UNIT 6 MATRICES AND DETERMINANTS

Structure

6.1 INTRODUCTION

Matrices and Determinants have become an important tool in the study of science and engineering.

In this unit, we will introduce matrices and various operations involving them. After introducing the notion of determinant of a square matric, some important properties of determinants are observed. These properties are useful in computing the determinants of a matric more efficiently. Determinants will be used to understand certain aspects of matrices. The notion of adjoint and inverse of a matric are also introduced and these concepts will be used in solving a system of linear equations.

Objectives

After studying this unit, you should be able to

- introduce the notion of matrices and operations involving matrices,
- introduce determinants of order *n* and give some useful properties of determinants,
- introduce the concept of inverse of a matrix, and
- understand certain aspects of matrices in terms of determinants.

6.2 MATRICES

In this section, we introduce the notion of a matrix and then study some aspects of matrices which are extremely useful to understand several engineering systems.

A *matrix* of order *m* by *n* is an array of *mn* numbers (real or complex), arranged in *m* ordered rows (each row containing *n* numbers) and *n* ordered columns (each column containing *m* numbers). Note that the order, *m* by *n*, of a matrix determines the number of rows, *m*, and the number of columns, *n*, associated with the matrix. Let the *i*th row of a *m* by *n* matrix consists of the *n* numbers.

$$a_{i1}\ a_{i2}\ \ldots a_{in},$$

for $i = 1$ to m. If we denote this matrix by A, then we usually write

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

or, more compactly as,

$$A = (a_{ij}),\ i = 1 \text{ to m and } j = 1 \text{ to } n.$$

Note that the first suffix i in a_{ij} refers to the row index and the second suffix j in a_{ij} refers to the column index. The number a_{ij} is called the *entry* or *element* in the *i*th row and *j*th column of the matrix A.

A matrix is called a *rectangular matrix* or a *square matrix* as $m \neq n$ or $m = n$. A square matrix with *n* rows and *n* columns is also referred to as a *n*-square matrix or a square matrix of order *n*. A matrix of order *m* by 1 is called a *column matrix*. Similarly, a matrix of order 1 by *n* is called a *row matrix*. For example, the following are matrices of orders mentioned against each matrix :

$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \end{bmatrix}$: matrix of order 2 by 3.

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$: square matrix of order 3.

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$: column matrix of order 3 by 1.

$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$: row matrix of order 1 by 3.

Amongst all the *m* by *n* matrices, a matrix with every entry 0 is called a *null* (or *zero*) matrix. A null matrix is denoted by 0. Note that we have a null matrix of every order and each of these is denoted by the same symbol, viz., 0. The following are example of null matrices :

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$: null matrix of order 3 by 2.

$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$: null matrix of order 2 by 4.

Amongst all the *n* square matrices, we also single out the following matrix :

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ . & & & & \\ . & & & & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} = (a_{ij}),$$

i.e., $a_{ij} = 0$ for $i \neq j$ and $a_{ij} = 1$ for $i = j$. This matrix is called the *identity matrix* (or *Unit Matrix*) and is denoted by I. Again, note that we have an identity matrix I for every n and each of these is denoted by the same symbol, viz. I. For example, the identity matrix of order 3 is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We remark that the context in which a null matrix or identity matrix is used usually indicates the order of these matrices. Otherwise, the order of a null or identity matrix should be specified.

A square matrix $A = (a_{ij})$ is called a *diagonal matrix* if $a_{ij} = 0$ for $i \neq j$ and a_{ii} are arbitrary. The diagonal matrices play an important role. The following is an example of a diagonal matrix of order 4 :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2-\mathrm{i} & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that some diagonal entries, a_{ii}, of a diagonal matrix may also be zero.

A square matrix $A = (a_{ij})$ is called an *upper triangular matrix* if $a_{ij} = 0$ for $i > j$ and a *lower triangular matrix* if $a_{ij} = 0$ for $i < j$. A matrix which is either upper triangular or lower triangular is called *triangular matrix*. Note that a diagonal matrix is both upper triangular and lower triangular. The following are examples of triangular matrices :

$$\begin{bmatrix} 1 & 2 & 3+i & 4 \\ 0 & -2 & 1 & 2 \\ 0 & 0 & 1+i & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix} \quad : \quad \text{upper triangular matrix of order 4}$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 2+i & i & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 5 & 6+i \end{bmatrix} \quad : \quad \text{lower triangular matrix of order 4.}$$

Either of these matrices is an example of a triangular matrix.

6.2.1 Equality of Matrices

Let $A = (a_{ij})$ and $B = (b_{ij})$ be any two matrices of the same order. Then, we define

$$\mathrm{A} = \mathrm{B} \Leftrightarrow \mathrm{a}_{\mathrm{ij}} = \mathrm{b}_{\mathrm{ij}} \text{ for all } i \text{ and } j.$$

Thus, two matrices are *equal* if

(i) their orders are same, and

(ii) the entries in the corresponding positions are same.

Observe that the following properties are satisfied by the above concept of equality of matrices.

(i) $\mathbf{A = B \Rightarrow B = A}$

(ii) $\mathbf{A = B}$ **and** $\mathbf{B = C \Rightarrow A = C}$

6.2.2 Sum of Matrices

The sum $A + B$ of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same order is defined by

$$A + B = (a_{ij} + b_{ij})$$

Thus, we add the corresponding entries of two matrices to find the sum of the given matrices. For example, if

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3+i & 4 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} i & -1 & 1 \\ -6 & 3+2i & 0 \end{bmatrix}$$

then $$A + B = \begin{bmatrix} 1+i & 1 & 1 \\ -3+i & 7+2i & -1 \end{bmatrix}$$

The following properties of addition of matrices, as defined above, are easy to prove :

(iii) **The sum of any two matrices of the same order is a matrix of the same order.**

(iv) $\mathbf{A + B = B + A}$ **for any two matrices *A* and *B* of the same order.**

(v) $\mathbf{(A + B) + C = A + (B + C)}$ **for any three matrices *A*, *B* and *C* of the same order.**

(iv) ***A* + 0 = 0 + *A* = *A* for any matrix *A*, where 0 is a null matrix of the same order as that of *A*.**

(vii) **For any matrix *A*, we can associate a matrix *B* such that** $\mathbf{A + B = B + A = 0}$**. In fact, if** $\mathbf{A = (a_{ij})}$**, then** $\mathbf{B = (-a_{ij})}$**. The matrix *B* thus obtained is usually denoted by (– *A*) and is called the *negative* of *A*.**

6.2.3 Scalar Multiplication

We define multiplication of a matrix $A = (a_{ij})$ by a scalar (number) α by the following rule :

$$\alpha A = A\alpha = (\alpha\, a_{ij}),$$

i.e., we multiply every entry of A by α to obtain αA. For example, if

$$A = \begin{bmatrix} 1 & 0 & -3 & 4 \\ i & 3+7i & 1 & -4 \end{bmatrix}$$

then $$4A = \begin{bmatrix} 4 & 0 & -12 & 16 \\ 4i & 12+28i & 4 & -16 \end{bmatrix}$$

and $$iA = \begin{bmatrix} i & 0 & -3i & 4i \\ -1 & -7+3i & i & -4i \end{bmatrix}$$

The following properties of scalar multiplication follow from the definition :

(viii) **αA is a matrix of the same order as that of A for every α.**

(ix) $\boldsymbol{\alpha (A + B) = \alpha A + \alpha B}$

(x) $\boldsymbol{(\alpha + \beta) A = \alpha A + \beta A}$

(xi) $\boldsymbol{(\alpha \beta) A = \alpha (\beta A)}$

(xii) $\boldsymbol{I A = A I = A}$

6.2.4 Transpose and Conjugate of a Matrix

Let $A = (a_{ij})$ be a m by n matrix. A new matrix, of order n by m, called the *transpose* of A and denoted by A^T, can be associated with the matrix A as follows :

$$A^T = (a_{ji}), \; j = 1 \text{ to } n \text{ and } i = 1 \text{ to } m.$$

Thus the ith row of A is the ith column of A^T and the jth column of A is the jth row of A^T. To obtain the transpose of a matrix, we have to interchange its rows and columns. For example, if

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 5 \end{bmatrix}$$

If a_{ij} are complex numbers, we can also define the *conjugate*, denoted by $\overline{A}$, of the matrix $A = (a_{ij})$ as follows :

$$\overline{A} = (\overline{a}_{ij})$$

i.e. we replace every a_{ij} by its conjugate $\overline{a}_{ij}$ to obtain $\overline{A}$. For example, if

$$A = \begin{bmatrix} 1+i & 2 \\ 3-i & 4+2i \end{bmatrix} \text{ then } \overline{A} = \begin{bmatrix} 1-i & 2 \\ 3+i & 4-2i \end{bmatrix}$$

We observe the following properties which follow from the definition itself :

(xiii) $\mathbf{(A^T)^T = A}$

(xiv) $\mathbf{(A + B)^T = A^T + B^T}$

(xv) $\mathbf{\overline{(\overline{A})} = A}$

(xvi) $\mathbf{\overline{(A + B)} = \overline{A} + \overline{B}}$

(xvii) $\boldsymbol{(\alpha A)^T = \alpha A^T}$

(xviii) $\boldsymbol{\overline{(\alpha A)} = \overline{\alpha}\, \overline{A}}$

6.2.5 Special Matrices

Using the notions of conjugate and transpose of a matrix, we now define some special types of matrices. These matrices have some properties which are not valid for arbitrary matrices.

A matrix A is called a *symmetric matrix*, if

$$A = A^T$$

Thus, $A = (a_{ij})$ is a symmetric matrix, if

$$a_{ij} = a_{ji} \text{ for all } i \text{ and } j$$

For example, the matrix

$$\begin{bmatrix} -1 & 2 & 4 \\ 2 & 0 & 3 \\ 4 & 3 & 5 \end{bmatrix}$$

is a symmetric matrix.

A matrix $A = (a_{ij})$ is called *skew-symmetric*, if

$$a_{ij} = -a_{ji} \text{ for all } i \text{ and } j,$$

i.e. $\quad A = -A^T$

For example, the matrix

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$

is a skew-symmetric matrix.

Note that $a_{ii} = 0$ for all $i = j$ in a skew-symmetric matrix because

$$a_{ii} = -a_{ii}$$

SAQ 1

(a) Verify properties (v), (ix), (x) and (xi) with the following data :

$$A = \begin{bmatrix} 2 & 3-i & 4 & -i \\ 1 & 0 & 2i & 4+i \\ 3 & 4 & i & 1+i \end{bmatrix}, B = \begin{bmatrix} 1 & 2+i & 3 & 2i \\ 0 & 1-i & i & 1 \\ 0 & 3 & 4i & -3 \end{bmatrix}, C = \begin{bmatrix} 0 & 2-i & 3i & 1 \\ -2 & 2i & 3 & -5 \\ 4 & 2 & 0 & 6i \end{bmatrix}$$

and $\alpha = 2$, $\beta = -3i$.

(b) Verify the properties (xiii) to (xviii) using the data given in SAQ 1(a).

6.3 MATRIX MULTIPLICATION

In this section, we introduce the important concept of matrix multiplication.

Let $A = (a_{ij})$ be a matrix of order m by n and $B = (b_{jk})$ be a matrix of order n by p so that i = 1 to m, j = 1 to n and k = 1 to p. Note that n is common in the orders of A and B, i.e. the number of columns of A equals the number of rows of B. If A and B have the orders as specified, then we say that A is *conformable or compatible with B* for multiplication. Further, the *product* of A with B, denoted by AB, is defined as follows :

$$AB = (c_{ik}),\ i = 1 \text{ to } m,\ k = 1 \text{ to } p,$$

where $$c_{ik} = a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + \ldots + a_{in}\, b_{nk}$$

Thus, each of the n numbers in ith row of A, viz, a_{ij} is multiplied with the corresponding number in the kth column of B, viz, b_{jk} and these n products are added to obtain c_{ik}, the element in the ith row and kth column of the matrix AB. Thus, we can write

$$c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}\ ,$$

Note that the matrix AB, as defined above, is of order m by p, where m is the number of rows of A and p is the number of columns of B.

As an example, let

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 1 & 0 \\ 5 & 2 & 1 \end{bmatrix}$$

Since the number of columns of A equals the number of rows of B, we can multiply A with B. AB is of order 2 by 3. Thus, we have

$$AB = \begin{bmatrix} 1 \times 4 + 2 \times 5 & 1 \times 1 + 2 \times 2 & 1 \times 0 + 2 \times 1 \\ 4 \times 4 + 5 \times 5 & 4 \times 1 + 5 \times 2 & 4 \times 0 + 5 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 5 & 2 \\ 41 & 14 & 5 \end{bmatrix}$$

It may be noted that AB may be defined whereas BA may not be defined, because B may not be conformable with A for multiplication. However, if A and B are square matrices of the same order then both AB and BA are defined.

Theorem 1

Prove the following properties of matrix multiplication :

(i) A (BC) = (AB) C

(ii) A (B + C) = AB + AC

(iii) (A + B) C = AC + BC

(iv) $(\alpha A) B = \alpha (AB) = A (\alpha B)$

(v) $(AB)^T = B^T A^T$

(vi) $\overline{AB} = \overline{A}\ \overline{B}$

(vii) $AI = IA = A$

Proof

Here we have assumed that the orders of the matrices A, B and C are such that the various matrices involved in the above properties are well defined. For example in (ii) if A is of order m by n then the orders of both B and C must be n by p for some p.

(i) Let $A = (a_{ij})$, $B = (b_{kl})$, and $C = (c_{rs})$ be $m \times n$, $n \times p$, and $p \times q$ matrices respectively, i.e. $1 \le i \le m, 1 \le j, k \le n, 1 \le l \le p$

$(i, l)^{th}$ element of $AB = \sum_{j=1}^{n} a_{ij}\, b_{jl}$

Hence for $1 \le i \le m, 1 \le s \le q$

$(i, s)^{th}$ element of $(AB)\, C$

$$= \sum_{k=1}^{p} \left(\sum_{j=1}^{n} a_{ij}\, b_{jk} \right) c_{ks}$$

$$= \sum_{k=1}^{p} \sum_{j=1}^{n} a_{ij}\, b_{jk}\, c_{ks} \qquad \ldots (6.1)$$

Also for $1 \le k \le n, 1 \le s \le q$

$(k, s)^{th}$ element of $BC = \sum_{k=1}^{p} b_{jk}\, c_{ks}$

Hence for $1 \le i \le m, 1 \le s \le q$

$(i, s)^{th}$ element of $A\, (BC)$

$$= \sum_{j=1}^{n} a_{ij} \left(\sum_{k=1}^{p} b_{jk}\, c_{ks} \right)$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{p} a_{ij}\, b_{kj}\, c_{ks}$$

$$= \sum_{k=1}^{p} \sum_{j=1}^{n} a_{ij}\, b_{jk}\, c_{ks} \qquad \ldots (6.2)$$

By rearranging the sum from Eqs. (6.1) and (6.2), we conclude that

$$A\, (BC) = (AB)\, C$$

(ii) Let $A = (a_{ij})$, $B = (b_{jk})$, and $C = (c_{jk})$ be $m \times n$, $n \times p$, and $n \times p$ matrices. Note that B and C are of the same order so that $B + C$ is defined as $B + C = (b_{jk} + c_{jk})$.

Hence $(i, k)^{th}$ element of

$$A(B+C)=\sum_{j=1}^{n} a_{ij}\,(b_{jk}+c_{jk})$$

$$=\sum_{j=1}^{n} a_{ij}\,b_{jk}+\sum_{j=1}^{n} a_{ij}\,c_{jk}$$

$= (i, k)^{\text{th}}$ element of $AB + (i, k)^{\text{th}}$ element of AC

$\therefore$ For $1 \le i \le m, 1 \le k \le p$

$A(B + C) = AB + AC$

Similarly, (iii) and (iv) can be proved.

(v) Let $A = (a_{ij})$, $B = (b_{jk})$ be two $m \times n$, $n \times p$ matrices, respectively.

Then $A^T = (c_{ji})$ where $c_{ji} = a_{ij}$ and $B^T = (d_{kj})$ where $d_{kj} = b_{jk}$ are $n \times m$ and $p \times n$ matrices, respectively.

Now AB is a $m \times p$ matric $\Rightarrow (AB)^T$ is a $p \times m$ matric.

Also $B^T A^T$ is a $p \times m$ matric.

Thus, both $(AB)^T$ and $B^T A^T$ are matrices of order $p \times m$.

$(k, i)^{\text{th}}$ entry of $(AB)^T$

$= (i, k)^{\text{th}}$ entry of AB

$$=\sum_{j=1}^{n} a_{ij}\,b_{jk}=\sum_{j=1}^{n} c_{ji}\,d_{kj}=\sum_{j=1}^{n} d_{kj}\,c_{ji}$$

$= (k, i)^{\text{th}}$ entry of $B^T A^T$.

$\therefore$ $(AB)^T = B^T A^T$.

Similarly, (vi) and (vii) can be proved.

Most of the properties that we have listed about matrix operations (addition, scalar multiplication, multiplication) are also valid in numbers. However, there are some important differences also. We bring these out in the following examples.

Example 6.1

Show that $AB \neq BA$ in general.

Solution

Let $A=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 1\end{bmatrix}, B=\begin{bmatrix}0 & 0\\ -1 & 1\\ 2 & 3\end{bmatrix}$

then $AB=\begin{bmatrix}4 & 11\\ 1 & 4\end{bmatrix}$, and $BA=\begin{bmatrix}0 & 0 & 0\\ -1 & -1 & -2\\ 2 & 7 & 9\end{bmatrix}$

In general, if A is of order $m \times n$ and B of order $n \times m$, then AB is of order $m \times m$ and BA of order $n \times n$. Thus, if $m \neq n$, AB and BA are of different orders and hence can never be equal. Even if $m = n$, $AB \neq BA$. For example if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

then $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Note that $AB \neq BA$.

Example 6.2

Consider matrices A and B of Example 6.1. Here we have $BA = 0$ where neither $B = 0$ nor $A = 0$.

It may also be noted that $BA = 0$ but $AB \neq 0$.

Example 6.3

Let $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

then $\quad B - C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Example 6.1 shows that

$$A\,(B - C) = 0$$

i.e. $\quad AB - AC = 0,$

i.e. $\quad AB = AC$

Thus, $\quad AB = AC$ but $B \neq C$

Example 6.4

Let $\quad A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and $B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

then $\quad A^2 = AA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $B^2 = BB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Thus, $\quad A^2 = B^2$ but $A \neq B$.

Example 6.5

Let $\quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

then $\quad A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $B^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Thus, $A^2 + B^2 = 0$ but neither $A = 0$ nor $B = 0$.

The above examples show that we have to be careful while working with various matrix operations. The usual rules of the arithmetic of numbers are not valid in the case of matrices.

Example 6.6

In this example, we observe that any system of linear equations can be compactly written as a *matrix equation*.

Consider the following system of *m* linear equations in *n* variables :

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}$$

Using the arithmetic of matrices, this system can be represented by the matrix equation

$$AX = B,$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{bmatrix}$.

SAQ 2

(a) If $A = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$, verify that $(A + I)(A - 4I) = 0$.

(b) Find the matrix X so that

$$X \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

(c) If $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Prove that $f(\alpha)\, f(\beta) = f(\alpha + \beta)$.

(d) Let A be a square matric. Show that $\frac{1}{2}(A + A^T)$ is a symmetric matrix and $\frac{1}{2}(A - A^T)$ is a skew symmetric matrix.

(e) If A and B are square matrices of same order, then prove that

(i) If A and B are symmetric, $A - B$ is also symmetric.

(ii) If A and B are skew symmetric, $A - B$ is also skew symmetric.

6.4 DETERMINANTS

Determinant of a Square Matrix

With every square matrix A, we associate a unique number called the determinant of the matrix and is denoted by *det A* or $| A |$ and is defined as follows :

If $A = (a_{11})$ is a square matrix of order 1, then det $A = a_{11}$, i.e. $| a_{11} | = a_{11}$.

If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is a square matrix of order 2, then

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\, a_{22} - a_{12}\, a_{21}$$

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is a square matrix of order 3, then

$$\det\ A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \qquad \text{. . . (1)}$$

We say that the determinant has been expanded with the help of the first row.

The expansion with the help of the second row will be

$$- a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \qquad \text{. . . (2)}$$

It can be verified that the value given by Eq. (2) is the same as given by Eq. (1). We can also expand it with the help of any column. For example expanding along the second column, we get its value equal to

$$- a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \qquad \text{. . . (3)}$$

For example,

(i) if $A = (-3)$, then $\det A = -3$

(ii) if $A = \begin{pmatrix} 4 & 2 \\ -1 & 7 \end{pmatrix}$, then $\det A = 4.7 - 2\,.\,(-1) = 28 + 2 = 30$

(iii) if $A = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 0 & 5 \\ 0 & 1 & 3 \end{pmatrix}$, then $\det A = 2 \begin{vmatrix} 0 & 5 \\ 1 & 3 \end{vmatrix} - 4 \begin{vmatrix} 0 & 5 \\ 0 & 3 \end{vmatrix} + 1 \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}$

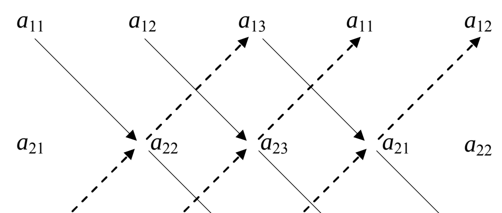
$$= 2\,(0 - 5) - 4\,(0 - 0) + 1\,(0 - 0)$$

$$= -10$$

Evaluation of a Determinant of a Square Matrix of Order 3 by Sarrus Rule

A more convenient method for evaluating a determinant of a square matrix of order 3 is given below. This is called **sarrus rule**.

Consider the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and write the elements as

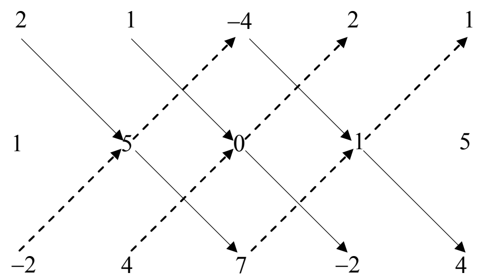


The value of the determinant is obtained by adding the products of elements along the diagonals parallel to *main diagonal* and subtracting the sum of the products of elements along the diagonals which run up from left to right. Observe that the first two columns are repeated in the above table to complete the process.

Remark

This method is only for determinants of order 3.

For example, to evaluate $\begin{vmatrix} 2 & 1 & -4 \\ 1 & 5 & 0 \\ -2 & 4 & 7 \end{vmatrix}$, write the elements as



The value of the given determinant

$$= (2.5.7 + 1.0 \,.\, (-2) + (-4) \,.\, 1.4)$$
$$- ((-2) \,.\, 5 \,.\, (-4) + 4.0.2 + 7.1.1)$$
$$= (70 + 0 - 16) - (40 + 0 + 7)$$
$$= 54 - 47 = 7$$

Minors and Cofactors

Minors

Let $A = (a_{ij})$ be a square matrix of order n and let A_{ij} denotes the square matrix of order $n - 1$, obtained from A by deleting the ith row and jth column, then the number det A_{ij} is called **minor** of the entry a_{ij} and is denoted by M_{ij}, i.e.

$$M_{ij} = |A_{ij}|, 1 \le i,\ j \le n.$$

For examples,

(i) let $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, then

$M_{11} = |\, d \,| = d$,

$M_{12} = |\, c \,| = c$,

$M_{21} = |\, b \,| = b$, and

$M_{22} = |\, a \,| = a.$

(ii) let $A = \begin{vmatrix} 4 & -2 & 8 \\ -1 & i & 0 \\ 3 & 1 & 2 \end{vmatrix}$, then

$$M_{11} = \begin{vmatrix} i & 0 \\ 1 & 2 \end{vmatrix} = 2i - 0 = 2i$$

$$M_{23} = \begin{vmatrix} 4 & -2 \\ 3 & 1 \end{vmatrix} = 4 - (-6) = 10, \text{ and}$$

$$M_{33} = \begin{vmatrix} 4 & -2 \\ -1 & i \end{vmatrix} = 4i - 2 \text{ etc.}$$

Cofactors

The number $(-1)^{i+j}$ det A_{ij} is called the cofactor of the entry a_{ij} and is denoted by C_{ij}, $1 \le i, j \le n$.

Note that $C_{ij} = \det A_{ij} = M_{ij}$ if $i + j$ is even, and

$$C_{ij} = -\det A_{ij} = -M_{ij} \text{ if } i + j \text{ is odd.}$$

For examples,

let $A = \begin{vmatrix} 4 & -2 & 8 \\ -1 & i & 0 \\ 3 & 1 & 2 \end{vmatrix}$,then

$$C_{12} = (-1)^{1+2} \begin{vmatrix} -1 & 0 \\ 3 & 2 \end{vmatrix} = -(-2 - 0) = 2$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 4 & -2 \\ 3 & 1 \end{vmatrix} = -(4 + 6) = -10, \text{ and}$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} -2 & 8 \\ i & 0 \end{vmatrix} = (0 - 8i) = -8i \text{ etc.}$$

Determinants and Cofactors

If $A = (a_{ij})$ be any $n \times n$ square matrix, then

$$\det A = \sum_{j=1}^{n} a_{ij} \, (-1)^{i+j} \det A_{ij}$$

$$= \sum_{j=1}^{n} a_{ij} \, C_{ij}$$

(expansion with the help of *i*th row)

i.e. det A = the sum of the product of element of a row (column) with their corresponding co-factor.

Example 6.7

Find the minors and the cofactors of each entry of the second row of the matrix A and hence evaluate det A where

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 2 & 5 \\ -1 & 3 & 7 \end{bmatrix}$$

Solution

$$M_{21} = \begin{vmatrix} 4 & 1 \\ 3 & 7 \end{vmatrix} = 28 - 3 = 25\,,$$

$$M_{22} = \begin{vmatrix} 2 & 1 \\ -1 & 7 \end{vmatrix} = 14 - (-1) = 15\,, \text{ and}$$

$$M_{23} = \begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} = 6 - (-4) = 10$$

$$\therefore \qquad C_{21} = (-1)^{2+1}\, M_{21} = (-1)\,.\,25 = -25\,,$$

$$C_{22} = (-1)^{2+2}\, M_{22} = 1.15 = 15\,, \text{ and}$$

$$C_{23} = (-1)^{2+3}\, M_{23} = (-1)\,.\,10 = -10$$

Now, on expanding det A with the help of second row, we get

$$\det A = a_{21}\, C_{21} + a_{22}\, C_{22} + a_{23}\, C_{23}$$

$$= 3\,.\,(-25) + 2.15 + 5\,.\,(-10) = -75 + 30 - 50 = -95$$

Example 6.8

Evaluate $\begin{vmatrix} 3 & -2 & 4 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$ by two methods.

Solution

$$\begin{vmatrix} 3 & -2 & 4 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 3\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} - (-2)\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 4\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= 3\,(0 - 2) + 2\,(6 - 1) + 4\,(4 - 0) \text{ (Expansion by first row)}$$

$$= -6 + 10 + 16 = 20$$

Also write the elements of the given determinant as (sarrus rule) :

$$\begin{array}{ccccc} 3 & -2 & 4 & 3 & -2 \\ 2 & 0 & 1 & 2 & 0 \\ 1 & 2 & 3 & 1 & 2 \end{array}$$

The value of the given determinant

$= (3.0.3 + (-2).1.1 + 4.2.2) - (1.0.4 + 2.1.3 + 3.2(-2))$

$= (0 - 2 + 16) - (0 + 6 - 12) = 14 + 6 = 20$

6.4.1 Properties of Determinants

The properties of determinants serve as useful tools for determining the values of the given determinants. We mention here these properties and verify them for a second or a third order determinant.

Theorem 2

Let $A = (a_{ij})$ be any square matrix of order n, then det A = det A^T, where A^T is the transpose of A.

Proof

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$\Rightarrow \quad \det A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (expand by R_1)

$\Rightarrow \quad |A| = (-1)^{1+1} a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + (-1)^{1+2} b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + (-1)^{1+3} c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$= a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$

$= a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_3 b_1 c_2 + a_2 b_3 c_1 - a_3 b_2 c_1 \quad \ldots$

(i)

Also $\quad \det A^T = |A^T| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ (expanded by R_1)

$\Rightarrow \quad |A^T| = (-1)^{1+1} a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + (-1)^{1+2} a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + (-1)^{1+3} a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

$= a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)$

$= a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1$

$= a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_3 b_1 c_2 + a_2 b_3 c_1 - a_3 b_2 c_1 \quad \ldots$ (ii)

From Eqs. (i) and (ii), we have

$|A| = |A^T|.$

(The value of a determinant remains unchanged if its rows are changed into columns and columns into rows.)

Theorem 3

If two rows (columns) of a determinant are interchanged, then the value of the determinant changes in sign.

Proof

Let $$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

then $$\det\ A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$\Rightarrow$ $$|A| = a_1\ (b_2 c_3 - b_3 c_2) - b_1\ (a_2 c_3 - a_3 c_2) + c_1\ (a_2 b_3 - a_3 b_2) \quad \ldots \text{(i)}$$

Let X be the matrix obtained from A by interchanging its first and second columns, then

$$\det\ X = \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} \qquad \text{(expand by } R_1\text{)}$$

$$= b_1\ (a_2 c_3 - a_3 c_2) - a_1\ (b_2 c_3 - b_3 c_2) + c_1\ (b_2 a_3 - b_3 a_2) \quad \ldots \text{(ii)}$$

From Eqs. (i) and (ii), we have

$$|X| = -|A|.$$

Cor. 1

If B be a matrix obtained from a square matrix A by passing one of its rows (or columns) over r rows (or columns), then

$$det\,B = (-1)^r\ det\ A.$$

Theorem 4

If two rows or columns of a determinant are identical, then the value of the determinant is zero.

Prove

Let A be a given matrix of order 3×3 which has two parallel lines identical, say second and third rows, i.e.

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{bmatrix}, \text{ and}$$

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$

Let B be the matrix obtained from A by interchanging the second and third rows, i.e.

$$B = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{pmatrix},$$

then $|B| = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{bmatrix}.$

Obviously, $|B| = |A|$ . . . (i)

Also by Theorem 2, $|B| = -|A|$ . . . (ii)

From Eqs. (i) and (ii), we get

$$|A| = -|A| \Rightarrow 2|A| = 0$$

$\Rightarrow$ $|A| = 0$, i.e. det $A = 0$.

Theorem 5

If each element of a row or a column of a determinant is multiplied by the same number, then the value of the determinant is also multiplied by that number.

Proof

Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, then

$$\det A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$\Rightarrow$ $|A| = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$ . . . (i)

Let X be the matrix obtained from A by multiplying every element of third row by the same number, say α, then

$$|X| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \alpha a_3 & \alpha b_3 & \alpha c_3 \end{vmatrix}$$

$$= a_1(\alpha b_2c_3 - \alpha b_3c_2) - b_1(\alpha a_2c_3 - \alpha a_3c_2) + c_1(\alpha a_2b_3 - \alpha a_3b_2)$$

$$= \alpha\,[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)] \quad . . . (ii)$$

From Eqs. (i) and (ii), we get

$$\det X = \alpha \det A.$$

Cor. 2

Let $A = (a_{ij})$ *be a square matrix of order n, then if B =* λA *where* λ *is a scalar, then det B =* λ^n *det A.*

Cor. 3

If A is a square matrix in which two rows (columns) are proportional, then det A = 0.

Theorem 6

If each element in any row (or column) of a determinant consists of sum of two terms, then the determinant can be expressed as the sum of two determinants.

Proof

Let $A = \begin{pmatrix} a_1 & b_1 + d_1 & c_1 \\ a_2 & b_2 + d_2 & c_2 \\ a_3 & b_3 + d_3 & c_3 \end{pmatrix}$, and

$B = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, and

$C = \begin{bmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{bmatrix}$.

Then $\det A = \begin{vmatrix} a_1 & b_1 + d_1 & c_1 \\ a_2 & b_2 + d_2 & c_2 \\ a_3 & b_3 + d_3 & c_3 \end{vmatrix}$ (expand by second column)

$$= -(b_1 + d_1)\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + (b_2 + d_2)\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - (b_3 + d_3)\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$= \left\{ -b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \right\}$$

$$+ \left\{ -d_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + d_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - d_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \right\}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$= \det B + \det C$.

Theorem 7

If the same multiple of elements of any row (or column) of a determinant are added to the corresponding elements of any other row (or column), then the value of the new determinant remains unchanged.

Proof

Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, and

$B = \begin{pmatrix} a_1 + \lambda c_1 & b_1 & c_1 \\ a_2 + \lambda c_2 & b_2 & c_2 \\ a_3 + \lambda c_3 & b_3 & c_3 \end{pmatrix}$,

then we shall show that $\det B = \det A$.

Now $\det B = \begin{vmatrix} a_1 + \lambda c_1 & b_1 & c_1 \\ a_2 + \lambda c_2 & b_2 & c_2 \\ a_3 + \lambda c_3 & b_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda c_1 & b_1 & c_1 \\ \lambda c_2 & b_2 & c_2 \\ \lambda c_3 & b_3 & c_3 \end{vmatrix} \quad \text{(using Theorem 5)}$$

$$= \det A + \lambda \begin{bmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{bmatrix} \quad \text{(using Theorem 4)}$$

$$= \det A + \lambda \,.\, 0 \quad \text{(using Theorem 3)}$$

$\therefore \quad \det B = \det A.$

Theorem 8

If A and B are square matrices of same order, then

$\det AB = \det A \,.\, \det B$.

Proof

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and

$B = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$.

Then $AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix}$

$$= \begin{bmatrix} ax + bz & ay + bt \\ cx + dz & cy + dt \end{bmatrix}$$

$$\Rightarrow \quad \det AB = \begin{vmatrix} ax + bz & ay + bt \\ cx + dz & cy + dt \end{vmatrix}$$

$$= (ax + bz)(cy + dt) - (ay + bt)(cx + dz)$$

$$= axcy + axdt + bzcy + bzdt - aycx - aydz - btcx - btdz$$

$$= axdt - aydz + bzcy - btcx$$

$$= ad\,(xt - yz) + bc\,(yz - xt)$$

$$= (ad - bc)(xt - yz)$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} x & y \\ z & t \end{vmatrix}$$

$$= \det A \,.\, \det B.$$

Theorem 9

The sum of the products of the elements of any row (or column) of a square matrix with cofactors of the corresponding elements of any other row (or column) of the given matrix is always zero.

Proof

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

then cofactor of $a_{11} = C_{11}$

$= (-1)^{1+1} \det |a_{22}| = a_{22}$

cofactor of $a_{12} = C_{12}$

$= (-1)^{1+2} \det |a_{21}| = -a_{21}$

Now, sum of the products of the elements of second row with the cofactors of corresponding elements of first row

$= a_{21} C_{11} + a_{22} C_{12}$

$= a_{21} a_{22} + a_{22} (-a_{21}) = 0$.

Notations

Let A be a matrix of order $n \times n$.

Let $R_1, R_2, R_3, \ldots, R_n$ denote its first, second, third, . . . , nth rows and $C_1, C_2, C_3, \ldots C_n$ denote its first, second, third, . . . , nth columns, respectively.

(i) The operations of interchanging ith and jth rows of A will be denoted by $R_i \leftrightarrow R_j$.

The operation of interchanging ith and jth columns of A will be denoted by $C_i \leftrightarrow C_j$.

(ii) The operation of multiplying each element of ith row of A by a scalar λ will be denoted by $R_i(\lambda)$.

The operation of multiplying each element of ith column of A by a scalar λ will be denoted by $C_i(\lambda)$.

(iii) The operation of adding to each element of ith row of A, λ times the corresponding elements of jth row ($j \neq i$) of A will be denoted by $R_i + \lambda R_j$.

The operation of adding to each element of ith column of A, λ times the corresponding elements of jth column ($j \neq i$) of A will be denoted by $C_i + \lambda C_j$.

Example 6.9

If w is one of the imaginary cube root of unity, find the value of

$$\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$$

Solution

Let $\Delta = \begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$

Adding the second and the third row to the first row, we have

$$\Delta = \begin{vmatrix} 1+w+w^2 & 1+w+w^2 & 1+w+w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} \qquad (1+w+w^2=0)$$

$$= 0$$

Example 6.10

Evaluate $\begin{vmatrix} 219 & 117 & 345 \\ 19 & 9 & 34 \\ 7 & 3 & 5 \end{vmatrix}$

Solution

Let $\quad \Delta = \begin{vmatrix} 219 & 117 & 345 \\ 19 & 9 & 34 \\ 7 & 3 & 5 \end{vmatrix}$

By $R_1 - 10\,R_2$ (i.e. subtract 10 times the 2nd row from the 1st row), we have

$$\Delta = \begin{vmatrix} 29 & 27 & 5 \\ 19 & 9 & 34 \\ 7 & 3 & 5 \end{vmatrix}$$

Now by $R_1 - R_3$ and $R_2 - 3R_3$, we have

$$\Delta = \begin{vmatrix} 22 & 24 & 0 \\ -2 & 0 & 19 \\ 7 & 3 & 5 \end{vmatrix} = 2\begin{vmatrix} 11 & 12 & 0 \\ -2 & 0 & 19 \\ 7 & 3 & 5 \end{vmatrix}$$

$$= 6\begin{vmatrix} 11 & 4 & 0 \\ -2 & 0 & 19 \\ 7 & 1 & 5 \end{vmatrix} \qquad \text{(by taking 3 common from 2nd column)}$$

$$= 6\,[-11 \times 19 - 4\,(-10 - 133)]$$

$$= 6\,(-209 + 572) = 6 \times 363 = 2178$$

Example 6.11

Show that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$

Solution

Let $\quad \Delta = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$

$$= \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + \begin{vmatrix} a & b & a+b+c \\ 2a & 2b & 4a+3b+2c \\ 3a & 3b & 10a+6b+3c \end{vmatrix}$$

The 2nd determinant is zero as first and second columns are proportional.

$$\Delta = \begin{vmatrix} a & a & a \\ 2a & 3a & 4a \\ 3a & 6a & 10a \end{vmatrix} + \begin{vmatrix} a & a & b \\ 2a & 3a & 3b \\ 3a & 6a & 6b \end{vmatrix} + \begin{vmatrix} a & a & c \\ 2a & 3a & 2c \\ 3a & 6a & 3c \end{vmatrix}$$

(The 2nd and the 3rd determinants are zero).

$$\Delta = \begin{vmatrix} a & a & a \\ 2a & 3a & 4a \\ 3a & 6a & 10a \end{vmatrix} = a^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix}$$

(Applying $C_2 - C_1$ and $C_3 - C_4$).

$$\Delta = a^3 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 7 \end{vmatrix} = a^3 \,(7-6)$$

$$= a^3.$$

SAQ 3

(a) Evaluate the following determinants.

(i) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

(ii) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

(iii) $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$

(b) Prove the following identities

(i) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-c)\,(c-a)\,(a-b)\,(a+b+c)$

(ii) $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (y - z)(z - x)(x - y)(yz + zx + xy)$

(iii) $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = bc + ca + ab + abc$

(iv) $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$

(v) $\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(c) If x, y, z are all different and

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$$

show that $xyz = -1$.

6.5 ADJOINT AND INVERSE OF A MATRIX

6.5.1 Adjoint of a Square Matrix

Definition 1

*Let $A = (a_{ij})$ be a square matrix of order n. Then the **adjoint of A**, written as adj A, is defined as the transpose of the matrix (C_{ij}) where C_{ij} is the cofactor of a_{ij} in $|A|$.*

For example, let $A = \begin{bmatrix} 4 & 3 \\ 1 & 6 \end{bmatrix}$ then $|A| = \begin{vmatrix} 4 & 3 \\ 1 & 6 \end{vmatrix}$.

Now $C_{11} = 6,\ C_{12} = -1$

$C_{21} = -3,\ C_{22} = 4$

Then $\quad \text{adj } A = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}^T$

$$= \begin{bmatrix} 6 & -1 \\ -3 & 4 \end{bmatrix}^T = \begin{bmatrix} 6 & -3 \\ -1 & 4 \end{bmatrix}.$$

Theorem 10

If A is a square matrix of order 3, then A (adj A) = | A | I_3 = (adj A) A.

Proof

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3, then

$$\text{adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\therefore \; A\,(\text{adj } A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

($\because$ The sum of products of the elements of a row (or column) with their corresponding cofactors = det A, and the sum of the products of the elements of a row (or column) with cofactors of the corresponding elements of another row (or column) is zero).

$$\therefore \qquad A\,(\text{adj } A) = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\, I_3 .$$

Similarly, $\quad$ (adj A) A = | A | I_3.

Therefore, $\quad$ A (adj A) = | A | I_3.

= (adj A) A.

Remark

The above result holds true for any square matrix of order n. In fact, we have

$$A\,(\text{adj } A) = |\,A\,|\, I_n = (\text{adj } A)\, A,$$

where A is any square matrix of order n.

Example 6.12

Find the adjoint of the matrix $A = \begin{bmatrix} 4 & -6 & 1 \\ -1 & -1 & 1 \\ -4 & 11 & -1 \end{bmatrix}$ and verify that

A (adj A) = (adj A) A = $|A|\, I_3$.

Solution

Here $\quad |A| = \begin{vmatrix} 4 & -6 & 1 \\ -1 & -1 & 1 \\ -4 & 11 & -1 \end{vmatrix} = 4\,(1-11) - (-6)\,(1+4) + 1\,(-11-4)$

$$= -40 + 30 - 15 = -25\,.$$

Cofactors of the elements of the first row are :

$$C_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 1 \\ 11 & -1 \end{vmatrix} = -10\,,$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} -1 & 1 \\ -4 & -1 \end{vmatrix} = -5\,,$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} -1 & -1 \\ -4 & 11 \end{vmatrix} = -15\,.$$

Cofactors of the elements of second row are :

$$C_{21} = (-1)^{2+1} \begin{vmatrix} -6 & 1 \\ 11 & -1 \end{vmatrix} = 5\,,$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 4 & 1 \\ -4 & -1 \end{vmatrix} = 0\,,$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 4 & -6 \\ -4 & 11 \end{vmatrix} = -20\,.$$

Cofactors of the elements of third row are :

$$C_{31} = (-1)^{3+1} \begin{vmatrix} -6 & 1 \\ -1 & 1 \end{vmatrix} = -5\,,$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 4 & 1 \\ -1 & 1 \end{vmatrix} = -5\,,$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 4 & -6 \\ -1 & -1 \end{vmatrix} = -10\,.$$

$$\therefore \qquad \text{adj } A = \begin{bmatrix} -10 & -5 & -15 \\ 5 & 0 & -20 \\ -5 & -5 & -10 \end{bmatrix}^T = \begin{bmatrix} -10 & 5 & -5 \\ -5 & 0 & -5 \\ -15 & -20 & -10 \end{bmatrix}.$$

Verification

$$A\,(\text{adj } A) = \begin{bmatrix} 4 & -6 & 1 \\ -1 & -1 & 1 \\ -4 & 11 & -1 \end{bmatrix} \begin{bmatrix} -10 & 5 & -5 \\ -5 & 0 & -5 \\ -15 & -20 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} -40+30-15 & 20-0-20 & -20+30-10 \\ 10+5-15 & -5-0-20 & 5+5-10 \\ 40-55+15 & -20+0+20 & 20-55+10 \end{bmatrix}$$

$$= \begin{bmatrix} -25 & 0 & 0 \\ 0 & -25 & 0 \\ 0 & 0 & -25 \end{bmatrix} = -25 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= (-25)\, I_3 = |A|\, I_3$$

Similarly, we can show that $(adj\ \mathrm{A})\ \mathrm{A} = |\mathrm{A}|\ \mathrm{I}_3$.

Hence $(\text{adj } A)\, A = |A|\, I_3 = A\, (\text{adj } A)$.

6.5.2 Inverse of a Square Matrix

Definition 2

*Let $A = (a_{ij})$ be a square matrix of order n, then A is called **inversible** (or **invertible**) iff there exists a square matrix B of order n such that $AB = I_n = BA$, I_n being the unit matrix of order n. B is called an inverse of A.*

Theorem 11

If a square matrix has an inverse, then it is unique.

Proof

Let A be a square matrix of order n, and let B, C (n-rowed square matrices) be two inverses of A, then (by definition).

$$AB = I_n = BA \qquad \text{. . . (i)}$$

and $$AC = I_n = CA \qquad \text{. . . (ii)}$$

Now $AB = I_n \Rightarrow C\,(AB) = CI_n$ (pre-multiplying by C)

$\Rightarrow$ $(CA)\, B = C$ $\quad (\because\ CI_n = C)$

$\Rightarrow$ $I_n B = C$ $\quad$ (by Eq. (ii))

$\Rightarrow$ $B = C$ $\quad (\because\ I_n B = B)$

Hence, inverse of a square matrix, if it exists, is unique.

Notation

If a square matrix A is inversible, then its unique inverse is denoted by A^{-1} and $AA^{-1} = I = A^{-1} A$.

Definition 3 : Non-singular Matrix

*Let A be a square matrix, then A is called **non-singular** iff $|A| \neq 0$, otherwise, A is said to be **singular**, i.e. a square matrix A is **singular** iff $|A| = 0$.*

Theorem 12

A square matrix is inversible iff it is non-singular.

Proof

Let $A = (a_{ij})$ be a square matrix of order n.

First, let A be inversible, then there exists a square matrix B of order n such that

$$AB = I_n = BA \Rightarrow |AB| = |I_n| = |BA|$$

$\Rightarrow$ $|A| . |B| = 1 = |B| . |A|$ $(\because |I_n| = 1)$

$\Rightarrow$ $|A| \neq 0 \Rightarrow A$ is non-singular.

Conversely, let A be non-singular, i.e. $|A| \neq 0$, we know that

$$A\,(\text{adj } A) = |A|\, I_n = (\text{adj } A)\, A$$

$\Rightarrow$ $A\left(\dfrac{1}{|A|} \text{adj } A\right) = I_n = \left(\dfrac{1}{|A|} \text{adj } A\right) A$ $(\because |A| \neq 0)$

Let $B = \dfrac{1}{|A|} \text{adj } A$, then $AB = I_n = BA \Rightarrow A$ is invertible and

$A^{-1} = B = \dfrac{1}{|A|} (\text{adj } A)$.

Cor. 4

If A is an $n \times n$ matrix and there exists a matrix B of order $n \times n$ such that $AB = I_n$ or $BA = I_n$, then A is invertible and $A^{-1} = B$.

Proof

First, let $AB = I_n \Rightarrow \det AB = \det I_n$

$\Rightarrow$ $\det A \det B = 1 \Rightarrow \det A \neq 0$

$\Rightarrow$ A is invertible, i.e. A^{-1} exists.

Now, $AB = I_n \Rightarrow A^{-1}(AB) = A^{-1} I_n$

$\Rightarrow$ $(A^{-1} A)\, B = A^{-1} \Rightarrow I_n\, B = A^{-1}$

$\Rightarrow$ $B = A^{-1}$.

Similarly, we can prove that result when $BA = I_n$.

Cor. 5

If A is non-singular, i.e. A is invertible, then $A^{-1} = I_n = A^{-1} A \Rightarrow A^{-1}$ is invertible and $(A^{-1})^{-1} = A$. (By definition of invertibility)

Cor. 6

A skew symmetric matrix of odd order is always non-invertible as its determinant is always 0.

Theorem 13

If *A*, *B*, *C* are square matrices of the same order such that *AB* = *AC* and *A* is invertible, then *B* = *C*.

Proof

Given $AB = AC$, pre-multiplying both sides by A^{-1}

$$A^{-1}(AB) = A^{-1}(AC), \text{ use associative law}$$

$\Rightarrow \quad (A^{-1}A)\,B = (A^{-1}A)\,C \Rightarrow IB = IC$

$\Rightarrow \quad B = C.$

Theorem 14

If A, B are invertible square matrices of the same order, then AB is also invertible and $(AB)^{-1} = B^{-1}\,A^{-1}$.

Proof

Given A and B are invertible square matrices of same order

$\Rightarrow \quad \det A \neq 0, \det B \neq 0$

$\Rightarrow \quad \det AB = \det A \,.\, \det B \neq 0$

$\Rightarrow \quad AB$ is invertible.

Let $(AB)^{-1} = D$, then $(AB)\,D = I = D\,(AB)$

$\Rightarrow \quad A\,(BD) = I \Rightarrow A^{-1}\,(A\,(BD)) = A^{-1}\,I$

$\Rightarrow \quad (A^{-1}A)\,(BD) = A^{-1}$

$\Rightarrow \quad I\,(BD) = A^{-1} \Rightarrow (IB)\,D = A^{-1}$

$\Rightarrow \quad BD = A^{-1}$

$\Rightarrow \quad B^{-1}\,(BD) = B^{-1}\,A^{-1}$

$\Rightarrow \quad (B^{-1}B)\,D = B^{-1}\,A^{-1}$

$\Rightarrow \quad ID = B^{-1}\,A^{-1} \Rightarrow D = B^{-1}\,A^{-1}.$

Theorem 15

If A is a non-singular matrix of order n, then det (adj A) = (det A)$^{n-1}$.

Proof

We know that $A\,(\text{adj}\,A) = |A|\,I_n = (\text{adj}\,A)\,A$

$\Rightarrow \quad \det\,(A\,(\text{adj}\,A)) = \det\,(|A|\,I_n)$

$\Rightarrow \quad \det A \,\det\,(\text{adj}\,A) = |A|^n \,\det I_n$

$\Rightarrow \quad |A|\,|\text{adj}\,A| = |A|^n \qquad (\because \det I_n = 1)$

$\Rightarrow \quad |\text{adj}\,A| = |A|^{n-1} \quad (\because |A| \neq 0)$ as A is non-singular.

Example 6.13

Let
$$A = \begin{pmatrix} 4 & -6 & 1 \\ -1 & -1 & 1 \\ -4 & 11 & -1 \end{pmatrix}$$

Show that A is invertible. Find A^{-1} and Adj A.

Solution

$$|A| = \begin{vmatrix} 4 & -6 & 1 \\ -1 & -1 & 1 \\ -4 & 11 & -1 \end{vmatrix} = 4\,(1-11) + 6\,(1+4) + (-11-4)$$

$$= -40 + 30 - 15 = -25$$

Hence $\quad |A| \neq 0$

$\therefore \quad A$ is invertible.

Now $\quad C_{11} = -10 \qquad C_{12} = -5 \qquad C_{13} = -15$

$C_{21} = 5 \qquad C_{22} = 0 \qquad C_{23} = -20$

$C_{31} = -5 \qquad C_{32} = -5 \qquad C_{33} = -10$

$$\therefore \quad \text{Adj } A = \begin{bmatrix} -10 & 5 & -5 \\ -5 & 0 & -5 \\ -15 & -20 & -10 \end{bmatrix}$$

and $$A^{-1} = \frac{1}{|A|}\,\text{Adj } A = \begin{bmatrix} \frac{2}{5} & \frac{-1}{5} & \frac{1}{5} \\ \frac{1}{5} & 0 & \frac{1}{5} \\ \frac{3}{5} & \frac{4}{5} & \frac{2}{5} \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & 4 & 2 \end{bmatrix}$$

SAQ 4

(a) Find the adjoint of the following matrices

(i) $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

(b) Find the inverse of the following matrices

(i) $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

6.6 SOLUTION OF LINEAR EQUATIONS WITH THE HELP OF INVERSE OF A MATRIX

We have seen in Example 6.6 that a system of linear equations can be written in the matrix form as $AX = B$.

We can solve the system of equations by finding A^{-1}.

Example 6.14

Solve the system of equations

$$2x - 3y + 3z = 1$$

$$2x + 2y + 3z = 2$$

$$3x - 2y + 2z = 3$$

Solution

This system of equation can be written in the form

$$\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \ldots$$

(i)

Let $A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

then $|A| = 2\,(4 + 6) + 3\,(4 - 9) + 3\,(-4 - 6)$

$= 20 - 15 - 30 = -25$

Also $C_{11} = 10$ $\quad C_{12} = 5$ $\quad C_{13} = -10$

$C_{21} = 0$ $\quad C_{22} = -5$ $\quad C_{23} = -5$

$C_{31} = -15$ $\quad C_{32} = 0$ $\quad C_{33} = 10$

$$\therefore \quad A^{-1} = -\frac{1}{25} \begin{bmatrix} 10 & 0 & -15 \\ 5 & -5 & 0 \\ -10 & -5 & 10 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix}$$

Premultiplying (i) by A^{-1}, we have

$$-\frac{1}{5}\begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

i.e. $$A^{-1}A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 2-9 \\ 1-2 \\ -2-2+6 \end{bmatrix}$$

i.e. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -7 \\ -1 \\ 2 \end{bmatrix}$$

$\therefore$ $x = \frac{7}{5}, y = \frac{1}{5}, z = -\frac{2}{5},$

SAQ 5

Solve the following equations by finding inverse of a matrix

(i) $2x + 3y + 3z = 5$

$x - 2y + z = -4$

$3x - y - 2z = 3$

(ii) $x + y - z = 0$

$x - 2y + z = 0$

$3x + 6y - 5z = 0$

(iii) $x - 3y + 4z = 0$

$2x + 3y - z = 0$

$3x + y + 3z = 0$

6.7 SUMMARY

Type of Matrices

$A = (a_{ij})$ be an $m \times n$ matrix, then A is said to be

- a row matrix iff $m = 1$,
- a column matrix iff $n = 1$,
- a zero matrix iff $a_{ij} = 0$ for all i and j,
- a square matrix iff $m = n$,
- a diagonal matrix iff $m = n$ and $a_{ij} = 0$, for $i \neq j$, and
- a unit matrix iff $m = n$, $a_{ij} = 0$ for $i \neq j$ and $a_{ij} = 1$ for all i.

Algebra of Matrices

- Two matrices are equal iff they are of the same type and corresponding entries are equal.
- If $A = (a_{ij})$ and $B = (b_{ij})$ are of the same order, then $A + B = (a_{ij} + b_{ij})$, $-A = (-a_{ij})$ and $\lambda A = (\lambda a_{ij})$, where λ is a scalar.
- If $A = (a_{ij})$, $B = (b_{jk})$ are two $m \times n$ and $n \times p$ matrices, respectively, then $AB = (c_{ik})$ is an $m \times p$ matrix where $c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$, for $1 \le i \le m, 1 \le k \le p$.

Properties of Matrices

If A, B, C, O (null) are matrices of the same order and λ, μ are scalars, then

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $\lambda\,(A + B) = \lambda A + \lambda B$
- $(\lambda + \mu)\, A = \lambda A + \mu A$
- $(\lambda\, \mu)\, A = \lambda\, (\mu A)$
- $(AB)\, C = A\,(B\, C)$ A, B, C are suitable matrices
- $\lambda\,(AB) = (\lambda A)\, B = A\,(\lambda B)$
- $(A + B)\, C = AC + BC$
- In general $AB \neq BA$
- AB may be zero when both A and B are non-zero matrices.

Transpose of a Matrix

- If $A = (a_{ij})$ is an $m \times n$ matrix, then $B = (b_{ji})$ where $b_{ji} = a_{ij}$ is an $n \times m$ matrix which is the transpose of A and is denoted by A^T.

Symmetric and Skew-symmetric Matrices

- A matrix $A = (a_{ij})$ is symmetric if $A^T = A$ and skew-symmetric if $A^T = -A$.

Determinant of a Square Matrix

- If $A = [a_{11}]$, then det $A = |A| = a_{11},$
- If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then det $A = a_{11}\, a_{22} - a_{21}\, a_{12}$,
- If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$|A| = a_{11} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

- In general if $A = (a_{ij})$ is a square matrix of order n, then

$$|A| = \Sigma\, a_{ij}\ |A_{ij}|\, (-1)^{(i+j)}$$

where A_{ij} is the minor of a_{ij} and is obtained from the matrix A by deleting the ith row and jth column. $C_{ij} = (-1)^{(i+j)}\, |A_{ij}|$ and C_{ij} is called the cofactor of a_{ij}.

Properties of Determinants

- det $I_n = 1$, where I_n is the identity matrix of order n.
- The value of a determinant remains unaltered if its rows and columns are interchanged.
- If two rows or columns of a determinant are interchanged, then the value of the determinant changes in sign.
- If two rows or columns of a determinant are identical, then the value of the determinant is zero.
- If each element of a row or column of a determinant is multiplied by the same number, then the value of the determinant is multiplied by the same number.
- If each element in any row or column of a determinant consists of two terms, then the determinant can be expressed as the sum of two determinants.
- If the same multiple of elements of any row or column of a determinant are added to the corresponding elements of any other row or column, then the value of the new determinant remains unaltered.

Adjoint of a Square Matrix

- Let $A = (a_{ij})$ be a square matrix of order n. Adj $A = (C_{ij})^T$ where C_{ij} is the cofactor of a_{ij} in $|A|$ and $A\,(\text{Adj } A) = |A|\, I_n = (\text{adj } A)\, A$.

Inverse of a Matrix

Let $A = (a_{ij})$ be a square matrix of order n. If there exists a matrix B of order n

- Such that $AB = BA = I_n$, then A is said to be invertible and B is called the inverse of A and is denoted by A^{-1}.
- A square matrix A is called non-singular iff $|A| \neq 0$ otherwise it is called singular.
- A square matrix A is invertible iff $|A| \neq 0$ and $A^{-1} = \dfrac{1}{|A|}(\text{adj } A)$ and

  $(AB)^{-1} = B^{-1}\, A^{-1}$.

6.8 ANSWERS TO SAQs

SAQ 2

(b) $X = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$

SAQ 3

(a) (i) $\Delta = 0$

(ii) $\Delta = 4a\,b\,c$

(iii) $\Delta = 0$

SAQ 4

(a) (i) $\begin{bmatrix} 3 & 3 & 0 \\ -11 & 1 & 8 \\ -1 & -1 & 4 \end{bmatrix}$

(ii) $\begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ -6 & -2 & 5 \end{bmatrix}$

(b) (i) $-\dfrac{1}{3}\begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$

(ii) $-\begin{bmatrix} -2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$

SAQ 5

(i) $x = 1, y = 2, z = -1$

(ii) $x = \frac{1}{3}, y = \frac{2}{3}, z = 1$

(iii) $x = y = z = 0$