UNIT 2 PRINCIPAL STRESSES

Structure

2.1 INTRODUCTION

In Unit 1, you have already been introduced to simple states of stress. Stress Analysis is an essential requirement in the evaluation of strength, stiffness, deformations and safety of solids so that one may produce functionally efficient and economic designs. There is a large number of ways in which stresses are induced in solids (a few samples ones you have already learnt), which will engage your attention in the subsequent units. In this unit, we shall be concerned with the analysis of a given state of stress (expressed in terms of stress components on selected planes) which will have a bearing on the analysis of strength and safety of solid components.

Objectives

After studying this unit, you should be able to

- define six stress components on mutually perpendicular planes at the requisite location,
- describe the principal plane and principal stress, and
- identify the plane of maximum shear stress.

2.2 STATE OF STRESS

From the point of functional utilization of a solid component we may determine the possible loads (forces) to which it may be subjected to, so that its equilibrium, compatibility and stability are satisfied on the whole. But a more critical analysis will imply the satisfaction of equilibrium at each and every point of the solid. The distribution of stresses over the volume of the solid is analyzed taking into these requirements. Once such a distribution has been arrived at it will give the state of stress at each and every point in the solid in terms of the stress components. Often one is not interested in the state of stress at each and every point in the solid, but is satisfied with the analysis of the state of stress at the critical locations of the solid. Description of the general state of stress involves the definition of six stress components namely, σ_x, σ_y, σ_z, τ_{xy}, τ_{yz} and τ_{zx} on the three mutually perpendicular planes of a small element at the requisite location. However, in the initial stages of the course, it is sufficient to master the concepts with reference to the state of stress in two dimensions. The general state of stress at any point in a two-dimensional element is given by the stress components σ_x, σ_y and τ_{xy} as shown in Figure 2.2. Of course, any element could only be three-dimensional, but the state of stress is two-dimensional due to the absence of any stress components in the pair of *z* planes. Hence, in considering equilibrium of forces, the dimension of the element in *z* direction is taken as unity; in whatever units the other two dimensions are expressed.

2.3 NORMAL AND SHEAR STRESSES

You have been already introduced to the concept, definition and description of normal stress and shear stress. In expressing shear stress components, we use two subscripts, such as τ_{xy}, τ_{yz}, τ_{zx} etc. Here, the first subscript denotes the direction of normal to the plane and the second subscript denotes the direction in which stress in *y* direction on *x* plane, i.e. plane normal to *x* direction. Logically, all the stress components should have double subscripts. However, as direction of the stress and direction of the normal to the plane are identically same in the case of normal stress component, only a single subscript is used, i.e. σ_x really represents σ_{xx} and so on. In the case of a shear stress component, two subscripts are necessary to define it correctly. The second subscript also indicates the plane on which its complementary component is acting.

We have already stated that among normal stresses, tension is considered positive while compression is considered negative. In the case of shear stresses, one of the components tends to rotate the element in the positive, i.e. anticlockwise direction and is considered positive, while its complementary component which tends to rotate the element in the clockwise direction is considered negative. Accordingly, in the state of stress, described in Figure 2.2, τ_{xy} is positive, while τ_{yx} is negative. This definition helps us to determine the sign of the shear stress on inclined planes also.

2.4 STRESS ON OBLIQUE SECTIONS

We have already seen that the equilibrium of a rigid body should be satisfied overall and in addition if we divide it into a number of small rigid bodies each of these small elemental bodies should also be in equilibrium individually. The satisfaction of equilibrium does not depend on the way elements are divided. Even if the solid is divided by inclined or even curved surfaces, equilibrium must be satisfied for each of the elements thus divided.

Consider the solid shown in Figure 2.1 where the solid is divided into small elements by inclined planes. The inclination (or orientation) of a plane is defined by its **aspect angle,** defined as the angle made by its normal to the longitudinal axis of the original bar. Let the aspect angle of the plane be θ. Since the width of the plane *b* is unaltered and length of the plane *h* is increased to $\frac{h}{\cos\theta}$, the area of the inclined plane is $\frac{A}{\cos\theta}$, where *A* is area of cross section of the original solid.

Figure 2.1

If σ_x be the stress acting on the plane normal to *x* axis (longitudinal axis), then the axial force $P = \sigma_x \times A$. This force acting on the inclined plane may be resolved into normal and tangential components.

Normal component on the plane $= P \cos\theta$

$$= \sigma_x \times A \cos\theta$$

Tangential component $= - P \sin\theta$

$$= - \sigma_x \times A \sin\theta$$

Normal stress on the plane $= \dfrac{\sigma_x \times A \cos\theta}{\dfrac{A}{\cos\theta}} = \sigma_x \times \cos^2\theta$ . . . (2.1)

Shear stress on the plane $= \dfrac{-\sigma_x \times A \sin\theta}{\dfrac{A}{\cos\theta}} = \sigma_x \times \cos\theta \sin\theta$ . . . (2.2)

If on the cross sectional area of the original solid shear stress τ_{xy} is applied, its components on the inclined plane may be evaluated as,

Normal stress component $= \tau_{xy} \cos\theta \sin\theta$ . . . (2.3)

Shear stress component $= \tau_{xy} \cos^2\theta$ . . . (2.4)

By a similar analysis (your exercise), you may verify the following:

If a normal stress of σ_y is applied on the solid, then the stress components on a plane whose normal is inclined at θ to the *x* axis are given by

Normal stress $= \sigma_y \sin^2\theta$ . . . (2.5)

Shear stress $= \sigma_y \sin\theta \cos\theta$ . . . (2.6)

If a shear stress of τ_{yx} is applied on the solid, the stress components on the inclined plane are given by (with $\tau_{yx} = \tau_{xy}$).

Normal stress $= \tau_{xy} \sin\theta \cos\theta$ . . . (2.7)

Shear stress $= \tau_{xy} \sin^2\theta$ . . . (2.8)

Figure 2.2 : General State of Stress in Two Dimensions

Since a general state of stress in two dimensions is defined by the stress components σ_x, σ_y and τ_{xy}, as shown in Figure 2.2, general expressions for normal and shear stress components may be obtained by algebraic sum of the respective components from Eqs. (2.1) to (2.8), as given below :

$$\sigma_n = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2\, \tau_{xy} \cos \theta \sin \theta$$

$$\tau_{nt} = - \sigma_x \cos \theta \sin \theta + \sigma_y \cos \theta \sin \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta)$$

On further simplification, we obtain

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \qquad \ldots (2.9)$$

$$\tau_{nt} = \left(\frac{\sigma_y - \sigma_x}{2} \right) \sin 2\theta + \tau_{xy} \cos 2\theta \qquad \ldots (2.10)$$

Application of Eqs.(2.9) and (2.10) will be elaborately dealt with in this unit. However, the significance of the equations should be stressed at this stage. When we are carrying out stress analysis on solids, we may be evaluating stress components on a set of mutually perpendicular planes, such as σ_x, σ_y and τ_{xy}. The magnitudes of these components may not always be sufficient to decide whether the solid is safe or not, and we require evaluation of stress components on other specific planes, or determination of planes on which the stress components have extreme value. Such an analysis can be easily carried out with the help of Eqs. (2.9) and (2.10).

Example 2.1

Figure 2.3 shows the projection of a rectangular prism ABCD, formed by adhesive bonding of two triangular prisms ABC and ACD. The state of stress in the prism is given by the components σ_x = 40 N/mm², σ_y = 0 and τ = 0.

Figure 2.3

If the tensile and shear strengths of the adhesive are 10 N/mm² and 12 N/mm², verify the safety of the joint and find out the value of σ_x at which the joint will fail.

Solution

The aspect angle θ of the bonding plane AC

$$= 90^\circ + \tan^{-1}\left(\frac{50}{75}\right) = 90^\circ + 33.69^\circ = 123.69^\circ$$

Known stress components are as follows:

$$\sigma_x = 40 \text{ N/mm}^2, \ \sigma_y = 0 \text{ and } \tau_{xy} = 0$$

Stress components on plane AC,

Normal stress, $\sigma_n = \sigma_x \cos^2 \theta$

$$= 40 \cos^2 123.69^\circ$$

$$= 12.3076 \text{ N/mm}^2$$

Shear stress $\tau_{nt} = -\frac{\sigma_x}{2} \sin 2\theta$

$$= -\frac{40}{2} \sin (2 \times 123.69^\circ)$$

$$= -18.462 \text{ N/mm}^2 > 12 \text{ N/mm}^2$$

The tensile stress on plane AC is well within the tensile strength of the bond. But the shear stress on the plane exceeds the shear strength of the bond and hence, the bond will fail in shear.

Let us find the normal stress σ_x that may be safely applied.

Shear strength of the bond = 12 N/mm²

Shear stress on bonding plane $= -\frac{\sigma_x}{2} \sin 2\theta$

$$\therefore \qquad -12 = -\frac{\sigma_x}{2} \sin 247.38^\circ$$

$$\sigma_x = \frac{12 \times 2}{\sin 247.38^\circ} = 26 \text{ N/mm}^2$$

So the maximum stress we may apply on the plane CB is 26 N/mm². Here, you may note that in strength analysis the sign of the shear stress has no significance, while the sign of the normal stress is important, since the tensile and compressive strengths may differ considerably.

SAQ 1

If the prism shown in Figure 2.3 is bonded along the diagonal DB, instead of AC, verify the safety of the joint and calculate the magnitude of σ_x at which the joint will fail.

2.5 PRINCIPAL STRESSES AND PRINCIPAL PLANES

In Section 2.4, we have seen that for a given state of stress at a point, the magnitude of normal stress and shear stress may vary with respect to the inclination of planes. If we are concerned with the safety of solids under stress, we are required to find on which planes extreme values of normal and shear stress components are present. Hence, it is essential to know,

(i) Maximum tensile stress,

(ii) Maximum compressive stress, and

(iii) Maximum shear stress.

In addition, we may also require to know the planes on which these values occur. The extreme values of normal stresses are called the **Principal Stresses** and the planes on which the principal stresses act are called the principal planes. In two-dimensional problems, there are two principal stresses, namely the **major principal stress** and the **minor principal stress** which are defined as the maximum and minimum values of the normal stresses respectively. Here, the maximum or minimum is to be considered algebraically. For example, if the principal stresses happen to be 20 N/mm² tensile and 75 N/mm² compressive, the tensile stress of 20 N/mm² is to be taken as the major principal stress denoted by the symbol σ_1 and the compressive stress of 75 N/mm² is to be taken as the minor principal stress (algebraically –75 N/mm²) and denoted by the symbol σ_2 . The corresponding planes are defined as major and minor principal planes.

2.5.1 Expressions for Principal Planes and Principal Stresses

In calculus, you have learnt that when a function reaches maximum or minimum its derivative with respect to the independent variable becomes zero. Since the normal stress on an arbitrary plane is a function of the aspect angle θ as given by the expression, $\sigma_n = \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$, the maxima and minima of σ_n occur on the planes for which $\frac{d\sigma_n}{d\theta}$ becomes zero, (similarly, τ_{nt} will be maximum on planes where $\frac{d\tau_{nt}}{d\theta} = 0$).

Let us now derive the expression,

$$\frac{d\sigma_n}{d\theta} = \frac{\sigma_x - \sigma_y}{2} (-2 \sin 2\theta) + \tau_{xy}\, 2 \cos 2\theta$$

$$= 2\left(\tau_{xy} \cos 2\theta + \frac{\sigma_y - \sigma_x}{2} \sin 2\theta\right)$$

$$= 2\,\tau_{nt}$$

i.e. $$\frac{d\sigma_n}{d\theta} = 2\,\tau_{nt} \qquad \text{. . . (2.11)}$$

Eq. (2.11) gives an important characteristic of the principal plane, namely, the absence of shear stress components on the plane. We can, therefore, alternatively define a principal plane as a plane on which only a normal stress component is acting. When dealing with a three-dimensional state of stress you will find that the third principal plane is neither maximum nor minimum. Hence, we will define principal planes as planes on which shear stresses are zero.

Equating $\frac{d\sigma_n}{d\theta}$ to zero, we get

$$\tau_{xy} \cos 2\theta + \frac{\sigma_y - \sigma_x}{2} \sin 2\theta = 0$$

or $$\frac{\sin 2\theta}{\cos 2\theta} = \frac{-2\tau_{xy}}{\sigma_y - \sigma_x} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Denoting the specific angles defining principal planes by ϕ_1 and ϕ_2,

or $$\tan 2\phi = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \text{. . . (2.12)}$$

Eq. (2.12) gives a condition for the determination of principal planes. Eq. (2.12) will have two solutions within the range $-\pi/2 < \phi < \pi/2$ and they will give the orientation of principal planes.

Further, the second derivatives, $\frac{d^2\sigma_n}{d\theta^2}$, will be negative for the solution ϕ_1 (aspect angle of the major principal plane) and positive for the solution ϕ_2 (aspect angle of the minor principal plane). Let us obtain these expressions too.

$$\frac{d^2\sigma_n}{d\theta^2} = \frac{d}{d\theta}\left(\frac{d\sigma_n}{d\theta}\right) = -2\left(\frac{\sigma_y - \sigma_x}{2} 2\cos 2\theta - 2\tau_{xy} \sin 2\theta\right)$$

$$\frac{d^2\sigma_n}{d\theta^2} = -4\left(\frac{\sigma_y - \sigma_x}{2} \cos 2\theta - 2\tau_{xy} \sin 2\theta\right) \qquad \text{. . . (2.13)}$$

After obtaining the solutions ϕ_1 and ϕ_2 of Eq. (2.12), their values may be substituted in the expression for $\frac{d^2\sigma_n}{d\theta^2}$ given in Eq. (2.13) and the major and minor principal planes may be identified. But in practical solutions this step is rarely required.

Instead, substitute the two solutions ϕ_1 and ϕ_2 in the expression for normal stress and obtain the values of principal stresses σ_1 and σ_2 and corresponding principal planes may be identified.

Now, let us derive the general expressions for the principal stresses. Since, we know that $\tan 2\phi = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$ on a principal plane, we may write as

$$\sin 2\phi = \frac{\tau_{xy}}{\sqrt{\tau^2{}_{xy} + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2}} \qquad \text{. . . (2.14)}$$

and $$\cos 2\phi = \frac{\frac{\sigma_x - \sigma_y}{2}}{\sqrt{\tau^2{}_{xy} + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2}} \qquad \text{. . . (2.15)}$$

Substituting Eqs. (2.14) and (2.15) in the expression for σ_n, we obtain

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\left(\frac{\frac{\sigma_x - \sigma_y}{2}}{\sqrt{\tau^2_{xy} + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2}}\right) + \frac{\tau_{xy}\tau_{xy}}{\sqrt{\tau^2_{xy} + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2}}$$

$$= \frac{\sigma_x + \sigma_y}{2} + \left(\frac{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2}{\sqrt{\tau^2_{xy} + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2}}\right) + \frac{\tau^2_{xy}}{\sqrt{\tau^2_{xy} + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2}}$$

$$= \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}}$$

Since $\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}}$ will have two roots namely $\pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}}$, we may write the final expression for major and minor principal stresses as follows :

$$\sigma_{1,\,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}} \qquad \text{. . . (2.16)}$$

Eqs. (2.2) and (2.6) may be used to readily determine the principal planes and principal stresses.

Let us now have an example for determination of principal stress and principal planes, given the state of stress.

Example 2.2

Evaluate the principal stresses and principal planes for the state of stress shown in Figure 2.4.

Figure 2.4

Solution

Given $\sigma_x = 60$ N/mm²

$\sigma_y = 20$ N/mm²

$\tau_{xy} = -26$ N/mm²

On substituting in Eq. (2.6), we get

$$\sigma_{1,\,2} = \frac{60+20}{2} + \sqrt{\left(\frac{60-20}{2}\right)^2 + (-26)^2} = 40 \pm 32.8$$

$\therefore$ $\sigma_1 = 72.8$ N/mm² and $\sigma_2 = 7.2$ N/mm²

Again substituting the values σ_x, σ_y and σ_{xy} in Eq. (2.2).

$$\tan 2\phi = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2\times(-26)}{60-20} = -\,1.3$$

Since θ is general angle, the specific angles representing the principal planes are designated as ϕ_1 and ϕ_2.

$\therefore$ $2\phi = -\,52.43^\circ, 127.57^\circ$

using $2\phi = -\,52.43^\circ$

$$\sigma_n = \frac{60+20}{2} + \frac{60-20}{2}\cos(-\,52.43^\circ) - 26\sin(-\,52.43^\circ)$$

$$= 72.8 \text{ N/mm}^2$$

Hence, we recognize that $\phi_1 = \dfrac{-\,52.43^\circ}{2}$ defines the major principal plane

and therefore, $\phi_2 = \dfrac{127.57^\circ}{2}$ should define the minor principal plane.

SAQ 2

(a) Derive an expression for the maximum shear stress in a general two dimensional state of stress and also an expression for the aspect angle of the corresponding plane.

(b) Evaluate the principal stresses and principal planes for the state of stress shown in Figure 2.5.

Figure 2.5

(c) Also find the normal and shear stress components on the planes whose aspect angles are given as 30º, 45º and 75º.

2.5.2 Maximum Shear Stress

We have the general expression for shear stress as

$$\tau_{nt} = \tau_{xy} \cos 2\theta - \left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta$$

Differentiating w.r.t. θ, and equating the derivative to zero,

$$\frac{d\tau_{nt}}{d\theta} = -2\,\tau_{xy} \sin 2\theta - (\sigma_x - \sigma_y) \cos 2\theta = 0$$

$$\therefore \qquad \tan 2\theta = \frac{-(\sigma_x - \sigma_y)}{2\tau_{xy}} \qquad \text{. . . (2.17)}$$

Since the planes on which maximum shear stresses occur are specific set of planes we may denote them distinctly by Ψ (instead of general aspect angle θ).

Comparing Eqs. (2.12) and (2.17), we conclude that $2\Psi = 2\phi \pm 90^\circ$ as $\tan 2\phi . \tan 2\Psi = -1$.

$$\therefore \qquad \Psi = \phi \pm 45^\circ \qquad \text{. . . (2.18)}$$

Eq. (2.18) indicates that the planes of maximum shear stress bisect the right angles between the major and minor principal planes.

The normals to the major and minor principal planes may now be defined as the major and minor principal axes. Once the principal stresses and principal planes are known, further analysis may be simplified by expressing the state of stress w.r.t. a new coordinate system with major and minor principal axes as coordinate axes themselves. These axes are usually called axes 1 and 2 respectively.

The general expressions for stress components on arbitrary planes whose aspect angle $\bar{\theta}$ may now be measured with axis-1 as reference axis.

Hence,

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\bar{\theta} \qquad \text{. . . (2.19)}$$

$$\tau_{nt} = \frac{\sigma_1 - \sigma_2}{2} \sin 2\bar{\theta} \qquad \text{. . . (2.20)}$$

Eq. (2.8) already defines that $\bar{\theta}$ should be ± 45° for τ_{nt} to be maximum.

Thus,

$$\tau_{\text{max, min}} = \frac{\sigma_1 - \sigma_2}{2} \sin (\pm 90^\circ)$$

$$\therefore \qquad \tau_{\max} = \frac{\sigma_1 - \sigma_2}{2}, \text{ and} \qquad \text{. . . (2.21)}$$

$$\tau_{\min} = -\frac{\sigma_1 - \sigma_2}{2}$$

Since the sign of maximum shear stress is not significant, expression for $\tau_{\min}$ is not generally used. Let us have a few examples.

Example 2.3

The state of stress at a critical point of a strained solid is given by $\sigma_x = 70$ kN/mm², $\sigma_y = -50$ kN/mm² and $\tau_{xy} = 45$ kN/mm². If the strength of the solid in tension, compression, and shear are given as 120 kN/mm², 90 kN/mm² and 75 kN/mm² respectively, verify the safety of the component.

Solution

Given $\sigma_x = 70$ kN/mm²

$\sigma_y = -50$ kN/mm²

$\tau_{xy} = 45$ kN/mm²

$$\therefore \quad \sigma_{1,2} = \frac{70 + (-50)}{2} \pm \sqrt{\left(\frac{70 - (50)}{2}\right)^2 + (45)^2}$$

$= 56.09, -36.09$ N/mm²

Maximum shear stress, $\tau_{max} = \dfrac{\sigma_1 - \sigma_2}{2} = \dfrac{56.09 - (-36.09)}{2} = 46.09$ N/mm²

$= 46.09$ N/mm²

All the stresses are within the strength limits of the solid and hence, the solid is safe.

Factor of safety in tension $= \dfrac{120}{56.09} = 2.139$

Factor of safety in compression $= \dfrac{90}{36.09} = 2.494$

Factor of safety in compression $= \dfrac{75}{46.09} = 1.6273$

Here, maximum tensile and compressive stresses are well within strength limits, maximum shear stress has reached the strength limit and therefore if the state of stress is proportionally raised the solid will fail in shear.

Example 2.4

A machine component is made of a material whose ultimate strength in tension; compression and shear are 40 N/mm², 110 N/mm² and 55 N/mm² respectively. At the critical point in the component the state of stress is represented by

$\sigma_x = 25$ kN/mm² and $\sigma_y = -75$ kN/mm².

Find the maximum value of the shear stress τ_{xy} which will cause failure of the component and also specify the mode of failure.

Solution

Given state of stress : $\sigma_x = 25$ kN/mm²

$\sigma_y = -75$ kN/mm².

We have to find what τ_{xy} is safe, if $\sigma_1 \leq 40$ kN/mm², $\sigma_2 \geq -110$ kN/mm² and $\tau_{max} > 40$ kN/mm².

The above three conditions are to be independently satisfied.

$$\text{Now,} \quad \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}} \leq 40$$

In the limiting case,

$$40 = \frac{25 + (-75)}{2} + \sqrt{\left(\frac{25 - (-75)}{2}\right)^2 + \tau^2{}_{xy}}$$

$$40 = -25 + \sqrt{50^2 + \tau^2{}_{xy}}$$

or $\qquad \tau^2{}_{xy} = [\,40 - (-25)\,]^2 - 50^2 = 1725$

$\therefore \qquad \tau_{xy} = \pm\, 41.533 \text{ N/mm}^2$

Note that the limiting case of $\sigma_1 = 40$ N/mm² will occur for both the τ_{xy} values of 41.533 N\mm² and – 41.533 N/mm². But the planes of failure will be different.

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}} \geq 110$$

In the limiting case,

$$-\,110 = \frac{25 + (-75)}{2} + \sqrt{\left(\frac{25 - (-75)}{2}\right)^2 + \tau^2{}_{xy}}$$

$$-\,110 = -\,25 - \sqrt{50^2 + \tau^2{}_{xy}}$$

$$-\,85 = -\sqrt{50^2 + \tau^2{}_{xy}}$$

$$-\,(85)^2 = -\,50^2 + \tau^2{}_{xy}$$

$$\tau_{xy} = \pm\sqrt{85^2 + 50^2} = \pm\, 68.7386 \text{ N/mm}^2$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2{}_{xy}} \leq 55$$

i.e. $\qquad 50^2 + \tau^2{}_{xy} = (55)^2$

$$\tau_{xy} = \pm\sqrt{55^2 - 50^2} = \pm\, 22.91 \text{ N/mm}^2$$

The permissible value of τ_{xy} is different for different limiting criteria, namely

$$| \tau_{xy} | \leq 41.53 \text{ if } \sigma_1 \leq 40$$
$$\leq 68.74 \text{ if } \sigma_2 \geq -\,110$$
$$\leq 22.91 \text{ if } \tau_{max} \geq 55$$

Hence, we find that the maximum safe value of τ_{xy} is only 22.91 N/mm² and the material will fail in shearing mode.

SAQ 3

If the state of stress at a point is defined by the stress component $\sigma_x = 9$ MPa, $\sigma_y = -7$ Mpa, $\tau_{xy} = 5$ MPa, find the principal stresses and principal planes. Also find the plane on which normal and shear stress components are equal in magnitude.

2.6 SUMMARY

This unit is a vital link in the analysis of solids so as to ensure safe design of different components of structures or machines or other systems. Here, you were exposed to a deeper insight into the implications of a given state of stress. You have learnt how to evaluate the stress components on different planes and also to find the extreme values of stress components.

2.7 ANSWERS TO SAQs

SAQ 1

On plane DB, $\sigma_n = 12.31$, $\tau_{nt} = 18.462$ both unsafe; when $\sigma_x = 32.494$, the joint will fail in tension.

SAQ 2

(a) $\text{Tan } 2\psi = \dfrac{\sigma_x - \sigma_y}{2\tau_{xy}}$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}}$$

(b) $\sigma_{1,2} = 49, -41$

$\phi_{1,2} = 71.565°, -18.435°$

(c) For $\theta = 30°$, $\sigma_n = 9.3827$ and $\tau_{nt} = 44.677$

For $\theta = 45°$, $\sigma_n = 31$ and $\tau_{nt} = 36$

For $\theta = 75°$, $\sigma_n = 48.677$ and $\tau_{nt} = -5.383$

SAQ 3

$\sigma_n = 93.57$ and $\tau_{nt} \leq 42.93$