UNIT 5 COMPLEX NUMBERS

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5.1 INTRODUCTION

The concept of imaginary numbers has its historical origin in the fact that the solution of the quadratic equation $ax^2 + bx + c = 0$ leads to an expression $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ which is not found meaningfull when $b^2 - 4ac < 0$. This is because of the fact that the square of a real number is never negative. So it created the need of the extension of the system of real numbers. Euler was the first mathematician who introduced the symbol *i* far $\sqrt{-1}$ with the properties $i^2 = -1$ and accordingly a root of the equation $x^2 + 1 = 0$. Also symbol of the form $a + ib$ where a and b are real numbers was called a complex number.

In this unit, we will study the system of complex numbers, the algebraic operations on complex numbers and the fundamental laws of these operations.

Objectives

After studying this unit, you should be able to

- identify a complex number,
- determine its complex conjugate, modulus and argument,
- describe the basic properties of complex numbers, its modulus and argument, and
- explain the De Moivres theorem and give some useful properties of the theorem.

5.2 COMPLEX NUMBERS

A number of the form $a + ib$, where a and b are real numbers and $i = \sqrt{-1}$ is called a complex number. The real number a is called the real part and the real number b is called the imaginary part of the complex number $a + ib$.

Let $a + ib$ be denoted by z, i.e. $z = a + ib$.

If $a = 0$, then $z = ib$ and z is said to be purely imaginary. If $b = 0$, then $z = a$ and z is said to be real.

$a - ib$ is said to be the conjugate of $z = a + ib$ and is denoted by $\bar{z}$.

5.2.1 Algebra of Complex Numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers. Then z_1 is said to be equal to z_2 if and only if $a = c$ and $b = d$.

Addition

$z_1 + z_2 = (a + ib) + (c + id)$ is defined as $(a + c) + i\,(b + d)$.

Subtraction

$z_1 - z_2 = (a + ib) - (c + id)$ is defined as $(a - c) + i\,(b - d)$.

Multiplication

$z_1 z_2 = (a + ib)\,(c + id)$ is defined as $(ac - bd) + i\,(bc + ad)$.

Division

Let $z_2 \neq 0$, i.e. $c \neq 0$ and $d \neq 0$.

We will prove that there exists a complex number $z = x + iy$ such that

$$z_1 = z \cdot z_2$$

This z is called the quotient of z_1 and z_2 and is denoted by $\dfrac{z_1}{z_2}$.

$$z_1 = z \cdot z_2$$

$$\Rightarrow \quad (a + ib) = (x + iy)\,(c + id)$$

$$= (cx - dy) + i\,(dx + cy)$$

$$\therefore \quad a = cx - dy, \quad b = dx + cy$$

Solving for x and y, we have

$$x = \frac{ac + bd}{c^2 + d^2}, \; y = \frac{bc - ad}{c^2 + d^2}$$

$$\therefore \quad \frac{z_1}{z_2} = \frac{ac + bd}{c^2 + d^2} + i\,\frac{bc - ad}{c^2 + d^2}$$

5.2.2 Complex Numbers Defined as Ordered Pair of Real Numbers

A complex number $z = a + ib$ is represented by an ordered pair (a, b) of real numbers and the set of all complex numbers is represented by C.

i.e. $\quad C = \{z : z = (a, b)\ \forall\ a, b \in R\}$

$\therefore$ Two complex numbers (a, b) and (c, d) are said to be equal iff $a = c$ and $b = d$.

The complex number (a, b) is said to be zero complex number iff $a = 0$ and $b = 0$.

The complex number $(-a, -b)$ is called the negative of the complex number (a, b) and vice-versa. We denote $(-a, -b)$ by $-(a, b)$ and $(a, -b)$ is the complex conjugate of (a, b).

Hence $\quad (a, b) + (c, d) = (a + c, b + d)$

$$(a, b)\quad(c, d) = (ac - bd, ad + bc)$$

$$(a, b) - (c, d) = (a, b) + (-c, -d)$$

$$= (a - c, b - d)$$

and $\quad \dfrac{(a, b)}{(c, d)} = \left(\dfrac{ac + bd}{c^2 + d^2}, \dfrac{bc - ad}{c^2 + d^2}\right)$

5.2.3 Basic Properties of Complex Numbers

If z_1, z_2, z_3 are three complex numbers, then it can be proved that

(i) $z_1 + z_2 = z_2 + z_1$

(ii) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(iii) $z_1\, z_2 = z_2\, z_1$

(iv) $(z_1\, z_2)\, z_3 = z_1\, (z_2\, z_3)$

(v) $z_1\, (z_2 + z_3) = z_1\, z_2 + z_1\, z_3$

(vi) $(z_1 + z_2)\, z_3 = z_1\, z_3 + z_2\, z_3$

(vii) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(viii) $\overline{z_1\, z_2} = \overline{z_1} \cdot \overline{z_2}$

(ix) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$

We will prove (viii).

$$z_1\, z_2 = (a + ib)\,(c + id)$$

$$= (ac - bd) + i\,(ad + bc)$$

$\therefore$
$$\overline{z_1\, z_2} = (ac - bd) - i\,(ad + bc)$$

$$= (a - ib)\,(c - id)$$

$$= \bar{z}_1\, \bar{z}_2$$

The other properties can be proved.

Example 5.1

Express $\dfrac{(1 + i)\,(2 + i)}{3 + i}$ in the form $a + ib$.

Solution

$$\frac{(1 + i)\,(2 + i)}{3 + i} = \frac{2 + i + 2i - 1}{3 + i}$$

$$= \frac{1 + 3i}{3 + i}$$

$$= \frac{(1 + 3i)(3 - i)}{(3 + i)(3 - i)}$$

$$= \frac{3 - i + 9i + 3}{9 + 1}$$

$$= \frac{6 + 8i}{10}$$

$$= \frac{3}{5} + \frac{4}{5} i$$

Example 5.2

Express $\dfrac{(6 + i)(2 - i)}{(4 + 3i)(1 - 2i)}$ in the form $a + ib$.

Solution

$$\frac{(6 + i)(2 - i)}{(4 + 3i)(1 - 2i)} = \frac{12 + 1 + i(2 - 6)}{4 + 6 + i(3 - 8)} = \frac{13 - 4i}{10 - 5i}$$

$$= \frac{(13 - 4i)(10 + 5i)}{(10 - 5i)(10 + 5i)} = \frac{150 + 25i}{100 + 25}$$

$$= \frac{6 + i}{5} = \frac{6}{5} + \frac{1}{5} i .$$

5.3 GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS

5.3.1 Argand Diagram

Mathematician Argand represented a complex number in a diagram known as Argand diagram. A complex number $x + iy$ can be represented by a point P whose coordinate are (x, y). The axis of x is called the real axis and the axis of y the imaginary axis. The distance OP is the modulus and the angle, OP makes with the x-axis, is the argument of $x + iy$.

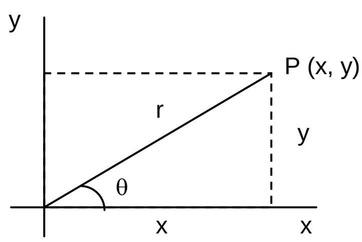

Figure 5.1

5.3.2 Modulus and Argument

Let $x + iy$ be a complex number.

Putting $x = r \cos \theta$ and $y = r \sin \theta$ so that $r = \sqrt{x^2 + y^2}$.

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}} \qquad \text{and} \qquad \sin\theta = \frac{y}{\sqrt{x^2 + y^2}}$$

the positive value of the root being taken.

Then r is called the *modulus* or absolute value of the complex number $x + iy$ and is denoted by $| x + iy |$.

The angle θ is called the argument or amplitude of the complex number $x + iy$ and is denoted by argument $(x + iy)$ and $\theta = \tan^{-1} \frac{y}{x}$.

It is clear that θ will have infinite number of values differing by multiples of 2π. The values of θ lying in the range $-\pi < \theta \le \pi$ is called the *principal value* of the argument.

Note : $r^2 = x^2 + y^2 = (x + iy)\,(x - iy) = z\,\bar{z}$.

The complex number in polar form is $z = r(\cos\theta + i\sin\theta)$.

Example 5.3

Find the modulus and principal argument of the complex number $\frac{1 + 2i}{1 - (1 - i)^2}$.

Solution

$$\frac{1 + 2i}{1 - (1 - i)^2} = \frac{1 + 2i}{1 - (1 - 1 - 2i)}$$

$$= \frac{1 + 2i}{1 + 2i} = 1$$

$$= 1 + 0i$$

$$\left| \frac{1 + 2i}{1 - (1 - i)^2} \right| = 1$$

Principal argument of $\frac{1 + 2i}{1 - (1 - i)^2}$

$$= \tan^{-1}\frac{0}{1} = \tan^{-1} 0 = 0^{\circ}$$

Example 5.4

Express $\frac{1 + 2i}{1 - 3i}$ in the form $r(\cos\theta + i\sin\theta)$.

Solution

$$\frac{1 + 2i}{1 - 3i} = \frac{(1 + 2i)\,(1 + 3i)}{(1 - 3i)\,(1 + 3i)}$$

$$= \frac{1 - 6 + 5i}{1 + 9}$$

$$= \frac{-5+5i}{10}$$

$$= -\frac{1}{2} + \frac{i}{2}$$

$$-\frac{1}{2} + \frac{i}{2} = r\,(\cos\theta + i\sin\theta) \qquad \ldots (5.1)$$

$$\therefore \qquad r\cos\theta = -\frac{1}{2} \qquad \ldots (5.2)$$

$$r\sin\theta = \frac{1}{2} \qquad \ldots (5.3)$$

Squaring Eqs. (5.2) and (5.3) and then adding, we get

$$r^2(\cos^2\theta + \sin^2\theta) = \frac{1}{4} + \frac{1}{4}$$

$$r^2 = \frac{1}{2} \text{ or } r = \frac{1}{\sqrt{2}}$$

Putting the value of r in Eqs. (5.2) and (5.3), we have

$$\frac{1}{\sqrt{2}}\cos\theta = -\frac{1}{2} \text{ or } \cos\theta = -\frac{1}{\sqrt{2}} \qquad \ldots (5.4)$$

$$\frac{1}{\sqrt{2}}\sin\theta = \frac{1}{2} \text{ or } \sin\theta = \frac{1}{\sqrt{2}} \qquad \ldots (5.5)$$

From Eqs. (5.4) and (5.5), we have

$$\theta = \frac{3\pi}{4}$$

Putting the values of r and θ in Eq. (5.1), we get

$$\frac{1+2i}{1-3i} = \frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right).$$

5.3.3 Properties of Modulus and Argument

Theorem 1

If z_1 and z_2 are two complex numbers, then

(i) $|z_1 z_2| = |z_1|\,|z_2|$

(ii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, **provided $z_2 \neq 0$**

(iii) $|z_1 + z_2| \leq |z_1| + |z_2|$

(iv) $|z_1 + z_2| \geq |z_1| - |z_2|$

(v) $|z_1 - z_2| \geq |z_1| - |z_2|$

Proof

(i) $|z_1\, z_2|^2 = (z_1\, z_2)\,\overline{(z_1\, z_2)}$

$$= (z_1\ z_2)\ (\bar{z}_1\ \bar{z}_2)$$

$$= (z_1\ \bar{z}_1)\ (z_2\ \bar{z}_2)$$

$$= |z_1|^2\ |z_2|^2$$

Since the modulus of a complex number is always non-negative,

$\therefore \qquad |z_1\ z_2| = |z_1|\,|z_2|$

(ii) Can be similarly proved.

(iii) $|z_1 + z_2|^2 = (z_1 + z_2)\ \overline{(z_1 + z_2)}$

$$= (z_1 + z_2)\ (\bar{z}_1 + \bar{z}_2)$$

$$= z_1\,\bar{z}_1 + z_2\,\bar{z}_2 + z_1\,\bar{z}_2 + z_2\,\bar{z}_1$$

$$= |z_1|^2 + |\,z_2|^2 + z_1\,\bar{z}_2 + z_2\,\bar{z}_1$$

$$= |z_1|^2 + |\,z_2|^2 + z_1\,\bar{z}_2 + \overline{(z_1\,\bar{z}_2)}$$

$$= |z_1|^2 + |\,z_2|^2 + 2 \text{ real part of } (z_1\,\bar{z}_2)$$

$$\le |z_1|^2 + |\,z_2|^2 + 2\,|z_1\,\bar{z}_2|$$

($\because$ Real part of $z_1\,z_2 \le |z_1\,z_2|$)

$|z_1|^2 + |\,z_2|^2 + 2\,|z_1|\,|\bar{z}_2| = |z_1|^2 + |\,z_2|^2 + 2\,|z_1|\,|\bar{z}_2|$ ($\because |\bar{z}_2| = |\,z_2|$)

$$= [|z_1| + |\,z_2|]^2$$

$\therefore \qquad |z_1 + z_2| \le |\,z_1| + |z_2|$

(iv) $|z_1 + z_2|^2 = (z_1 + z_2)\ \overline{(z_1 + z_2)}$

$$= (z_1 + z_2)\ (\bar{z}_1 + \bar{z}_2)$$

$$= z_1\,\bar{z}_1 + z_2\,\bar{z}_2 + z_1\,\bar{z}_2 + z_2\,\bar{z}_1$$

$$= |z_1|^2 + |\,z_2|^2 + 2\ \text{Re}\,|z_1\,\bar{z}_2|$$

$$\ge |z_1|^2 + |\,z_2|^2 - 2\,|z_1\,z_2|$$

$$\ge [|z_1| - |\,z_2|]^2$$

$\therefore \qquad |z_1 + z_2| \ge |\,z_1| - |z_2|$

(v) $|z_1| = |z_1 - z_2 + z_2\,|$

$\le |z_1 - z_2| + |z_2|$ by (iii)

$\therefore \qquad |z_1| - |\,z_2| \le |z_1 - z_2\,|$

i.e. $\qquad |z_1 - z_2| \ge |\,z_1| - |z_2|$

Theorem 2

Prove that

(i) The argument of the product of two complex numbers is the sum of their arguments.

(ii) The argument of the quotient of two complex numbers is the difference of their arguments.

Proof

(i) Let $z_1 = x_1 + i\,y_1 = r_1\,(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = x_2 + i\,y_2 = r_2\,(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers.

Then $|z_1| = r_1,\ |z_2| = r_2$,

and argument $z_1 = \theta_1$, argument $z_2 = \theta_2$.

Now
$$z_1 z_2 = [r_1\,(\cos\theta_1 + i\sin\theta_1)]\,[r_2\,(\cos\theta_2 + i\sin\theta_2)]$$
$$= r_1 r_2\,(\cos\theta_1 + i\sin\theta_1)\,(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2\,[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$\therefore$ Argument $z_1 z_2 = \theta_1 + \theta_2 =$ argument z_1 + argument z_2.

Cor.

$|z_1 z_2| = r_1 r_2 = |z_1|\,|z_2|$.

(ii)
$$\frac{z_1}{z_2} = \frac{r_1\,(\cos\theta_1 + i\sin\theta_1)}{r_2\,(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}\,\frac{(\cos\theta_1 + i\sin\theta_1)\,(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)\,(\cos\theta_2 - i\sin\theta_2)}$$
$$= \frac{r_1}{r_2}\,[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Hence the result.

Cor.

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}.$$

SAQ 1

(a) Express the following in the form $a + i\,b$, where a and b are real numbers

(i) $\dfrac{(3 + 4i)\,(2 + i)}{1 + i}$

(ii) $\dfrac{(1 + 2i)^3}{(1 + i)\,(2 - i)}$

(b) Find the modulus and principal argument of

(i) $\dfrac{(1+i)^2}{1-i}$

(ii) $-\sqrt{3}-i$

(c) Put the following complex numbers into polar form $r(\cos\theta + i\sin\theta)$

(i) $\dfrac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i}$

(ii) $\dfrac{(2+5i)(-3+i)}{(1-2i)^2}$

5.4 EXPONENTIAL AND CIRCULAR FUNCTIONS OF COMPLEX NUMBERS

Definition 1

If $z = x + iy$, we define

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \qquad \ldots (1)$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots \qquad \ldots (2)$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots \qquad \ldots (3)$$

From Eqs. (2) and (3), we have

$$\cos z + i\sin z = \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)$$

$$= 1 + \frac{iz}{1!} + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \ldots$$

$$= e^{iz}$$

$\therefore \qquad \cos z + i\sin z = e^{iz} \qquad \ldots (4)$

Similarly $\qquad \cos z - i\sin z = e^{-iz} \qquad \ldots (5)$

Hence from Eqs. (4) and (5), we have

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \ldots (6)$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2} \qquad \ldots (7)$$

5.4.1 De Moivres Theorem

If n is an integer, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Proof

We know that $\cos\theta + i\sin\theta = e^{i\theta}$

$$\therefore \qquad (\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta}$$

$$= \cos n\theta + i\sin n\theta$$

Cor.

If n is a fraction, then $\cos n\theta + i\sin n\theta$ *is one of the values of* $(\cos\theta + i\sin\theta)^n$.

Example 5.5

Express $\dfrac{(\cos\theta + i\sin\theta)^8}{(\sin\theta + i\cos\theta)^4}$ in the form $x + iy$.

Solution

$$\frac{(\cos\theta + i\sin\theta)^8}{(\sin\theta + i\cos\theta)^4} = \frac{(\cos\theta + i\sin\theta)^8}{(i)^4\left[\cos\theta + \frac{1}{i}\sin\theta\right]^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta - i\sin\theta)^4} = \frac{(\cos\theta + i\sin\theta)^8}{[(\cos\theta + i\sin\theta)^{-1}]^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta + i\sin\theta)^{-4}} = (\cos\theta + i\sin\theta)^8\,(\cos\theta + i\sin\theta)^4$$

$$= (\cos\theta + i\sin\theta)^{12}$$

Example 5.6

Prove that $(1 + \cos\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n\frac{\theta}{2}\cos\frac{n\theta}{2}$ where n is an integer.

Solution

$$1 + \cos\theta + i\sin\theta = 1 + 2\cos^2\frac{\theta}{2} - 1 + i\,.\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

and $$1 + \cos\theta - i\sin\theta = 1 + 2\cos^2\frac{\theta}{2} - 1 - i\,.\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)$$

$$= 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-1}$$

$\therefore$ $$(1 + \cos\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n$$

$$= \left[2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\right]^n + \left[2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-1}\right]^n$$

$$= 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^n + 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-n}$$

$$= 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}\right) + 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{n\theta}{2} - i\sin\frac{n\theta}{2}\right)$$

$$= 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2} + \cos\frac{n\theta}{2} - i\sin\frac{n\theta}{2}\right)$$

$$= 2^n\cos^n\frac{\theta}{2}\,.\,2\cos\frac{n\theta}{2}$$

$$= 2^{n+1}\cos^n\frac{\theta}{2}\cos\frac{n\theta}{2}$$

Example 5.7

If n is a positive integer, prove that $(\sqrt{3} + i)^n + (\sqrt{3} - i)^n = 2^{n+1}\cos\frac{n\pi}{6}$.

Solution

Let $\sqrt{3} + i = r(\cos\theta + i\sin\theta)$

Then $r = \sqrt{3+1} = 2$, $\tan\theta = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$

$\therefore$ $\theta = \frac{\pi}{6}$

Hence $\sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

and $\sqrt{3} - i = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

Hence $$(\sqrt{3} + i)^n + (\sqrt{3} - i)^n = \left[2\cos\left(\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^n + \left[2\cos\left(\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)\right]^n$$

$$= 2^n\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^n + 2^n\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^n$$

$$= 2^n \left(\cos \frac{n\pi}{6} + i \sin \frac{n\pi}{6}\right) + 2^n \left(\cos \frac{n\pi}{6} - i \sin \frac{n\pi}{6}\right)$$

$$= 2^n \left(\cos \frac{n\pi}{6} + i \sin \frac{n\pi}{6} + \cos \frac{n\pi}{6} - i \sin \frac{n\pi}{6}\right)$$

$$= 2^n \ . \ 2 \cos \frac{n\pi}{6} = 2^{n+1} \cos \frac{n\pi}{6}$$

Example 5.8

Find the different values of $(1+i)^{\frac{1}{3}}$.

Solution

$$1 + i = r (\cos \theta + i \sin \theta)$$

Then $\quad r = \sqrt{1+1} = \sqrt{2}$

and $\quad \tan \theta = \frac{1}{1} = 1, \text{ i.e. } \theta = \frac{\pi}{4}$

$\therefore \quad 1 + i = \sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$

$$(1+i)^{\frac{1}{3}} = \left[\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)\right]^{\frac{1}{3}}$$

$$= \left[\sqrt{2} \cos \left(2 n \pi + \frac{\pi}{4}\right) + i \sin \left(2 n \pi + \frac{\pi}{4}\right)\right]^{\frac{1}{3}}$$

$$= (2)^{\frac{1}{6}} \left[\cos \frac{1}{3}\left(2 n \pi + \frac{\pi}{4}\right) + i \sin \frac{1}{3}\left(2 n \pi + \frac{\pi}{4}\right)\right]$$

Putting $n = 0, 1, 2$, the three different values of $(1+i)^{\frac{1}{3}}$ are

$$(2)^{\frac{1}{6}} \left[\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}\right], \quad (2)^{\frac{1}{6}} \left[\cos \frac{9\pi}{12} + i \sin \frac{9\pi}{12}\right]$$

$$(2)^{\frac{1}{6}} \left[\cos \frac{17\pi}{12} + i \sin \frac{17\pi}{12}\right]$$

Example 5.9

Use De Moivers theorem to solve the equation $x^3 + 1 = 0$.

Solution

$$x^3 + 1 = 0 \Rightarrow x^3 = -1 = \cos \pi + i \sin \pi$$

$\therefore \quad x^3 = \cos (2 n \pi + \pi) + i \sin (2 n \pi + \pi)$

i.e. $x = [\cos(2n\pi + \pi) + i\sin(2n\pi + \pi)]^{\frac{1}{3}}$

$$= \cos(2n+1)\frac{\pi}{3} + i\sin(2n+1)\frac{\pi}{3}$$

$n = 0, 1, 2$

i.e. the roots are

$$\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right), (\cos\pi + i\sin\pi), \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$

i.e. $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}, -1, \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$

SAQ 2

(a) Find the value of

(i) $(1+i)^{\frac{1}{5}}$

(ii) $\left(\frac{1}{2} + \frac{\sqrt{-3}}{2}\right)^{\frac{3}{4}}$

(b) Use De Moivres theorem to solve following equations

(i) $x^4 + 1 = 0$

(ii) $x^{10} - x^5 + 1 = 0$ (**Hint :** Put $x^5 = y$)

(iii) $x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = 0$

(**Hint :** Multiply the equation by $x + 1$)

5.5 SUMMARY

- A number of the type $z = x + iy$ where x and y are real numbers and $i = \sqrt{-1}$ is called a complex number.
- x = real part of $z = \text{Re}(z)$

  y = imaginary part of $z = \text{Im}(z)$

- $z = x + i\,y$ is represented by a point $P(x, y)$ in XOY plane (Argands plane) and $|OP|$ is called the modulus of z and is denoted by $|z|$.

$$|z| = \sqrt{x^2 + y^2} = r \qquad \text{(Say)}$$

Then $x = r\cos\theta,\ y = r\sin\theta$ where $XOP! = \theta$ and is called the argument of z.

- $x - i\,y$ is called the complex conjugate of the complex number $z = x + i\,y$ and is denoted by $\bar{z}$, i.e. $\bar{z} = x - i\,y$.

- (i) $\bar{\bar{z}} = z$

  (ii) $|z_1 z_2| = |z_1||z_2|$ and $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$

  (iii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|},\ z_2 \neq 0$ and $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

  (iv) $|z|^2 = z\,\bar{z}$

  (v) $|z_1 + z_2| \leq |z_1| + |z_2|$

  (vi) $|z_1 - z_2| \leq |z_1| + |z_2|$

  and $\geq ||z_1| - |z_2||$

  (vii) amp $(z_1 z_2) =$ amp $(z_1) +$ amp (z_2), $z_1, z_2 \neq 0$

  (viii) amp $\left(\dfrac{z_1}{z_2}\right) =$ amp $(z_1) -$ amp (z_2), $z_1, z_2 \neq 0$

- De Moivres theorem

  (i) If n is any integer, then $(\cos\theta + i\sin\theta)^n = \cos n\,\theta + i\sin n\,\theta$.

  (ii) If n is a rational number, then $\cos n\,\theta + i\sin n\,\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

5.6 ANSWERS TO SAQs

SAQ 1

(a) (i) $\dfrac{13}{2} + \dfrac{9}{2}\,i$

(ii) $\dfrac{-7}{2} + \dfrac{1}{2}\,i$

(b) (i) $\sqrt{2},\ \dfrac{3\pi}{4}$

(ii) $2,\ \dfrac{-5\pi}{6}$

(c) (i) $2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

(ii) $r = \dfrac{\sqrt{290}}{5},\ \theta = \tan^{-1}\left(-\dfrac{1}{17}\right)$

SAQ 2

(a) (i) $2^{\frac{1}{10}}\left[\cos\frac{1}{5}\left(2n\,\pi+\frac{\pi}{4}\right)+i\sin\frac{1}{5}\left(2n\,\pi+\frac{\pi}{4}\right)\right],\ n=0,1,2,3,4$

(ii) $2^{\frac{3}{4}}\left[\cos\left(\frac{3n\pi}{2}+\frac{\pi}{4}\right)+i\sin\left(\frac{3n\pi}{2}+\frac{\pi}{4}\right)\right],\ n=0,1,2$

(b) (i) $\cos\,(2r+1)\,\frac{\pi}{4}+\ i\sin\,(2r+1)\,\frac{\pi}{4},\ r=0,1,2,3,4$

(ii) $x=\cos\,(6n+1)\,\frac{\pi}{15}\pm\ i\sin\,(6n+1)\,\frac{\pi}{15},\ n=0,1,2,3,4$

(iii) $x=\cos\,(2n+1)\,\frac{\pi}{7}+\ i\sin\,(2n+1)\,\frac{\pi}{7},\ n=0,1,2,3,4,5,6$