{m/sec}^2$.

Figure 6.15

6.7 SUMMARY

Motion is a common phenomenon in nature but by mechanics point of view, it is an extremely important phenomenon. In this unit, we have understood the cause of motion and found that a body at rest may be set into motion, on being acted upon by external forces. We have also discussed motion of a particle on a circular path as well as in simple harmonic motion.

In this unit, we learnt that whenever a constant force, F, acts on an object while it experiences a displacement, d, we say that the force does work W on the object. The amount of work done is a scalar quantity, and is calculated from $W = \boldsymbol{F} . \boldsymbol{d}$. The concepts of work, power and energy have been explained in detail in this unit along with different forms of energy.

6.8 ANSWERS TO SAQs

SAQ 1

Cannon Mass (M) = 20000 kg, Shell mass (m) = 100 kg.

Its velocity is = 800 m/s,

Let recoil velocity be V then $MV + mv = 0$ by principle of conservation of momentum

or $$V = -\frac{100 \times 800}{20000} = -4.0 \text{ m/s.}$$

(Direction of cannon velocity opposite to that of the shell.)

Final velocity of cannon (v) = 0, initial velocity u = 4.0 m/s.

Let a = acceleration of cannon,

t = time taken to stop cannon in its recoil distance of s (s = 0.4 m).

Then $v^2 = u^2 + 2\,as$

or $$a = \frac{1}{2 \times 0.4} \times \{0 - 4.0^2\} = -20 \text{ m/sec}^2 \text{ (retardation)}$$

Then retarding force P required = $M \times a = 20000 \times 20 = 4 \times 10^5$ N

Also $v = u + at$

$O = 4.0 - 20 \times t$

or $t = 0.2$ sec

SAQ 2

Mass of cannon (M) = 1000 kg; bullet (m) = 1 kg, its velocity (v) = 300 m/s.

Spring constant K = 12 N/mm = 12000 N/m.

By principle of conservation of momentum

$$MV + mv = 0$$

or $1000 \times V = -1 \times 300$

or $V = -0.3$ m/sec

The final velocity of cannon = 0 after it presses buffer.

Let s = spring displacement, then

$$v^2 = u^2 + 2\,as$$

$$0 = 0.3^2 + 2\,as \qquad \ldots \text{(a)}$$

Also by Newton's second law

$$P = m \times a \qquad \text{and} \qquad K = \frac{P}{s} \qquad \text{or} \qquad P = 12000\,s$$

Also $P = 1000 \times a$

Then $1000\,a = 12000\,s$

or $$s = \frac{a}{12} \qquad \ldots \text{(b)}$$

Substituting Eq. (b) in Eq. (a)

$$0.09 = -2 \times a \times \frac{a}{12} \quad \text{or} \quad a^2 = 0.54 \quad \text{or} \quad a = 0.735 \text{ m/sec}^2 \text{ (retardation)}$$

Substituting Eq. (a) from Eq. (c) into Eq. (b)

$$s = \frac{0.735}{12} = 0.0613 \text{ m} \quad \text{or} \quad 61.3 \text{ mm} \qquad \text{. . . (d)}$$

SAQ 3

Initial angular velocity = $\omega_0 = \frac{2\pi \times 1200}{60} = 40\pi$

Final angular velocity = $\omega_1 = \frac{2\pi \times 1800}{60} = 60\pi$

The free end of bar describes a circle of radius 1.6 m.

Initial linear velocity of $B = \omega_0 \, r = 40\,\pi \times 1.6 = 64\,\pi$ . . . (a)

Final linear velocity of B after 10 seconds = $\omega_1 r = 60\,\pi \times 1.6 = 96\,\pi$ . . . (b)

We have $\omega_1 = \omega_0 + \pi\, t$

or $60\,\pi = 40\,\pi + \alpha \times 10$ or $\alpha = 2\,\pi$ rad/sec^2

Angular velocity after 8 seconds = $\omega_1 = \omega_1 = 40\,\pi + 8 \times 2\,\pi = 56\,\pi$

The mid-point of bar describes a circle of 0.8 m radius.

$$\alpha_n = \text{normal acceleration} = -\,\omega^2 r = (56\,\pi)^2 \times 0.8$$

$$= -\,2508.8\,\pi^2 \text{ m/sec}^2$$

$$\alpha_t = \text{tangential acceleration}\, \alpha_r = 2\,\pi \times 0.8 = 1.6\,\pi \text{ m/sec}^2$$

SAQ 5

Angular speed of crank = $\omega = \frac{2\pi \times 150}{60} = 5\pi$

Periodic time $= T = \frac{2\pi}{\omega} = \frac{2\pi}{5\pi} = 0.40$ . . . (a)

Displacement of plunger = 0.15 m from its mean position

Velocity $V_x = \omega \sqrt{(r^2 - x^2)} = 5\pi \sqrt{(0.3^2 - 0.15^2)} = 1.3\pi \text{ m/s}$ . . . (b)

$$\alpha_x = \omega^2 x = (5\pi)^2 \times 0.15 = 3.75\pi^2 \qquad \text{. . . (c)}$$

$$\alpha_{\max} = (5\pi)^2 \times 0.3 = 7.5\pi^2 \qquad \text{. . . (d)}$$

$$\text{Force}_{(\max)} \frac{W}{g} \alpha_{\max} = \frac{300}{10} \times 7.5\pi^2 = 225\pi^2 \text{ N} \qquad \text{. . . (e)}$$

SAQ 6

Weight of wagon = 50 kN = 50000 N

Track resistance = 8 × 50 = 400 N

Initial velocity of wagon = 4 m/s

Initial height of wagon above $A = 40 \times \frac{1}{50} = 0.8$ m

Total distance travelled by wagon before coming to stop by
bumper = 40 + 100 + 0.6 = 56240 Nm

Work done in spring compression : If k is spring constant and δ compression then force in spring is $k\,\delta$ since work = force $\times$distance.

Initial KE of Wagon = $KE = KE = \frac{1}{2} mv^2 = \frac{50000}{2 \times 10} \times 4^2 = 40000$ Nm.

By principle of conservation of energy.

Initial KE of wagon + Work done by gravity = Work against friction
+ Work done m spring compression

i.e. $40000 + 50000 \times 0.8 = 56240 + 0.18\,K$

or $K = 23760/0.18 = 132000$ N/m

132 N/mm or 0.132 kN/mm.

The energy transferred from wagon to bumper during impact is released by pushing the wagon back by a distance of "s" m. Thus

$$\text{Energy stored in spring} = \frac{1}{2} \times 132000 \times 0.6^2 = 400 \times s$$

or $$S = \frac{23760}{400} = 59.4 \text{ m}$$

SAQ 7

Refer Figure 6.14.

Weight of wagon = 50 kN

Initial velocity = 36 km/hrs

$$\frac{36000}{60 \times 60} = 10 \text{ m/s}$$

$$\text{Initial KE of the wagon } = \frac{1}{2} mv^2 = \frac{1}{2} \times \frac{50000}{10} \times 10^2$$

Km = 2 kN/mm = 2000 kN/mm = 250,000 Nm = 250 kNm

KA = 1 kN/mm = 1000 kN/m

Let the total compression in auxiliary spring be δ, then total compression of main spring will be $0.2 + \delta$.

$$\text{Energy absorbed by main spring} = \frac{1}{2} k_m (0.2 + \delta)^2$$

$$= \frac{1}{2} \times 2000 \times (0.2 + \delta)^2 = 1000(0.2 + \delta)^2 \quad \ldots$$

(i)

$$\text{Energy absorbed by auxiliary spring} = 2 \times \frac{1}{2} K_A\, \delta^2 = 1000\, \delta^2$$

Law of conservation of energy gets $250 = 1000\,(0.2 + \delta)^2$

or $$0.04 + 0.4\,\delta + 2\delta^2 = 0.25$$

or $$\delta^2 + 0.2\delta + (0.02 - 0.125)$$

or $$\delta^2 + 0.2\delta - 0.105 = 0$$

Hence $$\delta = \frac{-0.2 \pm \sqrt{0.04 + 0.42}}{2} = \frac{-0.2 \pm 0.68}{2}$$

Omitting – ve sign as unfeasible

$$\delta = 0.24 \text{ m or } 24 \text{ mm} \qquad \ldots \text{(iii)}$$

Total deformation of main spring = 0.2 + 0.24 = 0.44 m

or = 440 mm . . .
(iv)

Energy absorbed by main bumper = $\frac{1}{2} \times 2000 \times (0.44)^2 = 193.6$ kNm . . . (v)

Hence % of energy absorbed by main spring $= \frac{193.6}{250} = 77.2\%$

or % energy absorbed by auxiliary spring = 22.8 %.

FURTHER READING

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