UNIT 3 INDEFINITE INTEGRALS

Structure

3.1 INTRODUCTION

So far we have concentrated only on that part of calculus which is based on the operation of the derivative, namely, **differential calculus**. The second major operation of the calculus is integral calculus. The name **integral calculus** originated in the process of summation, and the word **integrate** literally means **find the sum of**. Historically, the subject arose in connection with the determination of areas of plane regions. But in the seventeenth century it was realized that integration can also be viewed as the inverse of differentiation. Integral calculus consists in developing methods for the determination of integrals of any given function.

The relationship between the derivative and the integral of a function is so important that mathematicians have labelled the theorem that describes the relationship as the **Fundamental Theorem of Integral Calculus**.

In this unit, we shall introduce the notions of antiderivatives, indefinite integral and various methods and techniques of integration. The next unit will cover definite integrals which can be evaluated using these methods.

Objectives

After studying this unit, you should be able to

- compute the antiderivative of a given function,
- define the indefinite integral of a function,
- evaluate certain standard integrals by finding the antiderivatives of the integrals,

- compute integrals of various elementary and trigonometric functions,
- integrate rational functions of a variable by using the method of partial fractions, and
- evaluate the integrals of some specified types of irrational functions.

3.2 ANTIDERIVATIVES

So far, we have been occupied with the 'derivative problem' that is, the problem of finding the derivative of a given function. Some of the important applications of the calculus lead to the inverse problem, namely, given the derivative of a function, is it possible to find the function? This process is called **antidifferentiation** and the result of antidifferentiation is called an **antiderivative**. The importance of the antiderivative results partly from the fact that scientific laws often specify the rates of change of quantities. The quantities themselves are then found by antidifferentiation.

To get started, suppose we are given that $f\ '(x) = 9$, can we find $f(x)$? It is easy to see that one such function f is given by $f(x) = 9x$, since the derivative of $9x$ is 9.

Before making any definite decision, consider the functions

$$9x + 4,\ 9x - 10,\ 9x + \sqrt{3}$$

Each of these functions has 9 as its derivative. Thus, not only can $f(x)$ be $9x$, but it can also be $9x + 4$ or $9x - 10, 9x + \sqrt{3}$. Not enough information is given to help us determine which is the correct answer.

Let us look at each of these possible functions a bit more carefully. We notice that each of these functions differs from another only by a constant. Therefore, we can say that if $f\ '(x) = 9$, then $f(x)$ must be of the form $f(x) = 9x + c$, where c is a constant. We call $9x + c$ the antiderivative of 9.

More generally, we have the following definition.

Definition 1

Suppose f is a given function. Then a function F is called an antiderivative of f, if $F'(x) = f(x)\ \forall x$.

We now state an important theorem without giving its proof.

Theorem 1

If F_1 and F_2 are two antiderivatives of the same function, then F_1 and F_2 differ by a constant, that is

$$\boldsymbol{F_1(x) = F_2(x) + c}$$

Remark

From above Theorem, it follows that we can find all the antiderivatives of a given function, once we know one antiderivative of it. For instance, in the above example, since one antiderivative of 9 is $9x$, all antiderivative of 9 have the form $9x + c$, where c is a constant. Let us do one example.

Example 3.1

Find all the antiderivatives of $4x$.

Solution

We have to look for a function F such that $F'(x) = 4x$. Now, an antiderivative of $4x$ is $2x^2$. Thus, by Theorem 3.1, all antiderivatives of $4x$ are given by $2x^2 + c$, where c is a constant.

SAQ 1

Find all the antiderivatives of each of the following function

(i) $f(x) = 10x$

(ii) $f(x) = 11x^{10}$

(iii) $f(x) = -5x$

3.3 BASIC DEFINITIONS

We have seen, that the antiderivative of a function is not unique. More precisely, we have seen that if a function F is an antiderivative of a function f, then $F + c$ is also an antiderivative of f, where c is any arbitrary constant. Now we shall introduce a notation here : we shall use the symbol $\int f(x)\,dx$ to denote the class of all antiderivatives of f. We call it the indefinite integral or just the integral of f. Thus, if $F(x)$ is an antiderivative of $f(x)$, then we can write $\int f(x)\,dx = F(x) + c$.

This c is called the constant of integration. The function $f(x)$ is called the integrand, $f(x)\,dx$ is called the element of integration and the symbol $\int$ stands for the integral sign.

The indefinite integral $\int f(x)\,dx$ is a class of functions which differ from one another by constant. It is not a definite number; it is not even a definite function. We say that the indefinite integral is unique up to an arbitrary constant.

Thus, having defined an indefinite integral, let us get acquainted with the various techniques for evaluating integrals.

3.3.1 Standard Integrals

We give below some elementary standard integrals which can be obtained directly from our knowledge of derivatives.

Table 3.1

Sl. No.	Function	Integral
1	x^n	$\frac{x^{n+1}}{n+1} + c, n \neq -1$
2	$\sin x$	$-\cos x + c$
3	$\cos x$	$\sin x + c$
4	$\sec^2 x$	$\tan x + c$
5	$\operatorname{cosec}^2 x$	$-\cot x + c$
6	$\sec x \tan x$	$\sec x + c$
7	$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x + c$
8	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + c$ or $-\cos^{-1} x + c$
9	$\frac{1}{1+x^2}$	$\tan^{-1} x + c$ or $-\cot^{-1} x + c$
10	$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1} x + c$ or $-\operatorname{cosec}^{-1} x + c$
11	$\frac{1}{x}$	$\ln \lvert x \rvert + c$
12	e^x	$e^x + c$
13	a^x	$\frac{a^x}{\ln\lvert a\rvert} + c$

Now let us see how to evaluate some functions which are linear combination of the functions listed in Table 3.1.

3.3.2 Algebra of Integrals

You are familiar with the rule for differential which says

$$\frac{d}{dx}[a\,f(x) + b\,g(x)] = a\frac{d}{dx}[f(x)] + b\frac{d}{dx}[g(x)]$$

There is a similar rule for integration :

Rule 1

$$\int [a\,f(x) + b\,g(x)]\,dx = a\int f(x)\,dx + b\int g(x)\,dx$$

This rule follows from the two theorems.

Theorem 2

If f is an integrable function, then so is $K f(x)$ and $\int K\,f(x)\,dx = K\int f(x)\,dx$.

Proof

Let $\int f(x)\,dx = F(x) + c$

Then by definition, $\frac{d}{dx}[F(x)+c] = f(x)$

$\therefore \quad \frac{d}{dx}[K\{F(x)+c\}] = K\,f(x)$

Again, by definition, we have

$$\int K\,f(x)\,dx = K[F(x)+c] = K\int f(x)\,dx$$

Theorem 3

If f and g are two integrable functions, then $f+g$ is integrable, and we have $\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$.

Proof

Let $\int f(x)\,dx = F(x)+c, \int g(x)\,dx = G(x)+c_1$

Then $\frac{d}{dx}[\{F(x)+c\}+\{G(x)+c_1\}] = f(x)+g(x)$

Thus, $\int [f(x)+g(x)]\,dx = [F(x)+c]+[G(x)+c_1]$

$$= \int f(x)\,dx + \int g(x)\,dx$$

Rule (1) may be extended to include a finite number of functions, that is, we can write

Rule 2

$$\int [K_1 f_1(x) + K_2 f_2(x) + \ldots + K_n f_n(x)]\,dx$$

$$= K_1\int f_1(x)\,dx + K_2\int f_2(x)\,dx + \ldots + K_n\int f_n(x)\,dx$$

We can make use of Rule (2) to evaluate certain integrals which are not listed in Table 3.1.

Example 3.2

Let us evaluate $\int (2+4x+3\sin x+4e^x)\,dx$

$$= 2\int dx + 4\int x\,dx + 3\int \sin x\,dx + 4\int e^x\,dx$$

$$= 2x + 2x^2 - 3\cos x + 4e^x + c$$

Example 3.3

Suppose we want to evaluate $\int \frac{(1-x)^2}{x\sqrt{x}}\,dx$

Thus, $\int \frac{(1-x)^2}{x\sqrt{x}}\,dx$

$$= \int \frac{1-2x+x^2}{x^{\frac{3}{2}}}\,dx$$

$$= \int x^{-\frac{3}{2}}\, dx - \int 2x^{-\frac{1}{2}}\, dx + \int x^{\frac{1}{2}}\, dx$$

$$= -2x^{-\frac{1}{2}} - 4x^{\frac{1}{2}} + \frac{2}{3} x^{\frac{3}{2}} + c$$

And now some exercises for you.

SAQ 2

Write down the integrals of the following using Table 3.1 and Rule 2

(i) (a) x^8 (b) $x^{-\frac{5}{2}}$ (c) $4x^{-2}$ (d) 9

(ii) (a) $x^2 - x - 1$ (b) $\frac{1}{\sqrt{x}} - 3\sqrt{x}$ (c) $\left(x - \frac{1}{x}\right)^2$

(iii) (a) $e^x + e^{-x} + 4$ (b) $4\cos x - 3\sin x + e^x + x$ (c) $4\operatorname{sech}^2 x + e^x - 8x$

(iv) (a) $\frac{2}{\sqrt{1 - x^2}} + \frac{5}{x}$ (b) $\frac{2x^2 + 5}{x^2 + 1}$

(v) (a) $ax^3 + bx^2 + cx + d$ (b) $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$

(vi) (a) $\frac{\sin^4 x + \cos^4 x}{\sin^2 x \cos^2 x}$ (b) $(2 + x)(3 - \sqrt{x})$

3.4 METHODS OF INTEGRATION

We have seen in Section 3.3 that the decomposition of an integrand into the sum of a number of integrands, with known integrals, is itself an important method of integration.

We now give two general methods of integration, namely,

(i) Integration by substitution,

(ii) Integration by parts.

The method of substitution consists in expressing the integral $\int f(x)\, dx$ in terms of another simpler integral, $\int F(t)\, dt$, say where the variables x and t are connected by some suitable relation $x = \phi(t)$.

The method of integration by parts enables one to express the given integral of a product of two functions in terms of another, whose integration may be simpler.

3.4.1 Integration by Substitution

Consider the following integral

$$\int f'[g(x)]\, g'(x)\, dx \qquad \ldots (1)$$

Since $\frac{d}{dx} f[g(x)] = f'[g(x)]\, g'(x)$ (by Chain rule)

$\therefore$ $\int f'[g(x)]\, g'(x)\, dx$

$= f[g(x)] + c$

In Eq. (1), if we substitute $g(x) = t$

Then $g'(x) = \frac{dt}{dx}$

i.e. $g'(x)\, dx = dt$

Hence $f'[g(x)]\, g'(x)\, dx$

$= f'(t)\, dt$

$\therefore$ $\int f'[g(x)]\, g'(x)\, dx$

$= \int f'(t)\, dt$

$= f(t) + c$

$= f[g(x)] + c$

Let us now illustrate this technique with examples.

Example 3.4

Find $\int (x^2+1)^3\, 2x\, dx$

Solution

Let $t = x^2 + 1$

$dt = 2x\, dx$

Therefore, $\int (x^2+1)^3\, 2x\, dx$

$= \int t^3\, dt$

$= \frac{t^4}{4} + c$

Thus, $\int (x^2+1)^3\, 2x\, dx$

$= \frac{1}{4}(x^2+1)^4 + c$, since $t = x^2 + 1$

Example 3.5

Find $\int x^3 e^{x^4}\, dx$.

Solution

Let $t = x^4$

Then $dt = 4x^3\,dx$

Therefore, $\int x^3 e^{x^4}\,dx$

$$= \frac{1}{4}\int 4x^3 e^{x^4}\,dx$$

$$= \frac{1}{4}\int e^t\,dt$$

$$= \frac{1}{4}\left[e^t + c\right]$$

$$= \frac{1}{4}\left[e^{x^4} + c\right]$$

Some Typical Examples of Substitution

We now consider the integral $\int f(x)\,dx$, where the integrand $f(x)$ is in some typical form and the integral can be obtained easily by the method of substitution.

Various forms of integral can be obtained easily by the method of substitution. Various forms of integrals considered are as follows :

(a) $\int \boldsymbol{f(ax+b)\,dx}$

To integrate $f(ax+b)$, put $ax+b=t$

Therefore $a\,dx = dt$ or $dx = \frac{1}{a}\,dt$

Thus $\int f(ax+b)\,dx = \frac{1}{a}\int f(t)\,dt$

which can be evaluated, once the right hand side is known, for example, to find $\int \cos(ax+b)\,dx$, we put $ax+b=t$ and $a\,dx = dt$ or $dx = \frac{1}{a}\,dt$.

Then $\int \cos(ax+b)\,dx = \frac{1}{a}\int \cos t\,dt = \frac{1}{a}\sin t + c$

or $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

Similarly, we have the following results

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)\,a} + c,\ n \neq -1$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b) + c$$

$$\int e^{ax+b}\, dx = \frac{1}{a} e^{ax+b} + c$$

$$\int \sec^2 (ax + b)\, dx = \frac{1}{a} \tan (ax + b) + c \text{ etc.}$$

You can make direct use of the above results in solving exercises.

(b) $\int \boldsymbol{f(x)^n\ x^{n-1}\ dx}$

To integrate $f(x)^n\ x^{n-1}$, we let $x^n = t$.

Then $nx^{n-1}\, dx = dt$

and $\int f(x)^n\ x^{n-1}\, dx = \frac{1}{n} \int f(t)\, dt$

which can be found out once the right hand side is known.

For example, to find $\int x^2 \sin x^3\, dx$, put $x^3 = t$; then $3x^2\, dx = dt$, that is $x^2\, dx = \frac{1}{3}\, dt$.

Then, $\int x^2 \sin x^3\, dx$

$$= \frac{1}{3} \int \sin t\, dt$$

$$= -\frac{1}{3} \cos t + c$$

$$= -\frac{1}{3} \cos x^3 + c$$

(c) $\int \boldsymbol{\{f(x)\}^n\ f'(x)\, dx,\ n \neq -1}$

Putting $f(x) = t$; we see that $f'(x)\, dx = dt$ and

$$\int \{f(x)\}^n\ f'(x)\, dx = \int t^n\, dt = \frac{t^{n+1}}{n+1} + c$$

$$= \frac{\{f(x)\}^{n+1}}{n+1} + c$$

For example, $\int \cos^2 x \sin x dx$

$= -\int t^2\, dt$, where $t = \cos x$ (and hence $-dt = \sin x\, dx$)

Therefore, $\int \cos^2 x \sin x dx$

$$= -\frac{1}{3} t^3 + c$$

$$= -\frac{1}{3} (\cos x)^3 + c$$

(d) $\int \boldsymbol{\frac{f'(x)}{f(x)}\, dx}$

Putting $f(x) = t$, we have $f'(x)\,dx = dt$

and $$\int \frac{f'(x)}{f(x)}\,dx = \frac{dt}{t} = \ln|t| + c = \ln|f(x)| + c$$

i.e. the integral of a function in which the numerator is the differential co-efficient of the denominator, is equal to the logarithm of the denominator (plus a constant).

For example, applying this result, we have

$$\int \frac{\sin x}{\cos x}\,dx = -\int \frac{(-\sin x)}{\cos x}\,dx = c - \ln \cos x$$

Since $f(x) = \cos x$, in this case.

Therefore, $\int \tan x\,dx = c - \ln|\cos x|$

Similarly, you can obtain the following integrals

$$\int \cot x\,dx = \ln|\sin x| + c$$

$$\int \sec x\,dx = \ln|(\sec x + \tan x)| + c$$

$$\int \operatorname{cosec} x\,dx = \ln\left|\tan\left(\frac{x}{2}\right)\right| + c$$

Remember that logarithm of a quantity exists only when the quantity is positive. Thus, while making use of these formulas, make sure that the integrand to be integrated is positive in the domain under consideration.

(e) $\boldsymbol{\int f(a^2 \pm x^2)\,dx}$

Under this category we now give some results obtained by putting $x = a\,t$; and hence $dx = a\,dt$.

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left(\frac{x + \sqrt{a^2 + x^2}}{a}\right) + c$$

We usually use all these integrals given under (A) – (E) directly whenever required without actually proving them. Sometimes it may happen that two or more substitutions have to be used in succession. We now illustrate this point with the help of the following example.

Example 3.6

Calculate

(i) $\displaystyle\int \frac{2x}{1+x^2}\,dx$

(ii) $\displaystyle\int \sin^3 x \cos^2 x\,dx$

Solution

(i) $\displaystyle\int \frac{2x}{1+x^2}\,dx$

Put $1 + x^2 = t$

Then $2\,x\,dx = dt$

$$\therefore \quad \int \frac{2x}{1+x^2}\,dx = \int \frac{1}{t}\,dt$$

$$= \ln t + c$$

$$= \ln (1 + x^2) + c$$

(ii) $\displaystyle\int \sin^3 x \cos^2 x\,dx$

$$= \int \sin^2 x \cos^2 x \sin x\,dx$$

$$= \int \cos^2 x\,(1 - \cos^2 x) \sin x\,dx$$

Put $\cos x = t$

Then $-\sin x\,dx = dt$

$$\therefore \quad \int \sin^3 x \cos^2 x\,dx = -\int t^2\,(1 - t^2)\,dt$$

$$= -\int (t^2 - t^4)\,dt$$

$$= -\left[\frac{t^3}{3} - \frac{t^5}{5}\right] + c$$

$$= \frac{t^5}{5} - \frac{t^3}{3} + c$$

$$= \frac{(\cos x)^5}{5} - \frac{(\cos x)^3}{3} + c$$

So far we have developed the method of integration by substitution, by turning the chain rule into an integration formula. Let us do the same for the product rule. We know that the derivative of the product of two functions $f(x)$ and $g(x)$ is given by

$$\frac{d}{dx}[f(x)\,g(x)] = g(x)\,f'(x) + f(x)\,g'(x),$$

where the dashes denotes differentiation w. r. t. x. Corresponding to this formula, we have a rule called integration by parts.

3.4.2 Integration by Parts

Let us now discuss the method of integration by parts in detail. We begin by taking two functions $f(x)$ and $g(x)$. Let $G(x)$ be an antiderivative of $g(x)$, that is,

$$\int g(x)\,dx = G(x) \text{ or } G'(x) = g(x)$$

Then, by the product rule for differentiation, we have

$$\frac{d}{dx}[f(x)\,G(x)] = f(x)\,G'(x) + f'(x)\,G(x) = f(x)\,g(x) + f'(x)\,G(x)$$

Integrating both sides, we get

$$f(x)\,G(x) = \int f(x)\,g(x)\,dx + \int f'(x)\,G(x)\,dx$$

or $$\int f(x)\,g(x)\,dx = f(x)\,G(x) - \int f'(x)\,G(x)\,dx$$

Thus, $$\int f(x)\,g(x)\,dx = f(x)\int g(x)\,dx - \int f'(x)\,\{\int g(x)\,dx\}\,dx \quad \ldots (3.1)$$

The integration done by using the Eq. (3.1) is called integration by parts. In other words, it can be stated as follows :

The integral of the product of two functions

= first function × integral of the second function

– integral of (differential coefficient of the first × integral of the second).

We now illustrate this method through some examples.

Example 3.7

Integrate $x\,e^x$ with respect to x.

Solution

We use integration by parts.

Step 1

Take $f(x) = x$ and $g(x) = e^x$.

Then $f'(x) = 1$ and $\int e^x\,dx = e^x$

Step 2

By formula (3.1) we have

$$\int x\,e^x\,dx = x\,e^x - \int 1\,.\,e^x\,dx + c$$

or $$\int x\,e^x\,dx = x\,e^x - e^x + c$$

Sometimes we need to integrate by parts more than once. We now illustrate it through the following example.

Example 3.8

$$\int x^2 \cos x \, dx$$

Solution

$$\int x^2 \cos x \, dx = x^2 \int \cos x \, dx - \int 2x \{\int \cos x \, dx\} \, dx$$

$$= x^2 \sin x - 2 \int x \sin x \, dx + c_1 \qquad \ldots (3.2)$$

where c_1 is a constant of integration.

Integrating $\int x \sin x \, dx$, again by parts, we get

$$\int x \sin x \, dx = x \int \sin x \, dx - \int 1 . \{\int \sin x \, dx\} \, dx$$

$$= - x \cos x + \int \cos x \, dx$$

$$= - x \cos x + \sin x + c_2 \qquad \ldots (3.3)$$

where c_2 being the constant of integration. From Eqs. (3.2) and (3.3), we get

$$\int x^2 \cos x \, dx = x^2 \sin x - 2 (- x \cos x + \sin x + c_2) + c_2 ,$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + c ,$$

where we have written c for $c_1 - 2c_2$.

We now consider some examples of integrals which occur quite frequently and can be integrated by parts.

Example 3.9

Find $\int e^{ax} \cos bx \, dx$

Solution

Step 1

Choose $f(x) = e^{ax}$ and $g(x) = \cos bx$; then integrating by parts gives

$$\int e^{ax} \cos bx \, dx$$

$$= e^{ax} \frac{\sin bx}{b} - \int a e^{ax} \frac{\sin bx}{b} \, dx + c_1 \qquad \ldots (3.4)$$

Step 2

Integrating $\int e^{ax} \sin bx \, dx$ by parts again, we get

$$\int e^{ax} \sin bx \, dx = e^{ax} \frac{(- \cos bx)}{b} - \int a e^{ax} \frac{(- \cos bx)}{b} \, dx + c_2$$

$$= - \frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int e^{ax} \cos bx \, dx + c_2$$

Note that the second term on the right hand side is nothing but a constant multiple of the given integral.

Step 3

Substituting the value of $\int e^{ax} \sin bx\, dx$, in Eq. (3.4), we have

$$\int e^{ax} \cos bx\, dx = e^{ax} \frac{\sin bx}{b} - \frac{a}{b}\left[-\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int e^{ax} \cos bx\, dx + c_2\right] + c_1$$

$$= e^{ax} \frac{\sin bx}{b} + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx\, dx + c_3 \qquad \ldots (3.5)$$

where $\quad c_3 = c_1 - \frac{a}{b} c_2$

Step 4

Transposing the last term from the right of Eq. (3.5) to left, we get

$$\left(1 + \frac{a^2}{b^2}\right) \int e^{ax} \cos bx\, dx = \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx + c_3$$

Dividing by $\left(1 + \frac{a^2}{b^2}\right)$, we finally get

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2} (b \sin bx + a \cos bx) + c\,,$$

where $c = \frac{c_3}{a^2 + b^2}$, as the required integral.

Similarly, the integral of the type $\int e^{ax} \sin bx\, dx$ can be obtained.

And now some exercise for you.

SAQ 3

(a) Evaluate the following integrals :

(i) $\int \frac{dx}{9x^2 - 12x + 8}$

(ii) $\int \frac{x\, dx}{x^4 + x^2 + 1}$

(iii) $\int (\tan x)^5 \sec^2 x\, dx$

(iv) $\int \frac{dx}{e^x + 1}$

(v) $\int \frac{\cot x}{\ln \sin x}\, dx$

(vi) $\int \frac{1}{e^x - 1}\, dx$

(vii) $\int \frac{dx}{(e^x + e^{-x})^2}$

(viii) $\int x \sec^2 x^2 \, dx$

(ix) $\int \frac{(\sin^{-1} x)^2}{\sqrt{1 - x^2}}\, dx$

(x) $\int \frac{(1 + \ln x)^3}{x}\, dx$

(xi) $\int \frac{(\text{cosec}^2 x)}{(1 + \cot x)}\, dx$

(b) Evaluate

(i) $\int x^2 \ln x \, dx$

(ii) $\int x \,\text{cosec}^2\, x \, dx$

(iii) $\int e^{3x} \cos 4x \, dx$

(iv) $\int \sin^{-1} x \, dx$

(v) $\int x \tan^{-1} x \, dx$

(vi) $\int \frac{x \sin^{-1} x}{\sqrt{(1 - x^2)}}\, dx$

(vii) $\int \frac{x\, e^x}{(1 + x^2)}\, dx$

3.5 INTEGRATION OF RATIONAL FUNCTIONS

We know, by now, that it is easy to integrate any polynomial function, that is, a function *f* given by $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$. In this section, we shall see how a rational function can be integrated.

Definition

A function R is called a rational function if it is given by $R(x) = \frac{Q(x)}{P(x)}$,

where Q (x) and P (x) are polynomials. It is defined for all x for which P (x) $\neq 0$. *If the degree of Q (x) is less than the degree of P (x), we say that R (x) is a proper rational function. Otherwise, it is called an improper rational function.*

Thus $f(x) = \frac{x+1}{x^2+x+2}$ is a proper rational function, and $g(x) = \frac{x^3+x+5}{x-2}$ is an improper one. But $g(x)$ can also be written as

$g(x) = (x^2 + 3x + 6) + \frac{17}{x-2}$ (by long division).

Here we have expressed $g(x)$, which is an improper rational function, as the sum of a polynomial and a proper rational function. This can be done for any improper rational function.

3.5.1 Some Simple Rational Functions

Now we shall consider some simple types of proper rational functions, like $\frac{1}{x-a}, \frac{1}{(x-b)^k}$ and $\frac{x-m}{ax^2+bx+c}$. We shall illustrate the method of integrating these functions through some examples.

Example 3.10

Consider the function $f(x) = \frac{1}{(x+2)^4}$.

Solution

To integrate this function we shall use the method of substitution.

Thus, if we put $u = x + 2$ or $\frac{du}{dx} = 1$, and we can write

$$\int \frac{1}{(x+2)^4}\,dx = \int \frac{1}{u^4}\,du = \frac{u^{-3}}{-3} + c = -\frac{1}{3(x+2)^3} + c.$$

Example 3.11

Consider the function $f(x) = \frac{2x+3}{x^2-4x+5}$.

Solution

This has a quadratic polynomial in the denominator. Now

$$\int \frac{2x+3}{x^2-4x+5}\,dx = \int \frac{2x-4}{x^2-4x+5}\,dx + \int \frac{7}{x^2-4x+5}\,dx.$$

Perhaps you are wondering why we have split the integral into two parts.

The reason for this break-up is that now the integrand in the first integral on the right is of the form $\frac{g'(x)}{g(x)}$; and we know that $\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + c$.

Thus $\int \frac{2x-4}{x^2-4x+5}\,dx = \ln|x^2-4x+5| + c_1$.

To evaluate the second integral on the right, we write

$$\int \frac{1}{x^2-4x+5}\,dx = \int \frac{1}{(x^2-4x+4)+1}\,dx = \int \frac{1}{(x-2)^2+1}\,dx .$$

Now, if we put $u = x-2, \frac{du}{dx} = 1$ and

$$\int \frac{1}{x^2-4x+5}\,dx = \int \frac{1}{u^2+1}\,du = \tan^{-1} u + c_2 = \tan^{-1}(x-2) + c_2$$

This implies, $\int \frac{2x+3}{x^2-4x+5}\,dx = \ln|x^2-4x+5| + 7\tan^{-1}(x-2) + c$.

3.5.2 Partial Fraction Decomposition

You must have studied the factorisation of polynomials. For example, we know that

$$x^2 - 5x + 6 = (x-2)(x-3)$$

Here $(x-2)$ and $(x-3)$ are two linear factors of x^2-5x+6.

You must have also come across polynomial like x^2+x+1, which cannot be factorised into real factors. Thus, it is not always possible to factorise a given polynomial into linear factors. But any polynomial can, in principle, be factorised into linear and quadratic factors. We shall not prove this statement here. It is a consequence of the fundamental theorem of algebra. The actual factorization of a polynomial may not be very easy to carry out. But, whenever we can factorise the denominator of a proper rational function, we can integrate it by employing the method of partial fractions. The following examples will illustrate this method.

Example 3.12

Let us evaluate $\int \frac{5x-1}{x^2-1}\,dx$. Here the integrand $\frac{5x-1}{x^2-1}$ is a proper rational function.

Its denominator x^2-1 can be factored into linear factors as :

$$x^2-1 = (x-1)(x+1)$$

This suggests that we can write the decomposition of $\frac{5x-1}{x^2-1}$ into partial fraction as :

$$\frac{5x-1}{x^2-1} = \frac{5x-1}{(x-1)(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)}$$

If we multiply both sides by $(x-1)(x+1)$, we get

$$5x-1 = A(x+1) + B(x-1).$$

That is $\quad 5x-1 = (A+B)x + (A-B)$

By equalling the coefficients of x, we get $A + B = 5$. Equating the constant terms on both sides, we get $A – B = – 1$.

Solving these two equations in A and B, we get $A = 2$ and $B = 3$.

Thus, $$\frac{5x-1}{x^2-1} = \frac{2}{x-1} + \frac{3}{x+1}$$

Integrating both sides of this equation, we obtain

$$\int \frac{5x-1}{x^2-1}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{3}{x+1}\,dx$$

$$= 2 \ln|x-1| + 3 \ln|x+1| + c$$

Let us go to our next example now.

Example 3.13

Take a look at the denominator of the integrand in $\int \frac{x}{x^3 - 3x + 2}\,dx$.

It factors into $(x-1)^2\ (x+2)$. The linear factor $(x – 1)$ is repeated twice in the decomposition of $x^3 - 3x + 2$. In this case, we write

$$\frac{x}{x^3-3x+2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

From this point, we proceed as before to find A, B and C. We get

$$x = A\,(x-1)^2 + B\,(x+2)\,(x-1) + C\,(x+2)$$

We put $x = 1$, and $x = – 2$, and get $C = \frac{1}{3}$ and $A = -\frac{2}{9}$. Then to find B, let us put any other convenient value, say $x = 0$.

This gives us $\quad 0 = A – 2B + 2C$

or $$0 = -\frac{2}{9} - 2B + \frac{2}{3}$$

This implies $B = \frac{2}{9}$

Thus $$\int \frac{x}{x^3-3x+2}\,dx = -\frac{2}{9}\int \frac{1}{x+2}\,dx + \frac{2}{9}\int \frac{1}{x-1}\,dx + \frac{1}{3}\int \frac{1}{(x-1)^2}\,dx$$

$$= -\frac{2}{9}\ln|x+2| + \frac{2}{9}\ln|x-1| - \frac{1}{3}\,\frac{1}{(x-1)} + c$$

$$= \frac{2}{9}\ln\left|\frac{x-1}{x+2}\right| - \frac{1}{3\,(x-1)} + c$$

In our next example, we shall consider the case when the denominator of the integrand contains an irreducible quadratic factors (i.e. a quadratic factor which cannot be further factored into linear factors).

Example 3.14

To evaluate

$$\int \frac{6x^3 - 11x^2 + 5x - 4}{x^4 - 2x^3 + x^2 - 2x}\, dx$$

We factorise $x^4 - 2x^3 + x^2 - 2x$ as $x(x-2)(x^2+1)$.

Then we write $\int \frac{6x^3 - 11x^2 + 5x - 4}{x^4 - 2x^3 + x^2 - 2x} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{Cx + D}{x^2+1}$

Thus

$$6x^3 - 11x^2 + 5x - 4 = A(x-2)(x^2+1) + Bx(x^2+1) + (Cx + D)\,x\,(x-2)$$

Next, we substitute $x = 0$, and $x = 2$, to get $A = 2$ and $B = 1$. Then we put $x = 1$ and $x = -1$ (Some Convenient values) to get $C = 3$ and $D = -1$.

Thus, $\int \frac{6x^3 - 11x^2 + 5x - 4}{x^4 - 2x^3 + x^2 - 2x}\, dx = 2\int \frac{1}{x}\, dx + \int \frac{1}{x-2}\, dx + \int \frac{3x-1}{x^2+1}\, dx$

$$= 2 \ln|x| + \ln|x-2| + \frac{3}{2}\int \frac{2x}{x^2+1}\, dx - \int \frac{dx}{x^2+1}$$

$$= 2 \ln|x| + \ln|x-2| + \frac{3}{2} \ln \int |x^2+1| - \tan^{-1} x + c$$

Thus, you see, once we decompose integrand, which is a proper rational function, into partial fractions, then the given integral can be written as the sum of some integrals of the type discussed in previous examples.

All the functions which we integrated till now were proper rational functions. Now we shall take up an example of an improper rational function.

Example 3.15

Let us evaluate $\int \frac{x^3 + 2x}{x^2 - x - 2}\, dx$.

Solution

Since the integrand is an improper rational function, we shall first write it as the sum of a polynomial and a proper rational function.

$$\frac{x^3 + 2x}{x^2 - x - 2} = (x+1) + \frac{5x+2}{x^2 - x - 2}$$

Therefore, $\int \frac{x^3 + 2x}{x^2 - x - 2}\, dx = \int (x+1)\, dx + \int \frac{5x+2}{x^2 - x - 2}\, dx$

$$= \int x\, dx + \int dx + 4\int \frac{dx}{x-2} + \int \frac{dx}{x+1}$$

Hence $\int \frac{x^3 + 2x}{x^2 - x - 2}\, dx = \frac{x^2}{2} + x + 4 \ln|x-2| + \ln|x+1| + c$

Try to do the following exercise now.

SAQ 4

Evaluate

(i) $\int \frac{x}{x^2 - 2x - 3}\, dx$

(ii) $\int \frac{3x - 13}{x^2 + 3x - 10}\, dx$

(iii) $\int \frac{6x^2 + 22x - 23}{(2x - 1)(x^2 + x - 6)}\, dx$

(iv) $\int \frac{x^2 + x - 1}{(x - 1)(x^2 - x + 1)}\, dx$

(v) $\int \frac{x^2}{x^2 - a^2}\, dx$

(vi) $\int \frac{x^2 + 4}{x^2 + 2x + 3}\, dx$

3.6 INTEGRATION OF IRRATIONAL FUNCTIONS

The task of integrating functions gets tougher if the given function is an irrational one, that is, it is not of the form $\frac{Q(x)}{P(x)}$. In this section, we shall give you some tips for evaluating some particular types of irrational functions. In most cases, our endeavour will be to arrive at a rational function through an appropriate substitution. This rational function can then be easily evaluated by using the techniques developed in Section 3.5.

Integration of Functions Containing only Fractional Powers of *x*

In this case, we put $x = t^n$, where n is the lowest common multiple (l. c. m.) of the denominators of powers of x. This substitution reduces the function to a rational function of t.

Look at the following example.

Example 3.16

Let us evaluate $\int \frac{2x^{1/2} + 3x^{1/3}}{1 + x^{1/3}}\, dx$

Solution

We put $x = t^6$, as 6 is the l. c. m. of 2 and 3. We get

$$\int \frac{2x^{1/2} + 3x^{1/3}}{1 + x^{1/3}}\, dx = 6\int \frac{2t^3 + 3t^2}{1+t^2}\, 6t^5\, dt$$

$$= 6\int \frac{2t^8 + 3t^7}{1+t^2}\, dt = 6\int \left[2t^6 + 3t^5 - 2t^4 - 3t^3 + 2t^2 + 3t - 2 - \frac{3t-2}{1+t^2}\right] dt$$

$$= 6\left[\frac{2}{7}t^7 + \frac{1}{2}t^6 - \frac{2}{5}t^5 - \frac{3}{4}t^4 + \frac{2}{3}t^3 + \frac{3}{2}t^2 - 2t - \frac{3}{2}\ln(1+t^2) + 2\tan^{-1} t\right] + c$$

$$= \frac{12}{7}x^{7/6} + 3x - \frac{12}{5}x^{5/6} - \frac{9}{2}x^{2/3} + 4x^{1/2} + 9x^{1/3} - 12x^{1/6} - 9\ln|1+x^{1/3}| + 12\tan^{-1} x^{1/6} + c$$

Integrals of the Types

(i) $\int \sqrt{x^2 - a^2}\, dx$, (ii) $\int \sqrt{x^2 + a^2}\, dx$

(iii) $\int \sqrt{a^2 - x^2}\, dx$, (iv) $\int \sqrt{ax^2 + bx + c}\, dx$

(v) $\int (Px + q)\sqrt{ax^2 + bx + c}\, dx$

Now, let us evaluate the above integrals.

(i) Let $I = \int \sqrt{x^2 - a^2}\, dx$

Integrating by parts taking 1 as the second function, we have

$$I = x\sqrt{x^2 - a^2} - \int x \cdot \frac{1}{2} \cdot \frac{2x}{\sqrt{x^2 - a^2}}\, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2 - a^2}}\, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}}\, dx$$

$$= x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2}\, dx - a^2\int \frac{dx}{\sqrt{x^2 - a^2}}$$

$$= x\sqrt{x^2 - a^2} - I - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right) + c$$

$$\therefore \quad 2I = x\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right) + c$$

$$\therefore \quad I = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\left(\log x + \sqrt{x^2 - a^2}\right) + c$$

Similarly

(ii) $\int \sqrt{x^2 + a^2}\, dx = \frac{1}{2}x\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left(x + \sqrt{x^2 + a^2}\right) + c$, and

(iii) $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(iv) $\int \sqrt{ax^2 + bx + c}\ dx$

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$$

Put $\quad x + \frac{b}{2a} = t,\ \frac{c}{a} - \frac{b^2}{4a^2} = k^2$

Then the integral is reduced to any of the forms (i), (ii) or (iii).

(v) $\int (Px + q)\sqrt{ax^2 + bx + c}\ dx$

Choose constants A and B such that

$$Px + q = A\left[\frac{d}{dx}(ax^2 + bx + c)\right] + B$$

$$= A\,(2ax + b) + B$$

i.e. $\quad 2aA = p,\ Ab + B = q$

Thus the integral is reduced to

$$A\int (2ax + b)\sqrt{ax^2 + bx + c}\ dx + B\int \sqrt{ax^2 + bx + c}\ dx$$

$$= AI_1 + BI_2$$

$$I_1 = \int (2ax + b)\sqrt{ax^2 + bx + c}\ dx$$

Put $\quad ax^2 + bx + c = t$

$(2ax + b)\,dx = dt$

i.e. $\quad I_1 = \frac{2}{3}(ax^2 + bx + c)^{3/2} + c_2$

Similarly, $I_2 = \int \sqrt{ax^2 + bx + c}\ dx$ which can be worked out as in (iv).

$\therefore$ (v) can be determined.

Example 3.17

Evaluate $\int \sqrt{4 - x^2}\ dx$.

Solution

$$\int \sqrt{4 - x^2}\ dx = \int \sqrt{2^2 - x^2}\ dx$$

$$= \frac{1}{2}x\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2} + c.$$

Example 3.18

Evaluate $\int \sqrt{x^2 + 2x + 5}\ dx$

Solution

$$\int \sqrt{x^2 + 2x + 5}\ dx = \int \sqrt{(x+1)^2 + 4}\ dx$$

Put $x + 1 = t$, then $dx = dt$

$$\therefore \quad \int \sqrt{x^2 + 2x + 5}\ dx = \int \sqrt{t^2 + 4}\ dt = \int \sqrt{t^2 + 2^2}\ dt$$

$$= \frac{1}{2} t \sqrt{t^2 + 4} + \frac{1}{2} . 4 \log \left(t + \sqrt{t^2 + 4} \right) + c$$

$$= \frac{1}{2}(x+1) \sqrt{x^2 + 2x + 5} + 2 \log \left[(x+1) + \sqrt{x^2 + 2x + 5} \right] + c$$

Example 3.19

Evaluate $\int x \sqrt{1 + x - x^2}\ dx$

Solution

Let $$x = A \left[\frac{d}{dx} (1 + x - x^2) \right] + B$$

$$= A (1 - 2x) + B$$

$$\therefore \quad A = -\frac{1}{2},\ B = \frac{1}{2}$$

Thus $$\int x \sqrt{1 + x - x^2}\ dx = -\frac{1}{2} \int (1 - 2x) \sqrt{1 + x - x^2}\ dx + \frac{1}{2} \int \sqrt{1 + x - x^2}\ dx$$

$$= -\frac{1}{2} I_1 + \frac{1}{2} I_2$$

$$I_1 = \int (1 - 2x) \sqrt{1 + x - x^2}\ dx$$

Put $(1 + x - x^2)\ dx = dt$

Then $(1 - 2x)\ dx = dt$

$$\therefore \quad I_1 = \int t^{\frac{1}{2}}\ dt = \frac{2}{3} t^{\frac{3}{2}} + c_1$$

$$= \frac{2}{3} (1 + x - x^2)^{\frac{3}{2}} + c_1 \qquad \text{. . . (1)}$$

$$I_2 = \int \sqrt{1 - x - x^2}\ dx = \int \sqrt{\frac{5}{4} - \left(x - \frac{1}{2} \right)^2}\ dx$$

Put $x - \frac{1}{2} = t$, then $dx = dt$

$$\therefore \quad I_2 = \int \sqrt{\frac{5}{4} - t^2}\, dt$$

$$= \frac{1}{2} t \sqrt{\frac{5}{4} - t^2} + \frac{1}{2} \cdot \frac{5}{4} \sin^{-1} \frac{2t}{\sqrt{5}} + c_2$$

$$= \frac{1}{4}(2x-1)\sqrt{1 + x - x^2} + \frac{5}{8} \sin^{-1} \frac{2x-1}{\sqrt{5}} + c_2$$

Hence $\int x \sqrt{1 + x - x^2}\, dx$

$$= \frac{1}{3}(1 + x - x^2)^{\frac{3}{2}} + \frac{1}{8}(2x-1)\sqrt{1 + x - x^2} + \frac{5}{16} \sin^{-1} \frac{2x-1}{\sqrt{5}} + c$$

SAQ 5

Integrate the following functions :

(i) $\sqrt{x^2 + 4x + 6}$

(ii) $(x + 1)\sqrt{2x^2 + 3}$

(iii) $\sqrt{1 + 3x - x^2}$

3.7 INTEGRATION OF TRIGONOMETRIC FUNCTIONS

If the integrand is a rational function of sin x or cos x or both, it can be reduced to a rational function by substituting $t = \tan \frac{x}{2}$.

Then $\frac{dt}{dx} = \sec^2 \frac{x}{2} \cdot \frac{1}{2} = \frac{1 + t^2}{2}$

i.e. $dx = \frac{2dt}{1 + t^2}$

$$\sin x = \frac{2t}{1 + t^2}, \ \cos x = \frac{1 - t^2}{1 + t^2} \text{ as } \sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

and $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$.

Example 3.20

Evaluate $\int \frac{1 + \sin x}{\sin x\,(1 + \cos x)}\,dx$

Solution

Let $\tan \frac{x}{2} = t,$

Then $$\int \frac{1 + \sin x}{\sin x\,(1 + \cos x)}\,dx = \int \frac{1 + \frac{2t}{1 + t^2}}{\frac{2t}{1 + t^2}\left[1 + \frac{1 - t^2}{1 + t^2}\right]} \cdot \frac{2\,dt}{1 + t^2}$$

$$= 2 \int \frac{1 + t^2 + 2t}{2 + [1 + t^2 + 1 - t^2]}\,dt$$

$$= \frac{1}{2} \int \frac{1 + t^2 + 2t}{t}\,dt = \frac{1}{2} \int \left[\frac{1}{t} + t + 2\right] dt$$

$$= \frac{1}{2}\left[\log t + \frac{t^2}{2} + 2t + c\right]$$

$$= \frac{1}{2} \log \tan \frac{x}{2} + \frac{1}{4} \tan^2 \frac{x}{2} + \tan \frac{x}{2} + c$$

SAQ 6

Integrate the following :

(i) $\frac{1}{5 + 4 \sin x}$

(ii) $\frac{1}{2 + \cos \theta}$

(iii) $\frac{1}{1 + \sin x + \cos x}$

3.8 SUMMARY

The main points covered in this unit are

- Given the derivative of a function, the process to find the function is called **antidifferentiation** and the result of antidifferentiation is called an **antiderivative**.

- The indefinite integral $\int f(x)\, dx$ denotes the class of all antiderivatives of *f*.
- (a) $\int K\, f(x)\, dx = K \int f(x)\, dx$

  (b) $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$
- By the method of substitution :

  (a) $\int f(x)\, dx = \int f[\phi(t)]\, \phi'(t)\, dt$

  (b) $\int [f(x)]^n\, f'(x)\, dx = \dfrac{[f(x)]^{n+1}}{n+1}$, where $n + 1 \neq 0$

  (c) $\int \dfrac{f'(x)}{f(x)}\, dx = \ln |f(x)|$.
- By the method of integration by parts :

  Integral of product of two functions = first function × integral of second function – integral of (derivative of first function × integral of second function).
- A rational function *f* of *x* is given by $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial in *x*. It is called **proper** if the degree of $P(x)$ is less than the degree of $Q(x)$. Otherwise it is called **improper**.
- To integrate a proper rational function, we decompose the denominator into either linear or quadratic factors.
- Some rules to integrate irrational functions are

  (a) To integrate functions containing only fractional powers of *x*, put $x = t^n$, where *n* is l. c. m. of denominators of powers of *x*.

  (b) Rational functions of sin *x* or cos *x* or both can be reduced to a rational function of *t* by substituting $\tan \dfrac{x}{2} = t$ and then can be integrated.

3.9 ANSWERS TO SAQs

SAQ 1

(i) $5x^2 + c$

(ii) $x^{11} + c$

(iii) $\dfrac{-5x^2}{2} + c$

SAQ 2

(i) (a) $\dfrac{x^9}{9} + c$

(b) $-\frac{2}{3}x^{-3/2}+c$

(c) $-4x^{-1}+c$

(c) $9x+c$

(ii) (a) $\frac{x^3}{3}-\frac{x^2}{2}-x+c$

(b) $2x^{1/2}-2x^{3/2}+c$

(c) $\frac{x^3}{3}-2x-\frac{1}{x}+c$

(iii) (a) $e^x-e^{-x}+4x+c$

(b) $4\sin x+3\cos x+e^x+\frac{x^2}{2}+c$

(c) $4\tanh x+e^x-4x^2+c$

(iv) (a) $2\sin^{-1}x+5\ln|x|+c$

(b) $\int \frac{2(x^2+1)+3}{x^2+1}\,dx$

$= 2\int dx+3\int \frac{1}{x^2+1}\,dx$

$= 2x+3\tan^{-1}x+c$

(v) (a) $\frac{ax^4}{4}+\frac{bx^3}{3}+\frac{cx^2}{2}+dx+c_1$

(b) $\frac{x^2}{2}-2x+\ln|x|+c$

(vi) (a) $\int \frac{\sin^4 x+\cos^4 x}{\sin^2 x\cos^2 x}\,dx=\int \frac{(\sin^2 x+\cos^2 x)-2\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x}\,dx$

$=\int \frac{1-2\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x}\,dx$

$=\int \frac{1}{\sin^2 x}\,dx+\int \frac{1}{\cos^2 x}\,dx-2\int dx=-\cot x+\tan x-2x+c$

(b) $6x-\frac{4}{3}x^{\frac{3}{2}}+\frac{3}{2}x^2-\frac{2}{5}x^{\frac{5}{2}}+c$

SAQ 3

(a) (i) $\frac{1}{6}\tan^{-1}\frac{3x-2}{3}+c$

(ii) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+c$

(iii) $\frac{1}{6}\tan^6 x+c$

(iv) $x - \ln|(1 + e^x)| + c$

(v) $\ln|(\ln|\sin x|)| + c$

(vi) $\log|(e^x - 1)| - x + c$

(vii) $-\frac{1}{2}\frac{1}{(e^{2x}+1)} + c$

(viii) $\frac{1}{2}\tan x^2 + c$

(ix) Put $\sin^{-1} x = t$

So $\frac{1}{\sqrt{1-x^2}}\,dx = dt$ and $\int \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}}\,dx = \int t^2\,dt$

$$= \frac{t^3}{3} + c = \frac{1}{3}(\sin^{-1} x)^3 + c$$

(x) $\frac{1}{4}(1 + \log x)^4 + c$

(xi) $-\ln|(1 + \cot x)| + c$

(b) (i) $(3\ln|x| - 1)\frac{x^3}{9} + c$

(ii) $\ln|\sin x)| - x\cot x + c$

(iii) $\frac{e^{3x}(4\sin 4x + 3\cos 4x)}{25} + c$

(iv) $x\sin^{-1} x + \sqrt{1-x^2} + c$

(v) $\int x\tan^{-1} x\,dx$

$$= (\tan^{-1} x)\,.\,\frac{x^2}{2} - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + c$$

$$= \frac{1}{2}(x^2+1)\tan^{-1} x - \frac{1}{2}x + c$$

(vi) $-\phi\cos\phi + \sin\phi + c$, where $\phi = \sin^{-1} x$

(vii) $\frac{e^x}{1+x}$

SAQ 4

(i) $\frac{3}{4}\ln|x-3| + \frac{1}{4}\ln|x+1| + c$

(ii) $4\ln|x+5| - \ln|x-2| + c$

(iii) $\dfrac{6x^2 + 22x - 23}{(2x-1)(x^2 + x - 6)} = \dfrac{6x^2 + 22x - 23}{(2x-1)(x+3)(x-2)} = \dfrac{A}{2x-1} + \dfrac{B}{x+3} + \dfrac{C}{x-2}$

$$6x^2 + 22x - 23 = A(x+3)(x-2) + B(x-2)(2x-1) + C(2x-1)(x+3)$$

$$x = 2 \Rightarrow C = 3$$

$$x = -3 \Rightarrow B = -1$$

$$x = \frac{1}{2} \Rightarrow A = 1$$

$$\therefore \int \frac{6x^2 + 22x - 23}{(2x-1)(x^2 + x - 6)}\, dx = \frac{1}{2} \ln|2x-1| - \ln|x+3| + 3\ln|x-2| + c$$

(iv) $\dfrac{x^2 + x - 1}{(x-1)(x^2 - x + 1)}\, dx = \dfrac{A}{x-1} + \dfrac{Bx + C}{x^2 - x + 1}$

$$\therefore\ x^2 + x - 1 = A(x^2 - x + 1) + (Bx + C)(x-1)$$

$$x = 1 \Rightarrow A = 1$$

$\therefore$ We have

$$x^2 + x - 1 = x^2 - x - 1 + Bx^2 + (C - B)x - C$$

Thus $\quad 1 = 1 + B$

$\therefore \quad B = 0$

Also $\quad -1 = 1 - C$

$\therefore \quad C = 2$

$$\therefore \int \frac{x^2 + x - 1}{(x-1)(x^2 - x + 1)}\, dx = \int \frac{dx}{x-1} + 2\int \frac{dx}{x^2 - x + 1}$$

$$= \ln|x-1| + \frac{4}{\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{3}} + c$$

(v) $\displaystyle\int \frac{x^2}{x^2 - a^2}\, dx$

$$= \int \frac{x^2 - a^2 + a^2}{x^2 - a^2}\, dx$$

$$= \int dx + \frac{a^2}{x^2 - a^2}\, dx$$

$$= \int dx + \frac{a^2}{2a} \int \left\{\frac{1}{x-a} - \frac{1}{x+a}\right\} dx$$

$$= x + \frac{a}{2} \ln\left|\frac{x-a}{x+a}\right| + c$$

(vi) $\displaystyle\int \frac{x^2 + 4}{x^2 + 2x + 3}\, dx$

$$= \int \left\{1 - \frac{2x-1}{x^2 + 2x + 3}\right\} dx$$

$$= \int dx - \int \frac{2x-1}{x^2+2x+3}\,dx$$

$$= \int dx - \int \frac{2x-2}{x^2+2x+3}\,dx + \frac{3}{x^2+2x+3}\,dx$$

$$= x - \ln|x^2+2x+3| + \frac{3}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}} + c$$

SAQ 5

(i) $\dfrac{x+2}{2}\sqrt{x^2+4x+6} + \log\left(x+2+\sqrt{x^2+4x+6}\right) + c$

(ii) $\dfrac{1}{6}(2x^2+3)^{\frac{3}{2}} + \dfrac{x}{2}\sqrt{2x^2+3} + \dfrac{3\sqrt{2}}{4}\log\left(x+\sqrt{x^2+\dfrac{3}{2}}\right) + c$

(iii) $\dfrac{2x-3}{4}\sqrt{1+3x+x^2} + \dfrac{13}{8}\sin^{-1}\left(\dfrac{2x-3}{\sqrt{13}}\right) + c$

SAQ 6

(i) $\dfrac{2}{3}\tan^{-1}\dfrac{\left(5\tan\dfrac{x}{2}+4\right)}{3} + c$

(ii) $\dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\tan\dfrac{\theta}{2}\right) + c$

(iii) $\log\left(1+\tan\dfrac{x}{2}\right) + c$