UNIT 2 APPLICATIONS OF DERIVATIVES

Structure

2.1 INTRODUCTION

In Unit 1, we have defined the derivative of a function. In this unit, we shall study some applications of the derivatives of a function. In Section 2.2, we propose to study the application of derivative to geometry. In Section 2.4, we will define increasing and decreasing functions and in Section 2.5, we shall discuss how derivatives can be used to determine the points where a differentiable function has maxima and minima and how to solve problems involving maximisation or minimisation of some functions. We shall also discuss some fundamental theorems of differential calculus.

Objectives

After studying this unit, you should be able to

- find the tangent and normal to a given function at given points,
- find the angle of intersection between two curves,
- determine whether a function is decreasing or increasing,
- locate the points where a function has a maximum or a minimum,
- solve some problems when it is required to minimize or maximize a function, and

- identify whether the derivative of a function can vanish once within an interval.

2.2 APPLICATION TO GEOMETRY

2.2.1 Geometrical Meaning of the Derivative

In Unit 1, we defined the derivative of a function $y = f(x)$ as

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We now examine the geometrical meaning of this definition. Let $P(x, y)$ and $Q(x + h, y + k)$ be two neighouring points on the curve $y = f(x)$ (Figure 2.1).

Figure 2.1

Draw PM, QN the ordinates through P and Q respectively. Let the secant PQ makes an angle θ with the positive x-axis. Draw PR perpendicular to QN. Then from Figure 2.1, we see that

$$MP = f(x) = y,\ NQ = f(x+h) = y + k$$

$$RQ = NQ - NR = NQ - MP = f(x+h) - f(x)$$

$$MN = h = PR$$

Hence $$\frac{f(x+h) - f(x)}{h} = \frac{RQ}{PR} = \tan\theta$$

$$= \text{slope of the secant } PQ.$$

In the limit $h \to 0$, the point $Q \to P$ along the curve $y = f(x)$ and the secant PQ becomes the tangent at P. If the tangent at P makes an angle ψ with the positive direction of the x-axis, we get

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{Q \to P} \tan\theta = \tan\psi$$

$$= \text{slope of the tangent at } P. \qquad \ldots (2.1)$$

Eq. (2.1) shows that

$$\frac{dy}{dx} = \tan\psi$$

represents the slope of the tangent to the curve $y = f(x)$ at any point (x, y). The above discussion shows that at any point of the graph of a differentiable function $y = f(x)$, the slope of the tangent is defined and is equal to the derivative of the function at that point. This means that if $y = f(x)$ is differentiable, then the graph of $y = f(x)$ has a tangent at every point.

Remark

(i) If $\frac{dy}{dx} = 0$ at a point P, then tan $\psi = 0$. Hence $\psi = 0$. This means the tangent to the curve at P is parallel to the x-axis. Conversely, if the tangent at a point is parallel to the x-axis then $\frac{dy}{dx} = 0$ at that point (Figure 2.2(a)).

Figure 2.2(a) : $\frac{dy}{dx} = 0$, Tangent Parallel to *x*-axis

(ii) If $\frac{dy}{dx}$ is infinite at a point P then tan $\psi = \infty$ or $\psi = \frac{\pi}{2}$, so that the tangent is parallel to the y-axis at that point (Figure 2.2(b)).

Figure 2.2(b) : $\frac{dy}{dx} = 0$, Infinite, Tangent Parallel to *y*-axis

2.2.2 Equations of the Tangent and Normal at a Point

You know that the equation of the line through (x_1, y_1) with slope m is

$$y - y_1 = m\,(x - x_1)$$

If this line is the tangent to $y = f(x)$ at (x_1, y_1), then we must have

$$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

where $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ is the value of the derivative at (x_1, y_1). Hence the equation of the tangent to the curve $y = f(x)$ at (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x_1, y_1) \qquad \ldots (2.2)$$

The normal at $P\ (x_1, y_1)$ is perpendicular to the tangent at P. The normal at $P\ (x_1, y_1)$ therefore has the slope

$$-\frac{1}{\left(\dfrac{dy}{dx}\right)_{(x_1,y_1)}} = -\left(\frac{dx}{dy}\right)_{(x_1,y_1)}$$

[Recall that if two straight lines with slopes m and m' are perpendicular to each other then $mm' = -1$ or $m' = -\frac{1}{m}$]. Hence the equation of the normal at (x_1, y_1) to $y = f\,(x)$ is

$$y - y_1 = -\frac{1}{\left(\dfrac{dy}{dx}\right)_{(x_1,y_1)}}\,(x - x_1)$$

or

$$(x - x_1) + \left(\frac{dy}{dx}\right)_{(x_1,y_1)}(y - y_1) = 0. \qquad \ldots (2.3)$$

Example 2.1

Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point (x_1, y_1).

Solution

From $y^2 = 4ax$, we find

$$2y\,\frac{dy}{dx} = 4a$$

or

$$\frac{dy}{dx} = \frac{4a}{2y}$$

This gives

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{2a}{y_1}$$

The equation of the tangent is (by Eq. (2.2))

$$y - y_1 = \frac{2a}{y_1}\,(x - x_1)$$

or

$$yy_1 - y_1^2 = 2ax - 2ax_1$$

or

$$yy_1 = 2ax - 2ax_1 + y_1^2$$

Since (x_1, y_1) is a point on the parabola, we have $y_1^2 = 4ax_1$. Using this we can write the equation of the tangent as

$$yy_1 = 2ax - 2ax_1 + 4ax_1$$

or

$$yy_1 = 2a\,(x + x_1)$$

Similarly, the equation of the normal by Eq. (2.3) is

$$(x - x_1) + \frac{2a}{y_1}\,(y - y_1) = 0$$

Example 2.2

Find the points where the tangent to the circle $x^2 + y^2 = 25$ is parallel to the line $2x - y + 6 = 0$.

Solution

From $\quad x^2 + y^2 = 25$, we get

$$2x + 2y\frac{dy}{dx} = 0$$

Hence $\quad \dfrac{dy}{dx} = -\dfrac{x}{y}$

So the slope of the tangent at any point on the circle is $\left(-\dfrac{x}{y}\right)$. The slope of the given line is 2. So the tangent to the circle is parallel to the given line if

$$-\frac{x}{y} = 2 \text{ or } x = -2y$$

Solving this and $x^2 + y^2 = 25$, we get the required points as $(-2\sqrt{5}, \sqrt{5})$ and $(2\sqrt{5}, -\sqrt{5})$.

2.2.3 Angle of Intersections between Two Curves

Two curves may intersect at one or more points. The x-coordinates of the points of intersection of two curves given by the equations $y = f_1(x)$ and $y = f_2(x)$ can be obtained by solving the equation $f_1(x) = f_2(x)$. The y-coordinates of the points of intersection can be obtained by putting these values of x in any of the equations $y = f_1(x)$ or $y = f_2(x)$.

If A is the point of intersection of two curves, then the angle between the two curves at A is defined to be the acute angle between the tangents to the two curves at A. Let C_1 and C_2 be two curves intersecting at A and let α be the acute angle (Figure 2.3) between the tangents to C_1 and C_2 at A.

Figure 2.3

Let m_1 = slope of the tangent $AT_1 = \dfrac{dy}{dx}$ for the first curve $C_1 : y = f_1(x)$ at A and

m_2 = slope of $AT_2 = \dfrac{dy}{dx}$ for the second curve $C_2 : y = f_2(x)$ at A.

Hence $$\tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{f_1' - f_2'}{1 + f_1' f_2'},$$

where the derivatives on the right hand side are to be evaluated at the point of intersection.

Remark

If $\alpha = 0$, the two curves have a common tangent. The two curves are said to touch each other if they have a common tangent.

Remark

If $\alpha = \frac{\pi}{2}$, the two curves intersect orthogonally. In this case, $\tan\alpha = \infty$ or $m_1 m_2 = -1$.

Example 2.3

Show that the ellipse $\frac{x^2}{18} + \frac{y^2}{8} = 1$ and the hyperbola $x^2 - y^2 = 5$ cut orthogonally.

Solution

We first find the points of intersection of the two curves. To do this, we solve simultaneously

$$\frac{x^2}{18} + \frac{y^2}{8} = 1 \qquad \ldots (1)$$

and $$x^2 - y^2 = 5 \qquad \ldots (2)$$

From Eq. (1) $$y^2 = 8\left(1 - \frac{x^2}{18}\right)$$

Substituting this in Eq. (2), we get

$$x^2 - 8\left(1 - \frac{x^2}{18}\right) = 5$$

or $$x^2 + \frac{4}{9}x^2 = 13$$

or $$\frac{13x^2}{9} = 13$$

or $$x^2 = 9$$

$$x = \pm 3$$

Putting these values of x in Eq. (2), we get

$$9 - y^2 = 5 \text{ or } y^2 = 4 \text{ or } y = \pm 2$$

Hence the points of intersections are

$$A\,(3, 2),\ B\,(3, -2),\ C\,(-3, 2) \text{ and } D\,(-3, -2)$$

Differentiating Eq. (1), we get

$$\frac{x}{18} + \frac{y}{8}\frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = -\frac{4}{9}\frac{x}{y}$$

Let m_1 and m_2 denote the slope of the tangents to the curves (1) and (2) respectively.

Then $$m_1 = \frac{-4}{9}\left(\frac{3}{2}\right) = \frac{-2}{3} \text{ at } A.$$

Similarly

$$m_1 = \frac{2}{3} \text{ at } B$$

$$m_1 = \frac{2}{3} \text{ at } C$$

$$m_1 = \frac{-2}{3} \text{ at } D$$

Differentiating (2), we get

$$2x - 2y\frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = \frac{x}{y}$$

So $$m_2 = \frac{3}{2} \text{ at } A$$

$$m_2 = -\frac{3}{2} \text{ at } B$$

$$m_2 = -\frac{3}{2} \text{ at } C$$

and $$m_2 = \frac{3}{2} \text{ at } D$$

Now $m_1\, m_2$ at $A = \left(-\frac{2}{3}\right)\left(\frac{3}{2}\right) = -1$.

Hence the two curves cut orthogonally at A.

Similarly, you can verify that the two curves cut orthogonally at all the other points of intersection.

SAQ 1

(a) Find the equation of the tangent and the normal at any point of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(b) Find where the tangent is parallel to the x-axis for the curve

$$y^3 = x^2\,(2 - x).$$

(c) Show that the curves

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ and } \frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$$

intersect orthogonally.

(d) Find the angles of intersection of $xy = 10$ and $x^2 + y^2 = 29$.

2.3 DERIVATIVE AS A RATE MEASURE

The derivative of a function $y = f(x)$ with respect to the independent variable x is nothing but the rate measure of change of y with respect to change in x. Let there be a small change Δx in the value of x and let Δy be the corresponding small change in the value of y.

Then
$$\begin{aligned} y + \Delta y &= f(x + \Delta x) \\ \Rightarrow \Delta y &= f(x + \Delta x) - y \\ &= f(x + \Delta x) - f(x) \end{aligned}$$

Average change in y per unit change in x

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For small values of Δx, this average rate of change in the value of y will approximate very closely to the rate of change in the value of y. Hence when $\Delta x \to 0$, $\frac{\Delta y}{\Delta x}$ represents the rate of change of y with respect to x, i.e. the rate of change of y w. r. t the change in x.

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$$

i.e. $f'(x)$ represents the rate measure of $f(x)$ w. r. t x.

Example 2.4

A balloon which remains spherical has a diameter $\frac{3}{2}(2x+3)$. Determine the rate of change of volume w. r. t x.

Solution

Radius of the sphere $= \frac{1}{2} \cdot \frac{3}{2}(2x+3) = \frac{3}{4}(2x+3)$

$$V = \text{volume} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \left[\frac{3}{4}(2x+3)\right]^3$$

$$= \frac{9}{16}\pi (2x+3)^3$$

Rate of change of volume w. r. t x

$$\frac{dV}{dx} = \frac{9}{16}\pi \cdot 3(2x+3)^2 \cdot 2$$

$$= \frac{27}{8}\pi (2x+3)^2$$

Example 2.5

A man 2 metres high walks at a uniform speed of 6 km/h away from a lamp post 6 metres high. Find the rate at which the length of his shadow increases.

Figure 2.4

Solution

Let S be the source of light and BS is the pole. Let HP be the position of the man at any time t. Let x be the distance of the man from the pole and y be the length of the shadow.

From similar triangles AHP and ABS, we have

$$\frac{AH}{AB} = \frac{HP}{BS} \Rightarrow \frac{y}{y+x} = \frac{2}{6} = \frac{1}{3}$$

$$\Rightarrow 3y = y + x \Rightarrow 2y = x \Rightarrow y = \frac{1}{2}x$$

$\therefore \quad \frac{dy}{dt} = \frac{1}{2}\frac{dx}{dt}$, we are given $\frac{dx}{dt} = 6 \text{ km/h}$.

$\therefore \quad \frac{dy}{dt} = \frac{1}{2} \cdot 6 = 3 \text{ km/h}$

2.3.1 Motion in a Straight Line

Suppose a particle P is moving in a straight line. We take a point O on the straight line as origin and set up a coordinate system on the line, that is, we consider it as the number line. The directed distance of the particle from the origin is a function of line. Let the particle P be at the point s at time t. Then $s = f(t)$.

As you already know, the velocity of the particle is the rate of change of its distance (from the origin). Let the position of the particle be the point $(s + \Delta s)$ at time $t + \Delta t$, so that $s + \Delta s = f(t + \Delta t)$. Then average velocity of the particle between the points

$$s \text{ and } s + \Delta s = \frac{\Delta s}{\Delta t}$$

The velocity v (instantaneous velocity) of the particle at the point s (at time t) is the limiting value of $\frac{\Delta s}{\Delta t}$ as $\Delta t \to 0$.

$$\therefore \qquad v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{(s + \Delta s) - s}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = f'(t)$$

Similarly, if v is the velocity of the particle at time t and $v + \Delta v$ is the velocity at $t + \Delta t$, then average acceleration of the particle between the points s and $s + \Delta s$.

$$= \frac{\Delta v}{\Delta t} = \frac{(v + \Delta v) - v}{\Delta t} = \frac{f'(t + \Delta t) - f'(t)}{\Delta t}$$

The acceleration a at the point s is the limiting value of the average acceleration of the particle between the points s and $s + \Delta s$.

$$= \lim_{\Delta t \to 0} \frac{f'(t + \Delta t) - f'(t)}{\Delta t} = f''(t)$$

Thus when the distance travelled by a particle (from the origin) is given as a function of time, say

$s = f(t)$, then $f'(t) = \left(\frac{ds}{dt}\right)$ represents the velocity and $f''(t) = \left(\frac{d^2 s}{dt^2}\right)$ represents its acceleration at time t.

Example 2.6

A particle is moving along a straight line according to the formula $s = 12t - 3t^2$, where s is in metres and t is in seconds. Find its velocity and acceleration.

Solution

We have $s = 12t - 3t^2$. Differentiating with respect to t.

$$\frac{ds}{dt} = 12 - 6t$$

This gives the velocity at time t.

Differentiating once again with respect to t.

$$\frac{d^2 s}{dt^2} = -6$$

This gives the acceleration at time t.

We note that the acceleration is the same at all times t.

The negative sign of the acceleration means that the velocity is decreasing. Sometimes it (negative acceleration) is called retardation.

Example 2.7

A particle is moving in a straight line according to the formula $s = t^3 - 9t^2 + 3t + 1$, where s is measured in metres and t in seconds. When the velocity is – 24 m/s, find the acceleration.

Solution

We have

$$s = t^3 - 9t^2 + 3t + 1$$

The velocity is given by $\frac{ds}{dt} = 3t^2 - 18t + 3 = 3(t^2 - 6t + 1)$.

If this is equal to – 24, then

$$3(t^2 - 6t + 1) + 24 = 0$$

That is, $\quad t^2 - 6t + 9 = 0$

$\therefore \quad t = 3.$

The acceleration is given by

$$\frac{d^2s}{dt^2} = 3(2t - 6) = 6(t - 3)$$

$$\therefore \quad \left(\frac{d^2s}{dt^2}\right)_{t=3} = 0$$

Thus, when the velocity is – 24 m/s, the acceleration is 0 m/s^2.

2.3.2 Motion Under Gravity

One particular instance of motion in a straight line is the motion of a falling body under gravity. The acceleration of the falling body due to gravity has been calculated as g = 32 feet/second^2 or 9.8 metres/second^2, towards the centre of the earth. In this sub-section, we use differentiation to some practical problems concerning this motion.

Example 2.8

The motion of a stone thrown vertically upwards satisfies an equation of the form $s = at^2 + bt$ when s and t are measured in metres and seconds respectively. If the maximum height reached by the stone is 4.9 metres and if its acceleration is – 9.8 m/s^2, find its height after half a second.

Solution

We have $s = at^2 + bt$. Differentiating $\frac{ds}{dt} = 2at + b$.

Differentiating once again,

$$\frac{d^2s}{dt^2} = 2a$$

It is given that the acceleration is – 9.8 m/s^2.

$\therefore \qquad 2a = -9.8$

or $\qquad a = -4.9$

$\therefore \qquad \dfrac{ds}{dt} = b - 9.8t$

$\dfrac{ds}{dt}$ becomes zero when $t = \dfrac{b}{9.8}$ and negative for greater values of t. (If b were negative, there would be no upward motion at all). The maximum height reached is given by

$$s = at^2 + bt \text{ with } t = \frac{b}{9.8}.$$

This is, when $s = -4.9\dfrac{b^2}{(9.8)^2} + \dfrac{b^2}{9.8} = \dfrac{b^2}{9.8}\left(1 - \dfrac{1}{2}\right) = \dfrac{b^2}{19.6}$

From the given data $\dfrac{b^2}{19.6} = 4.9$

$\therefore \qquad b^2 = 4.9 \times 19.6$

$\therefore \qquad b = 9.8$ (as already mentioned, b cannot be negative)

$\therefore$ The equation of motion of the stone becomes $s = 4.9t^2 + 9.8t$.

When $\quad t = \dfrac{1}{2}, s = -\dfrac{4.9}{4} + \dfrac{9.8}{2} = -\dfrac{4.9}{4} + 4.9$

$$= 4.9\left(1 - \frac{1}{4}\right) = \frac{14.7}{4} = 3.675$$

The stone is at a height of 3.675 m after half a second.

2.4 INCREASING AND DECREASING FUNCTIONS

Definition 1

Let I be an open interval contained in the domain of a real function f. f is called an increasing (decreasing) function on I if whenever $x_1 < x_2$ in I it is true that $f(x_1) \le f(x_2)$ $[f(x_1) \ge f(x_2)]$. Symbolically, it is written as f (x) is an increasing function on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \le f(x_2)$ and f (x) is a decreasing function on I if $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$.

(a) **(b)**

Figure 2.5

The graph in Figure 2.5(a) is the graph of an increasing function $f(x)$ in (a, b) and the graph in Figure 2.5(b) is the graph of a decreasing function $f(x)$ in (a, b).

Remark

It is possible that a function f is neither increasing nor decreasing on a given interval I. The function $f(x)$ in Figure 2.6 is neither increasing nor decreasing in (a, b).

Figure 2.6

If $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ [or $f(x_1) > f(x_2)$], then f is said to be strictly increasing (or decreasing).

Definition 2

A function f is said to be increasing at a point x_0 if there is an interval $I = (x_0 - h, x_0 + h)$ containing x_0 such that for $x_1, x_2 \in I$.

$$x_0 < x_2 \Rightarrow f(x_0) \le f(x_2)$$

and $$x_1 < x_0 \Rightarrow f(x_1) \le f(x_0)$$

Definition 3

A function f is said to be decreasing at a point x_o if there is an interval $I = (x_0 - h, x_0 + h)$ containing x_0 such that for $x_1, x_2 \in I$.

$$x_0 < x_2 \Rightarrow f(x_0) \ge f(x_2)$$

and $$x_1 < x_0 \Rightarrow f(x_1) \ge f(x_0)$$

Example 2.9

Prove that the function $f(x) = 3x + 5$ is an increasing function.

Solution

Let $x_1, x_2 \in R$ such that $x_1 < x_2$, i.e. $x_1 - x_2 < 0$ then

$$f(x_1) - f(x_2) = (3x_1 + 5) - (3x_2 + 5)$$

$$= 3(x_1 - x_2) < 0$$

$$\therefore \quad f(x_1) < f(x_2)$$

i.e. $f(x)$ is an increasing function.

Note that $f(x)$ is a strictly increasing function.

Example 2.10

$f(x) = x^2$ is a strictly decreasing function in $I = (-\infty, 0)$.

Solution

Let $x_1, x_2 \in I$ s. t $x_1 < x_2$. Multiplying by $x_1 \in I$, i.e. x_1 is negative then we have $x_1^2 > x_1 x_2$.

Similarly $x_1 x_2 > x_2^2$ (Multiplying $x_1 < x_2$ by x_2)

i.e. $x_1^2 > x_2^2$

i.e. $f(x_1) > f(x_2)$

$\therefore$ f is a strictly decreasing function.

Theorem 1

A differentiable real function $f(x)$ is strictly increasing on an open interval I if $f'(x) > 0$ for all x in I.

Explanation

If $f'(x)$ is positive then $f(x+h) - f(x)$ and h have the same sign for small values of h. When h is +ve, then $x + h > x \Rightarrow f(x+h) > f(x)$ and when h is – ve, then $x + h < x \Rightarrow f(x+h) < f(x)$. This means $f(x)$ is strictly increasing function as shown in Figure 2.7.

Figure 2.7

On similar lines, we have that a differentiable real function is strictly decreasing on an interval I if $f'(x) < 0$ for all x in I.

Remark

When the interval is not mentioned, we must prove it for its domain, i.e. R.

Example 2.11

Prove that the function sin x is strictly increasing in the interval $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing in the interval $\left(\frac{\pi}{2}, \pi\right)$.

Solution

Let $f(x) = \sin x$

Then $f'(x) = \cos x$

We know that cos x is +ve in $\left(0, \frac{\pi}{2}\right)$ and –ve in $\left(\frac{\pi}{2}, \pi\right)$.

$\therefore$ sin x is strictly increasing in $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$.

Example 2.12

Find the intervals in which the function $f(x) = 2x^3 - 3x^2 - 36x + 7$ is (a) strictly increasing (b) strictly decreasing.

Solution

$$f(x) = 2x^3 - 3x^2 - 36x + 7$$

$$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$$

$$= 6(x - 3)(x + 2)$$

$\therefore$ $\quad f'(x) = 0$ if $x = 3$ or $x = -2$

The points $x = -2$ and $x = 3$ divide the real line in the disjoint intervals.

$$I_1 = (-\infty, -2), I_2 = (-2, 3) \text{ and } I_3 = (3, \infty)$$

In I_1, $f'(x)$ is +ve.

In I_2, $f'(x)$ is –ve.

In I_3, $f(x)$ is +ve.

Hence $f(x)$ is strictly increasing in I_1 and I_3 and strictly decreasing in I_2.

SAQ 2

(a) Prove that the function $x + \frac{1}{x}$ is strictly increasing on (– 1, 1).

(b) Which of the functions are strictly decreasing on $\left(0, \frac{\pi}{2}\right)$

(i) cos x,

(ii) tan x.

2.5 MAXIMA AND MINIMA

We shall study how to locate the points where a differentiable function has its maximum or minimum.

We shall also consider some practical problems involving the theory of maximum and minimum of a function.

Definition 4

We say that a function f has a maximum at a point x = c, if for all x in a sufficiently small neighbourhood of c,

$$f(x) \leq f(c)$$

Definition 5

We say that a f has a minimum at a point x = d, if for all x in a sufficiently small neighbourhood of d,

$$f(x) \geq f(d)$$

Example 2.13

In Figure 2.8, the graph of some function f in the interval $[a, b]$ is shown.

Let A be the point $x = c$. As you see, at this point f has a maximum because at $x = c$ the value $f(c)$ of the function (the ordinate AP) is greater than the value of the function $f(x)$ for all values of x in a small neighbourhood of A.

Figure 2.8

The dotted line near the ordinate AP indicates that all other ordinates at points near to A are smaller than AP.

Similarly the point B, where $x = d$, the function f has a minimum because in the neighbourhood of $x = d$ the value of the function is greater than $f(d)$. The ordinate BQ is smaller than any other ordinates at points in a small neighbourhood of B. This is indicated by the dotted lines.

From the figure, it is clear that A is not the only point where there is a maximum and B is not the only point where there is a minimum. The function has maximum at A_1, A_2 and it has minimum at B_1, B_2. Also note that the value of the function at A_1 (which is a maximum) is less than the value of the function at B (which is a minimum).

Thus there can be several maxima and minima in an interval.

In ordinary language, the expression maximum and minimum are used to mean the greatest value and the least value respectively. For us the meaning of maximum (or minimum) is not the same. All we need for a function f to have a maximum at $x = c$ is that f at $x = c$ should be greater than all other values of f in some small neighbourhood of $x = c$.

Note that the greatest value is automatically a maximum but a maximum need not be the greatest value of the function in its domain. Similarly, a minimum is not necessarily the smallest value of the function in its domain.

In order to distinguish from the greatest value, the maximum is sometimes called a *relative* or *local maximum* and the greatest value is called *absolute* or *global maximum*. Similarly, a minimum is called *relative* or *local minimum* and the least value is called the *absolute* or *global minimum*. The term extremum (plural extrema) or extreme value is used to mean both maximum and minimum.

Example 2.14

Let $f(x) = x^2, -1 \le x \le 2$. The graph of *f* is a parabola (Figure 2.9). Since $f(x) \ge 0$ for all x in $-1 \le x \le 2$ and $f(0) = 0$, we conclude that *f* has a minimum at $x = 0$. Since this is the only minimum, $x = 0$ is the absolute minimum. The function has maxima at the two end points $x = -1$ and $x = 2$. At the point $x = 2$, there is an absolute maximum.

Figure 2.9

2.5.1 Necessary Condition for a Maximum and Minimum

We give a method for finding possible points of the maxima and minima of a function. We have the following theorem which we state without proof.

Theorem 2

If *f* is differentiable at $x = x_1$ and if *f* has a maximum or a minimum at $x = x_1$ then $f'(x_1) = 0$.

Note

The theorem does not say that if $f'(x_1) = 0$ then *f* has a maximum or minimum at $x = x_1$. In fact, even $f'(x_1) = 0$, there may not be any maximum or minimum as the following example will show.

Example 2.15

Let $\quad f(x) = x^3$

Then $\quad f'(x) = 3x^2$ and $f'(0) = 0$

But $\quad f(x) < 0$ if $x < 0$ and $f(x) > 0$ if $x > 0$.

Therefore, we cannot find any neighbourhood of $x = 0$ where

$$f(0) \geq f(x) \text{ or } f(0) \leq f(x)$$

Hence $x = 0$ is neither a maximum nor a minimum even though $f'(0) = 0$. Thus $f'(x_1) = 0$ is a **necessary** condition for f to have a maximum or a minimum at $x = x_1$ but it is not **sufficient**.

We know from Section 2.2 that if $\frac{dy}{dx} = 0$ at a point $x = x_1$ then the tangent to the curve $y = f(x)$ is parallel to the x-axis. We also know that if f has a maximum or a minimum at a point $x = x_1$ and if $f'(x_1)$ exists, then $f'(x_1) = 0$. Combining these two results, we get the following obvious geometric fact :

If f is differentiable at a point $x = x_1$ and if f has a maximum or a minimum at $x = x_1$, then the tangent to the curve $y = f(x)$ at $\dot{x} = x_1$ is parallel to the x-axis.

2.5.2 Rule for Finding Maxima and Minima

The derivative $f'(x)$ gives us the points of local minima or points of local maxima. How do we distinguish whether the point x_0 satisfying $f'(x) = 0$ is a point of local maximum or a point of local minima? We have seen that if x_0 is a point of local maximum then $f'(x) > 0$ at a nearby point to the left of x_0 and < 0 at a nearby point to the right of x_0. On the other hand, if x_0 is a point of local minimum, then $f'(x) < 0$ at a nearby point of the left of x_0 and > 0 at a nearby point to the right of x_0. Thus, we have the following working rule for finding the points of local maxima or local minima.

Theorem 3 : First Derivative Test

Let $f(x)$ be a differentiable function on I. Then

(a) x_0 is a point of local maximum of $f(x)$ if

(i) $f'(x_0) = 0$

(ii) $f'(x_0) > 0$ at every point close to the left of x_0 and $f'(x_0) < 0$ at every point close to the right of x_0.

(b) x_0 is a point of local minimum of $f(x)$ if

(i) $f'(x_0) = 0$

(ii) $f'(x_0) < 0$ at every point close to the left of x_0 and $f'(x_0) > 0$ at every point close to the right of x_0.

(c) If $f'(x_0) = 0$ but $f'(x)$ does not change sign as x increases through x_0, then x_0 is neither a point of local maxima nor local minima. Such a point is called the point of inflection.

The First Derivative Test helps us in finding the points of local maximum or local minimum. But it takes time to verify how $f'(x)$ is changing sign as x passes through the points given by $f'(x) = 0$. We have another test known as the second derivative test which enables us to find the points of local maxima or local minima.

Consider a point x_0 such that $f'(x_0) = 0$ and $f''(x_0) < 0$. It being assumed that the second derivative exists at x_0. This suggests that f' is strictly decreasing at x_0 as its derivative is negative. Therefore $f'(x)$ is positive to the left of x_0 and negative to the right of x_0 in a small interval around x_0. This in turn implies that $f(x)$ is strictly increasing upto x_0 and then decreasing in this small interval, i.e. x_0 is a point of local maximum.

Figure 2.10

Thus if $f'(x_0) = 0$ and $f''(x_0) < 0$, x_0 is a point of local maximum.

Similarly, if $f'(x_0) = 0$ and $f''(x_0) > 0$, x_0 is a point of local minimum.

Theorem 4 : Second Derivative Test

Let $f(x)$ be a differentiable function on an interval I and $x_0 \in I$ and $f''(x)$ be continuous at x_0. Then

x_0 is a point of local maximum if both $f'(x_0) = 0$ and $f''(x_0) < 0$.

x_0 is a point of local minimum if both $f'(x_0) = 0$ and $f''(x_0) > 0$.

So we have the following rule for finding points of local maximum or local minimum.

Step 1

Find all the points where f' is zero.

Step 2

At each of these points find the sign of f''.

Step 3

If f'' is –ve, the point is a point of local maximum. If f'' is +ve, then the point is a point of local minimum.

Note : This test fails if $f''(x_0)$ is also zero. In that case, we go back to the first derivative test.

Example 2.16

Find the maximum and minimum of $f(x) = x^2 - x$ with the help of the first derivative test.

Solution

To solve this problem, we follow the steps discussed above.

Step 1

We observe that f is differentiable at all points.

Step 2

$f'(x) = 2x - 1,\ f'(x) = 0$ gives $x = \frac{1}{2}$

Step 3

So the only critical point is $x = \frac{1}{2}$.

Hence, if f has any maximum or minimum, it must be at $x = \frac{1}{2}$.

Step 4

We test the change of sign of $f''(x)$ while crossing $x = \frac{1}{2}$ from left to right. We have $f'(x) = 2x - 1 = 2\left(x - \frac{1}{2}\right)$

(i) $f'(x) < 0$ if $x < \frac{1}{2}$, and

(ii) $f'(x) > 0$ if $x > \frac{1}{2}$.

Therefore, f has a minimum at $x = \frac{1}{2}$.

Example 2.17

Investigate the maximum and minimum of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$ with the help of the first derivative test.

Solution

(i) The function is differentiable at all points.

(ii) $f'(x) = 5x^4 - 20x^3 + 15x^2$

$= 5x^2(x^2 - 4x + 3)$

$= 5x^2(x - 3)(x - 1)$

(iii) The critical points are given by $f'(x) = 0$.

or $5x^2(x - 1)(x - 3) = 0$

or $x = 0, x = 1, x = 3$

(iv) We now test the critical points,

The critical point $x = 0$.

When $x < 0$, we have

$x^2 > 0,\ x - 1 < 0$ and $x - 3 < 0$

Hence $f'(x) = 5x^2(x - 1)(x - 3) > 0$ . . . (1)

$$(+)\ (-)\ (-)$$

When x crosses over $x = 0$ but remains very near $x = 0$, we have $x > 0$, $x < 1$, and $x^2 > 0$, $x - 1 < 0$ and $x - 3 < 0$.

So $\qquad f'(x) = 5x^2(x-1)(x-3) > 0\,.$ $\qquad$ . . . (2)

From Eqs. (1) and (2), we see that $f''(x)$ remains positive when x goes from left to the right of the critical point $x = 0$ and does not change its sign. Hence, at $x = 0$, the function has neither a maximum or minimum. The function increases at $x = 0$.

Critical point $x = 1$.

When $\qquad x < 1,\ x^2 > 0,\ x - 1 < 0,\ x - 3 < 0$

Hence $\qquad f'(x) = 5x^2(x-1)(x-3) > 0$ $\qquad$ . . . (3)

$$(+)\ (-)\ (-)$$

When x crosses over $x = 1$ but remains very near $x = 1$, we have

$$x^2 > 0,\ x - 1 > 0,\ x - 3 < 0$$

so that $\qquad f'(x) = x^2(x-1)(x-3) < 0$ $\qquad$ . . . (4)

Thus, from Eqs. (3) and (4), $f''(x)$ changes its sign from positive to negative. Hence, at $x = 1$ there is a maximum.

Similarly, it can be seen that $x = 3$, the function has a minimum.

Example 2.18

Find the points of maximum and minimum for the function of Example 2.17 by using second derivative test.

Solution

We have

$$f(x) = x^5 - 5x^4 + 5x^3 - 1$$

$$f'(x) = 5x^4 - 20x^3 + 15x^2$$

$$= 5x^2(x-1)(x-3)$$

The critical points are $x = 0$, $x = 1$, $x = 3$.

$$f''(x) = 20x^3 - 60x^2 + 30x$$

$$x = 1,\ f''(1) = 20 - 60 + 30 = -10 < 0$$

Therefore, at $x = 1$, the function f has a maximum.

At $\qquad x = 3,\ f''(3) = 20.3^3 - 60.3^2 + 30.3 = 540 - 540 + 90 = 90 > 0$

Therefore, at $x = 3$, the function f has a minimum. At $x = 0$, $f''(0) = 0$. (However, by observing the change of sign of $f'(x)$ at $x = 0$, we have seen in Example 2.17 that at $x = 0$, there is neither a maximum nor a minimum.)

2.5.3 Greatest and Least Values of a Function in a Closed Interval

Let f be continuous in a closed interval $[a, b]$. Then we know that f always has a greatest value and a least value in this interval $[a, b]$. The greatest (or least) value may be attained either at an interior point of the interval or at the end points $x = a$ (or $x = b$) of the interval.

It is easy to verify that

(i) f has a maximum at $x = a$ if $f'(a+) < 0$.

(ii) f has a minimum at $x = a$ if $f'(a+) > 0$.

(iii) f has a minimum at $x = b$ if $f'(b-) < 0$.

(iv) f has a maximum at $x = b$ if $f'(b-) > 0$.

Therefore, if we are required to find the greatest (or least) value of a continuous function in a closed interval $[a, b]$, we should consider the following steps :

(i) Find all the maxima (or minima) in the open interval $a < x < b$. To do this we follow the procedure described in Sub-section 2.5.2. Find the values of the function at the maximum (or minimum) points so obtained.

(ii) Find the value $f(a)$ and $f(b)$ of the function at the end points $x = a$ and $x = b$ of the interval $[a, b]$.

(iii) Now pick the greatest (or least) of all the values of the function so obtained. This value will be the greatest value (or the least value) of the function in the closed interval $[a, b]$.

Example 2.19

Determine the greatest and the least value of the function

$$f(x) = x^5 - 5x^4 + 5x^3 - 1$$

in the interval [0. 2].

Solution

In Example 2.18, we saw that the critical points of this function are

$$x = 0, x = 1 \text{ and } x = 3$$

Since $x = 3$ is outside the given interval [0, 2] and $x = 0$ is an end point, thus the only critical point in the open interval $0 < x < 2$ is at $x = 1$.

Now $\quad f(1) = 1 - 5 + 5 - 1 = 0.$

The values of the function at the end points are

$$f(0) = -1$$

$$f(2) = 2^5 - 5.2^4 + 5.2^3 - 1$$

$$= 32 - 80 + 40 - 1$$

$$= -9$$

Thus the greatest value of $f(x)$ in [0. 2] is 0 and the least value is – 9.

2.5.4 Maxima and Minima : Problems

The theory of maxima and minima developed in the previous sections provides a powerful tool for solving problems that require minimizing or maximizing certain functions. In this section, we illustrate how this is done by solving some problems. We shall summarise the technique at the end.

Example 2.20

Find two positive numbers such that their sum is 10 and their product is as large as possible.

Solution

Let one of the numbers be x. Then the other number must be $10 - x$. The product of the two numbers is

$$f(x) = x(10 - x) = 10x - x^2 \qquad \ldots (1)$$

Since both the numbers are positive, we have

$$x > 0 \text{ and } 10 - x > 0 \text{ or } x < 10.$$

We have to choose x in such a way that $f(x)$ is maximum.

From Eq. (1), we find

$$f'(x) = 10 - 2x = 2(5 - x)$$

Therefore $f'(x) = 0$ at $x = 5$

Now $f'(x) > 0$ if $x < 5$

and $f'(x) < 0$ if $x > 5$

Hence $f(x)$ has a maximum at $x = 5$.

Hence the two required numbers are $x = 5$ and $10 - x = 5$.

Example 2.21

A cylindrical container is to be made with capacity 1000 cubic metre. The material for the side costs Rs. 200/- per square metre and for the ends Rs. 150/- per square metre. Find the radius of the base so that the cost of material for making the container is least.

Solution

Let the cylinder be of radius r and height h,

Its volume is

$$V = \pi r^2 h = 1000 \text{ (given)}$$

or

$$h = \frac{1000}{\pi r^2} \qquad \ldots (1)$$

The area of the side is

$$A_1 = 2\pi r h$$

Cost of the material for side = Rs. 200. $A_1 = 2\pi r h \; . \; 200$ . . . (2)

Area of top end $= \pi r^2$

Area of bottom end $= \pi r^2$

So that sum of the areas of the two ends $= 2\pi r^2$

Cost of material of the two ends $= 2\pi r^2 . 150$ . . . (3)

Hence total cost [adding Eqs. (2) and (3)]

$$C = 2\pi r^2 . 150 + 2\pi r h . 200$$

$$= 300 \pi r^2 + 400 \pi r h$$

$$= 300 \pi r^2 + 400 \pi r . \frac{1000}{\pi r^2}$$

(Substituting the value of h from Eq. (1))

$$= 300 \pi r^2 + \frac{400000}{r}$$

Hence $$\frac{dC}{dr} = 600 \pi r - \frac{400000}{r^2}$$

$= 0$ (when C is minimum)

This gives $$r^3 = \frac{400000}{600\pi} = \frac{2000}{3\pi}$$

$$r = 5.96$$

So radius of the base is 5.96 cm. . . . (4)

Note that $\frac{d^2C}{dr^2} = 600\pi + \frac{800000}{r^3} > 0$ for the value of r. Hence the cost C is minimum.

Example 2.22

The cost of running an engine is proportional to the square of its speed and is Rs. 48/- per hour for a speed of 16 m.p.h. Other expenses amount to Rs. 300/- per hour. What is the most economical speed?

Solution

Let v km.p.h be the speed. If C is the cost of running the engine then $C = Kv^2$, where K is a constant. Now $C = 48$ when $v = 16$. Therefore

$$K = \frac{C}{v^2} = \frac{48}{16 \times 16} = \frac{3}{16}$$

The running cost per hour is $= \left(300 + \frac{3}{16} v^2\right)$.

If the distance travelled is s km, then number of hours the engine run is $\frac{s}{v}$.

Hence the total cost of running is

$$y = \frac{s}{v}\left(300 + \frac{3}{16} v^2\right).$$

For the most economical speed, we must have

$$\frac{dy}{dv} = 0 \text{ and } \frac{d^2 y}{dv^2} > 0 .$$

Now $$\frac{dy}{dv} = s\left(-\frac{300}{v^2} + \frac{3}{16}\right) = 0$$

gives $$v = 40$$

Again $$\frac{d^2 y}{dv^2} = \frac{600\,s}{v^3} > 0 \text{ for } v = 40 .$$

Hence, the most economical speed is 40 km.p.h.

SAQ 3

(a) Find the maximum and minimum of

(i) $x + \dfrac{4}{x+2}$.

(ii) $x^3 - 3x + 3$ on the interval $\left[-3, \dfrac{3}{2}\right]$.

(iii) $y = (x-1)\,(x-2)^2$.

(b) Show that $f(x) = \sin x\,(1 + \cos x)$ has a maximum at $x = \dfrac{\pi}{3}$.

(c) Find the extreme values of $f(x) = \sin 2x$ for $0 \leq x \leq 2\pi$.

(d) Show that the maximum rectangle that can be inscribed in a circle is a square.

(e) Show that of all the rectangles of given area the square has the smallest perimeter.

2.6 ROLLE'S THEOREM

Consider the function sin x. It takes the value zero at the points $0, \pm\pi, \pm 2\pi, \ldots$ Its derivative is cos x which vanishes at the points $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$ We note that between any two points where sin x vanishes, there is a point where its derivative vanishes.

Consider another function $x^2 - 4x + 3$. This vanishes at the points $x = 1$ and $x = 3$. The derivative of this function is $2x - 4$ which vanishes at $x = 2$. Hence, again $x = 2$ is a point lying in between 1 and 3.

In the above two examples, we have observed that between any two points where a function vanishes, there is a point where its derivative is also zero. Rolles theorem asserts that the observed result is a general truth which is as follows :

Rolle's Theorem

Let f be a real function defined in a closed interval $[a, b]$ such that

(i) $f(a) = f(b)$

(ii) f is continuous in the closed interval $[a, b]$.

(iii) f is differentiable in the open interval (a, b),
then there is a point $c \in (a, b)$ such that $f'(c) = 0$.

The conclusion of Rolle's theorem also holds true if we replace the condition $f(a) = f(b)$ by the condition $f(a) = f(b) = 0$ keeping other conditions the same. Rolle's theorem has a very simple geometrical interpretation. If the graph of a function is an unbroken curve intersecting the x-axis at the points a and b and if the curve has a tangent at every point except, possibly, at the end points, then there must be at least one point $(c, f(c))$ on the curve different from the end points at which the tangent is parallel to the x-axis [Figures 2.11(a) and (b)].

(a) **(b)**

Figure 2.11

Example 2.23

The polynomial function $f(x) = x^3 - x$ is continuous and differentiable for all real x. If we take $a = -1$ and $b = 1$, we have $f(-1) = 0 = f(1)$. Therefore, the conditions of Rolle's theorem are satisfied on $[-1, 1]$. Thus, there must be at least a number c such that $-1 < c < 1$ and $f'(c) = 3c^2 - 1 = 0$. In fact we have two values

$$c = \pm\sqrt{\left(\frac{1}{3}\right)}, \text{ where } f'(c) = 0$$

Both the roots are between – 1 and 1.

Example 2.24

The function

$$y = \begin{cases} x, \text{ if } 0 \le x \le 1 \\ 0, \text{ if } x = 1 \end{cases}$$

is zero at $x = 0$ and $x = 1$ and $\frac{dy}{dx}$ exists on the open interval (0, 1). But the function is not continuous on [0, 1]. The end point $x = 1$ is a point of discontinuity. Therefore, Rolle's theorem is not applicable on the interval [0, 1]. You can observe that $\frac{dy}{dx}$ is different from zero $\left(\frac{dy}{dx} = 1\right)$ on (0, 1).

SAQ 4

(a) Show that the conical tent of given capacity will require the least amount of canvas if its height is $\sqrt{2}$ times its base radius.

(b) An open storage bin with square base and vertical sides is to be constructed from a given amount of material. Determine its dimensions if its volume is to be maximum neglecting the thickness of material and waste in constructing it.

(c) Find the height of a right cylinder with greatest lateral surface area that may be inscribed in a given sphere of radius *R*.

(d) Given a point on the axis of the parabola $y^2 = 2px$ at a distance *a* from the vertex, find the abscissa of the point of the curve closest to it.

(e) Can Rolle's theorem be applied to each of the following functions? Find 'c' in case it can be applied.

(i) $f(x) = \sin^2 x$ on the interval $[0, \pi]$.

(ii) $f(x) = x^2 + 4$ on $[-2, 2]$.

(iii) $f(x) = \sin x + \cos x$ on $\left[0, \frac{\pi}{2}\right]$.

(iv) $f(x) = x^3 - 2x$ on $[0, 1]$.

2.7 MEAN-VALUE THEOREM

Let us consider a function $f(x)$ which satisfies all the conditions of Rolle's theorem except the condition, $f(a) = f(b)$. Then the conclusion of the theorem need not hold true. That is, there need not be a point $(c, f(c))$ on the graph where the tangent is parallel to the x-axis. However, there appears to be a point where the tangent is parallel to the chord that joins the points $(a, f(a))$ and $(b, f(b))$ of the graph $y = f(x)$. A generalization of Rolle's theorem, called the Mean-value Theorem, says that this will always happen, if the curve $y = f(x)$ is continuous on $[a, b]$ and differentiable on (a, b).

Figure 2.12

Theorem 5 : The Mean Value Theorem

Let $f : [a, b] \to R$ be a function such that

(i) f is continuous on $[a, b]$

(ii) f is differentiable on (a, b).

Then there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example 2.25

Let $y = x^3$ and $-2 \le x \le 2$.

Find a number c such that the tangent to the curve at the point $P : (c, f(c))$ is parallel to the chord joining $A : (-2, -8)$ and $B : (2, 8)$.

Solution

We see that $y = x^3$ satisfies the conditions of Mean-value theorem.

The slope of the tangent to $y = x^3$ is $\dfrac{dy}{dx} = 3x^2$.

The slope of AB is

$$\frac{f(2) - f(-2)}{2 - (-2)} = \frac{8 + 8}{4} = 4$$

By Mean-value theorem we have a point 'c' such that $-2 < c < 2$ and $f'(c) = 3c^2 = 4$ which implies that $c = \pm \dfrac{2}{\sqrt{3}}$. There are two values of c between -2 and 2 where the tangent to the curve $y = x^3$ is parallel to the chord AB.

Figure 2.13

Example 2.26

Check whether Mean value theorem is applicable to the function $y = 1 - x^{\frac{2}{3}}$ over the interval $[-1, 1]$.

Solution

The function is continuous in $[-1, 1]$.

The derivative $y'(x) = -\dfrac{2}{3} x^{-\frac{1}{3}}$ exists at all non-zero points of $[-1, 1]$ and does not exist at $x = 0$. Moreover, the slope of AB is

$$\frac{f(1) - f(-1)}{1 + 1} = 0$$

and $y'(c) = -\dfrac{2}{3}(c)^{-\frac{1}{3}} \neq 0$ for any finite c.

Mean value theorem does not hold in this case because the derivative $f'(x)$ fails to exist at a point of $(-1, 1)$, $y = f(x)$ does not have a tangent at $x = 0$.

SAQ 5

Verify the condition of mean value theorem in the following examples. In each case, find c in the interval as stated by the mean value theorem

(i) $\sin x$ on $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$

(ii) $x^3 - 2x^2 - x + 3$ on $[0, 1]$.

2.8 CURVE SKETCHING

In this section, we use the results of differential calculus to sketch some curves. Then results will be used to find

(i) in which intervals is the curve increasing.

(ii) in which intervals is the curve decreasing.

(iii) at which points the curve takes turn.

We can use these points together with the observations of symmetry to sketch the curve.

Example 2.27

Draw the graph of the curve $f(x) = x^2 - 4x$.

Solution

$$f(x) = x^2 - 4x$$

f being a polynomial function is differentiable for all values of R so the curve is continuous.

$$f'(x) = 2x - 4 = 2(x - 2)$$

For $x \geq 2$, f is an increasing function and for $x \leq 2$, f is a decreasing function. Also $f'(x) = 0$ for $x = 2$, i.e. the tangent to the curve is parallel to x-axis for
$x = 2$. We construct a table of values of x only as under

x	– 2	– 1	0	1	2	3	4	5
y	12	5	0	– 3	– 4	– 3	0	5

Plot the points and the graph as shown in Figure 2.14.

Figure 2.14

Example 2.28

Draw the graph of the curve $y = \sin^2 x$.

Solution

The function is continuous and differentiable for all value of $x \in R$. Also, it is symmetrical about y-axis as the equation remains unchanged when x is changed to $-x$.

$$f'(x) = 2\sin x \cos x = \sin 2x$$

Hence

$$f'(x) > 0$$

when

$$0 < x < \frac{\pi}{2},\ \ \pi < x < \frac{3\pi}{2},\ \ 2\pi < x < \frac{5\pi}{2}, \ldots$$

and $\quad f'(x) < 0$

when $\quad \frac{\pi}{2} < x < \pi,\ \ \frac{3\pi}{2} < x < 2\pi, \ldots$

i.e. $f(x)$ is increasing in the intervals $\left[0, \frac{\pi}{2}\right], \left[\pi, \frac{3\pi}{2}\right], \left[2\pi, \frac{5\pi}{2}\right], \ldots$ and decreasing in $\left[\frac{\pi}{2}, \pi\right], \left[\frac{3\pi}{2}, 2\pi\right], \ldots.$

Also,

$$f'(x) = 0$$

when

$$2x = 0, \pi, 2\pi, 3\pi, \ldots$$

i.e. $\quad x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$

A sketch of the curve is as follows :

Figure 2.15

2.9 SUMMARY

The important points covered in this unit are

- Given the curve $y = f(x)$, $\frac{dy}{dx}$ represents the slope of the tangent to the curve at the point (x, y).
- The equation of the tangent to the curve $y = f(x)$ at the point (x_1, y_1) is $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} (x - x_1)$.
- The equation of the normal to the curve $y = f(x)$ at the point (x_1, y_1) is $(x - x_1) + (y - y_1)\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = 0$.
- The time rate of change of displacement is called velocity. The time rate of change of velocity is called acceleration, i.e.
  $v = \frac{ds}{dt}, a = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$.
- Let f be a real function and I be a subset of R, then f is called an increasing (decreasing) function on I iff $x_1, x_2 \in I$,
  $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ $(f(x_1) \geq f(x_2))$.
- If $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ $(f(x_1) > f(x_2))$ then f is called a strictly increasing (strictly decreasing) function on I.
- $f(x)$ has a maximum at $x = x_0$ if $f'(x_0) = 0$ and $f''(x_0) < 0$.
- $f(x)$ has a minimum at $x = x_0$ if $f'(x_0) = 0$ and $f''(x_0) > 0$.
- We have studied Rolles theorem and learned the geometrical meaning of it.
- Learned the Mean value theorem and its applications.

2.10 ANSWERS TO SAQs

SAQ 1

(a) Equation of the tangent is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

and equation of the normal is $\frac{xy_1}{b^2} - \frac{yx_1}{a^2} = x_1\ y_1 \left(\frac{1}{b^2} - \frac{1}{a^2}\right)$.

(b) Tangent is parallel to x-axis at $x = \frac{4}{3}, y = \frac{3^2}{3}$.

(d) $\tan \alpha = -\frac{21}{20}$ at the point (5, 2).

SAQ 2

(a) Local minimum value $f(0) = 2$.

Local maximum value $f(-4) = -6$.

(b) (i) Minimum = – 15

Maximum = 5

(ii) $x = 2$ is minima.

$x = \frac{4}{3}$ is a maxima.

SAQ 3

(c) Extreme values are 1, – 1.

SAQ 4

(b) $h = \frac{a}{2}$, where h is the vertical side and a is the side of the square base.

(c) $h = 2\,r$, where h is the height and r is the radius of the base.

(d) $(a, \sqrt{2\,pa}),\ (a, -\sqrt{2\,pa})$.

(e) (i) Yes. Rolle's theorem is applicable, $c = \frac{\pi}{2}$.

(ii) Yes, $c = 0$.

(iii) Yes, $c = \frac{\pi}{4}$.

(iv) No.

SAQ 5

(i) $c = \cos^{-1}\left(\frac{3\pi}{2}\right)$.

(ii) $c = \frac{1}{3}$.