# UNIT 1 STRESSES AND STRAINS

**Structure**

## 1.1 INTRODUCTION

The concepts of Strength of Material course involve finding various types of stresses in structural members (composing a structure, a machine, or any equipment) due to the action of external forces (loads) that it is supported to resist. For any design problem, analysis of internal effects, i.e. force, stress, strain and deformation etc. is a essential intermediate step. In fact, it is very first step towards studying the behaviour of material under the influence of external loads.

In this unit, we will study the concepts of stress and strain which are of utmost importance for understanding the behaviour of materials. The unit includes

description pertaining to various types of stress, deformation caused due to these stresses, composite bars and relationship between different elastic constants.

## Objectives

After studying this unit, you should be able to

- grasp the concepts of stress, strain and deformation,

- elaborate the effect of forces on deformable solids and the relationship between stress and strain for typical materials,

- explain the elastic properties of solids and their deformations due to direct forces,

- explain the methods of analyzing composite bars,

- explain the effect of temperature on bodies,

- understand how thermal stresses are developed in materials,

- determine thermal stresses in uniform, stepped, tapered and compound bars, and

- define the elastic constants and their interrelationships.

When you have clearly learnt these concepts, you will be able to use your knowledge to understand the behaviour of solids under simple loadings and will be able to design such members. You will also be able to verify whether such members are safe under the loads they are subjected to. In addition, you will also be able to find the load carrying capacity of members whose geometric and elastic properties are known.

## 1.2 BASIC CONCEPTS

### 1.2.1 Rigid and Deformable Solids

The subject matter of Strength of Materials is also under the title Applied Mechanics, and Mechanics of Solids. Applied Mechanics is the general title in which Fluid Mechanics and Solid Mechanics are branches. What we study in Strength of Materials is the effect of forces acting on solids of different geometrical and elastic properties, and hence more people now choose to call it solid mechanics.

Let us now learn two terms used to describe two types of solids. A **rigid solid** is one which does not undergo any change in its geometry, size or shape. On the other hand, a deformable solid is one in which change in size, shape or both will occur when it is subjected to a force. The geometrical changes produced are called **deformations** and hence the name deformable solid. Many of the common solid articles we handle in our day-to-day life exhibit perceptible deformations when subjected to loads. In many other solids, though we may not perceive any deformations with our naked eye, with measuring instruments of sufficient precision we can see that they also get deformed under applied forces. A more careful scrutiny will reveal that all solids are deformable and the idea of rigid solids is only a conceptual idealization.

Look at the common swing shown in Figure 1.1, which you would have enjoyed playing with during your earlier days. When a child sits on the swing we may not perceive any appreciable deformations. But if the iron bars be replaced by

bamboo sticks and the iron chain be replaced by rubber strings, we can observe considerable deformations as shown in Figure 1.2. We may then conclusively say that whenever forces are applied on solids, deformations are introduced in them. All solids are deformable solids; hence, the subject which once was called **Mechanics of Deformable Solids** is now simply called Mechanics of Solids.

**Figure 1.1**

**Figure 1.2**

You may wonder as to why we should have the term rigid solid at all, if in reality, there are no such solids. It has been already mentioned that the term rigid solid is only a conceptual idealization. There are a few important uses for such idealization. Consider the analysis of forces in the members of some of the plane trusses. You were given the geometry of a structure (which in its totality may be considered as a solid of complex geometry), on which some forces have been presented to act. While analyzing these structures (solids) for finding out the member forces, the forces in the members have been resolved into their horizontal and vertical components, assuming that the member orientations have not changed due to the application of external forces. In reality, the orientations of most of the members change, as the structure deforms under the loads. However, these deformations and hence, the resulting changes in member orientations are so small that the error in computed member forces are negligibly small. You may think that even these errors may be eliminated by taking into account the deformations too. But an attempt to do so will reveal the complexity of the calculations involved. The corrections thus effected are called **secondary effects** and may be neglected in most of the cases. In the analysis of other types of

structures also when the overall equilibrium is considered the entire structure or any part under consideration may be treated as a rigid solid.

In many cases, known as determinate problems, the analysis of forces within a solid can be calculated treating it as rigid.

### 1.2.2  Stresses and Strains

When we apply forces on solids, deformations are produced if the solid is prevented from motion (with acceleration = force/mass of solid) either fully or partially. If the solid is not restrained, it may undergo displacements without change in shape or size and these displacements are termed as rigid body displacements. If the solid is restrained by some other force, known as **reaction,** which keeps the solid in equilibrium, the force will be transmitted through the medium of the solid to the restraining support.

Consider the solid shown in Figure 1.3(a) which is pulled by the force *P* at the right hand side and restrained by a support at the left hand side end. If the solid is to remain in static equilibrium we know that the support should exert an equal pulling force in the direction opposite to that of force *P*. This restraining force, termed as support reaction, is shown as *R* in Figure 1.3(a). Now consider the solid being divided into a number of small bits or elements of different lengths as shown in Figure 1.3(b). The force *P* applied at the L.H.S. end of element 1 is transmitted to the support through the medium of elements 2, 3 and 4. The same thing is true even if you divide the solid into hundreds or even millions of elements.

**(a)**

**(b)**

**Figure 1.3**

Now, to keep the element 1 in static equilibrium, element 2 should exert a force *R* (= *P*) on element 1. At the same time, element 1 exerts a force *P* on element 2 so that equilibrium exists at the point or surface of contact. The element 2 is again pulled by the equilibrant *R* exerted by element 3. Thus equilibrium exists at each and every point or surface of contact and each of the elements are kept in equilibrium.

The force *P* (or *R*) shown in the figures is only the resultant of forces applied over some area (small or even full). On any area the force may not be distributed uniformly. However, we may consider an infinitesimally small area $\Delta A$ over which a small part of the total force $\Delta P$ may be considered to be uniformly acting.

Here, the ratio $\dfrac{dP}{dA} = \left( \text{or rather} \underset{\Delta A \to 0}{\text{Lt}} \dfrac{\Delta P}{\Delta A} \right)$ may be called as pressure (suctional or compressive) at the point. Considering a small element having the area $dA$ on one of its faces as shown in Figure 1.4, we realize that this pressure is accompanied by the equilibrium pressure on the opposite face of the element.

**Figure 1.4**

Such a set of internal pressures acting on solids is defined as stress. Depending on which set of planes such pressures are applied, there can be a number of such sets and they may be called stress components. (More about this we shall learn later.)

To take the simplest case, if the total force $P$ is uniformly distributed over the cross section $A$ as shown in Figure 1.3 (a), then the magnitude of stress in the solid is $P/A$. This entity, stress is denoted by the symbol 'σ' and usually expressed in N/mm² units, also denoted as Mega Pascal (MPa) as both are numerically equal.

Let us now turn our attention to deformations. The set of forces $P$ and $R$ (Figure 1.3(a)) are pulling the solid in opposite directions. Consequently the solid gets elongated and let this elongation be denoted by the symbol 'δ'. If the forces be applied on the individual elements, each element will elongate (Figure 1.5(a)) and it has been experimentally established that

   (i)    the magnitude of the individual elongations are proportional to the length of each of the elements, and

   (ii)   the total elongation of all the elements add up exactly to δ.

We may conclude that the ratio of elongation to length of the elements $\dfrac{\delta_1}{L_1}, \dfrac{\delta_2}{L_2} \ldots$

is invariable and equal to $\dfrac{\delta}{L}$. This ratio is defined as **strain** (more specifically as longitudinal strain) and denoted by ε. If the forces applied are in the opposite direction as shown in Figure 1.5 (b), the solid will get its length shortened and the deformation and strain in such a case are considered negative.

**(a)**

**(b)**

**(c)**

**Figure 1.5**

## 1.2.3  Tensile, Compressive and Shear Stresses

We have observed that forces produce elongations or contractions on solids depending on how they act. The forces that produce elongation on solids are called **tensile forces** and the stresses induced by them are called **tensile stresses.** Tensile forces and tensile stresses are both considered positive. Similarly, the forces which cause shortening of length are called **compressive forces** and the stresses induced by them are called **compressive stresses.** Both compressive forces and compressive stresses are considered negative.

**Figure 1.6**

We may also observe that both tensile stresses and compressive stresses act normal to the surface on which they act. For this reason, they are both classified as normal stresses. However, there are many instances of load applications when the stresses are induced in directions other than that of the normal to the surface.

A few such cases are shown in Figure 1.6. A small element is shown enlarged in scale. Beside the element the coordinate system is also given. The plane BEFC is normal to the *x* axis and is, hence, defined as *x* plane. Similarly, the planes DCFG and ABCD are defined as *y* and *z* planes respectively.

You may also observe that the stress on *x* plane is normal (tensile), stress on *y* plane is inclined and that on *z* plane is parallel to the plane itself. The stresses that are acting parallel to the plane on which they are applied are called **shear stresses.** On further consideration we may resolve the stress on the *y* plane into components parallel and normal to the surface. Hence, on any plane, there may be normal stresses, shear stresses or both. Further, while the direction of normal to a plane is uniquely defined, there are infinitely large numbers of directions in which shear stresses may be applied on a plane.

Whatever may be the direction of shear stress, it can be further resolved into components in two mutually perpendicular directions, and having done this, you would have completely defined the stresses acting on a plane. Defining such states of stress on all the planes will furnish the complete state of stress on the small element (or at a point).

Let us now consider a few examples of solids in which shear stresses are induced. Figure 1.7 shows how two steel plates are connected together by a riveted joint. When the connected plates are pulled, the pull from plate A is transmitted to plate B through the rivet. The rivet may be considered to consist of two parts. Part C of the rivet carries the force *P* from plate A by diametric compression or bearing, while part D of the rivet transmits the load *P* to plate B by bearing. The transfer of force from part C to its counterpart D takes place at the common interface namely its cross sectional area at middle length. On this area, the force *P* is applied parallel and hence, the rivet is in shear.

Again consider two wooden blocks connected together by some glue and also connected to a rigid base similarly, as shown in Figure 1.8. If a horizontal force *P* is applied on the *x* plane of the block A at the top, this force is transmitted to the base through a horizontal surface which is the common interface between the base and the block D. As the force is parallel to the surface on which it is acting, it is shear force and the stress induced by it also should be shear stress. On further scrutiny, you will find that the transmission of force from one block to another in the assembly is only as shear force and not only on interfaces between two adjacent blocks, but on any horizontal.

**(a)**

**Figure 1.7**

**(b)**

**(c)**

**(d)**

**Figure 1.7**

**Figure 1.8**

## 1.2.4 Complementary Shear Stresses

Consider the equilibrium of the solid element of dimensions $l$, $b$ and $h$ and acted upon by shear stress components $\tau_{xy}$ and $\tau_{yx}$ as shown in Figure 1.9. By inspection, we may recognize that the components $\tau_{xy}$ on a pair of vertical planes and the pair

of $\tau_{yx}$ on two horizontal planes satisfy independently the force equilibrium equations :

$$\Sigma \ F_x = 0 \quad \text{and} \quad \Sigma \ F_y = 0$$

**Figure 1.9**

Let us consider the moment equilibrium equations. $\tau_{xy}$ components on planes CDHG and ABFE have resultant forces $Q_{xy} = \tau_{xy} \ . \ b. \ h$ which form a couple equal to $(\tau_{xy} \ . \ b \ . \ h) \times l$. Similarly, the moment of the couple formed by the shear stress component $\tau_{xy} = (\tau_{yx} \ . \ l \ . \ b) \ h$. For equilibrium,

$$(\tau_{xy} . \ b \ . \ h) \ l + (\tau_{yx} \ . \ l \ . \ b) \ h = 0$$

or $\qquad\qquad\qquad \tau_{xy} = - \tau_{yx}.$

That is $\tau_{xy}$ is always accompanied by $- \tau_{yx}$ and this pair of shear stresses are called **complementary shear stresses.**

When a piece of carrot is cut into two pieces, the force exerted by the knife acts on surface parallel to the force and hence the carrot is sheared into two. Likewise, you may think over the various cases of load applications in our day-to-day life and identify the cases in which shear stresses are involved.

# 1.3 MECHANICAL BEHAVIOUR OF MATERIALS

## 1.3.1 Stress-Strain Curves

As we have earlier observed, the behaviour of solids, when subjected to loads, depends upon its geometrical and mechanical properties. The mechanical properties of solids have to be investigated through laboratory tests. Let us consider one such test which you might have conducted in the Strength of Materials laboratory.

**Tension Test on a Mild Steel Rod**

A mild steel rod of known diameter and length of about 400 mm to 500 mm is firmly gripped in a universal testing machine and a deformeter is fixed on the rod over a measured gauge of 150 mm to 200 mm. The rod is subjected to axial tensile load. The load at any instant is indicated by the load dial of the UTM while the elongation of the bar over the gauge length is measured through the deformeter.

The load is gradually applied on the bar in suitable increments and the corresponding elongations of the bar are measured at each stage. The test is carried out until the specimen ultimately fails by rupture. The load values at each stage are divided by the cross sectional area of the bar to get the stress values and the corresponding elongations of the bar divided by the gauge length given the strain values. A graph is drawn relating stress and corresponding strain. A typical stress-strain curve for mild steel is shown in Figure 1.10.

**Figure 1.10  Stress-Strain Curve for Mild Steel**

## 1.3.2  Hooke's Law and Elastic Limit

From the stress-strain curve of a material, a number of essential mechanical properties of the material can be studied. When you look at the stress-strain curve (Figure 1.10), you can identify segments of different nature. The initial segment of the curve is steep and straight and the strain in this range is proportional to the stress. Hence, the terminal stage of this range is called proportional limit. At this stage, we find some small fluctuations followed by a horizontal segment, indicating that the strain in the solid is continuously increasing without any increase in stress. Here the material yields at the end of proportional limit and exhibits plasticity (i.e. being strained without increase in stress) on further pulling. This range of strain is called plastic range, and is about 12 to 13 times the proportional range of strain. The stage where proportional range ends and plastic range begins is called yield point.

When the end of plastic range is reached, further loading results in increase of both stress and strain. But here the stress-strain curve is not as steep as in the initial range and is also non-linear. The material having exhibited plasticity once again hardens to take additional stress and hence this range is called **strain hardening** range. Straining further leads to ultimate failure by rupture.

To understand the stress-strain relationship of the solid more clearly, further tests are carried out by loading, unloading and reloading of the solid at various stages and results of such an investigation are shown in Figure 1.11.

**Figure 1.11**

From the results of such a study the following observations may be made.

(i) When the loading is done within the proportional limit of the solid, the strain (deformation) caused is fully recovered when the stress is removed. (If the load is partially removed, a proportional amount of strain gets relieved.)

(ii) When the solid is loaded beyond yield limit and then unloaded; only a part of the strain is relieved and a considerable part of strain still persist thus effecting permanent changes in the geometry of the solid. The strain remaining unrelieved is known as **permanent set**.

(iii) When we reload a solid which is already subjected to permanent set, the stress-strain curve for reloading is linear up to the maximum stage to which it was subjected to earlier.

From these observations, what does a designer learn? If a structure or any component is loaded beyond proportional limit, the geometry gets permanently changed and if repeatedly loaded beyond proportional limit, it will go completely out of shape after a few repetitions. On the other hand, you can load it within their proportional limit any number of times and the original geometry recover on the removal of the load. Hence, in any design, one has to be careful to avoid loading the component beyond the proportional limit. Within this limit the solid is said to be elastic. The property called elasticity consists of two aspects, normally complete removal of strains on unloading and linearity of stress-strain relationship. Hence, the part of stress-strain curve up to yield point is also called as elastic limit. To ensure that the solid is always within this elastic limit, designers permit stresses in solid much lower than the elastic limit. The ratio of the maximum stress a material can withstand to the maximum stress which a designer chooses to allow is called the **factor of safety**. In short,

$$\text{Factor of Safety} = \frac{\text{Maximum stress the material can withstand}}{\text{Maximum stress permissible}}$$

If the numerator is yield point stress, a factor of safety is 2 usually allowed and if it is ultimate stress, factor of safety may be 3 or more.

All materials do not behave as mild steel. The stress-strain curves of a few materials are given in Figure 1.12. Irrespective of the variety of shapes seen in these curves, one can identify in each of them a segment where the curve may be considered linear. This limit is taken as the elastic limit for the material.

15

The overall behaviour of the material and its complete stress-strain relationship is discussed here only in order to caution the designer to keep the stress level well within the elastic limit. Hence, for practical purposes the stress-strain relationship is taken as linear and during the rest of this course you will not be dealing with nonlinearity of material behaviour. Hence, for a designer, stress is always proportional to the strain and the proportionality constant is defined as **Elastic Modulus**.

**Figure 1.12 Stress-Strain Relationship for a Few Materials**

Thus,                Stress, $\sigma$ = Elastic Modulus, ($E$) × Strain, ($\varepsilon$)

or                          $\sigma = E \cdot \varepsilon$                                . . . (1.1)

From Eq. (1.1), we get

$$E = \frac{\sigma}{\varepsilon}$$                                . . . (1.2)

Eq. (1.2) indicates that the Elastic Modulus of a material may be calculated by measuring the slope of the stress-strain curve (within the elastic limit). The fact that Eq. (1.1) holds good within elastic limit which has already been taught to you as Hooke's Law, which we may restate as follows :

*Within elastic limit the strain produced in a solid is always proportional to the stress applied.*

### 1.3.3  Elastic Constants

As there are different types of stresses possible, such as linear (normal) stress, shear stress, volumetric (bulk) stress, and correspondingly different modulii of elasticity namely Young's modulus, shear modulus and bulk modulus are defined as elastic constants of a material. The term $E$ is used in Eqs. (1.1) and (1.2) in the context of normal stresses and corresponding strains is known as Young's Modulus.

In addition to the three modulii of elasticity an important elastic constant, used in defining the mechanical properties of a solid, is known as Poisson's ratio. While testing the material for stress-strain relationship, you observe that strains are produced not only in the direction of the applied stress, but also in direction perpendicular (lateral) to it. On further investigations, you will find that the

lateral strain is always proportional to the longitudinal strain and the proportionality constant is negative. This proportionality constant is defined as Poisson's Ratio and denoted by $\upsilon$.

$\therefore$ $\qquad$ Poisson's Ratio, $\upsilon = \dfrac{- \text{Lateral Strain}}{\text{Longitudinal Strain}}$

While applying tensile force on a rod we found that the deformations produced involved change of volume of the solid as well as its shape. Such deformations may be found in many other cases of loading. Deformations involving change of volume and shape are more common. However, we may identify two special cases of deformations, dilatation and distortion.

Dilatation is defined as change in volume (bulk) of the solid without change of shape. If normal stress components of equal magnitude and same nature (all tensile or all compressive) are applied on all the three mutually perpendicular planes of a solid element as shown in Figure 1.13, the solid will undergo only volume change without any change in shape. Such a state of stress (as may be obtained in the case of solid subjected to hydrostatic pressure) is called volumetric stress and the change of volume produced per unit volume of the solid is called the **volumetric stress**. The ratio between volumetric stress and volumetric strain is defined as Bulk Modulus.

**Figure 1.13 Bulk Stress and Dilatation**

Thus, $\qquad$ Bulk Modulus, $K = \dfrac{\text{Volumetric Stress}}{\text{Volumetric Strain}}$ $\qquad$ . . . (1.3)

in which $\qquad$ Volumetric strain $= \dfrac{\text{Change of Volume}}{\text{Original Volume}}$ $\qquad$ . . . (1.4)

The state of stress, with equal values of stress components in all the direction is also known as **spherical state of stress**.

The state of deformation involving change of shape without any change of volume is defined as **Distortion**. Though distortion may be introduced in many ways let us consider a simple case. When shear stresses are applied on a solid as shown in Figure (1.14), we find that angular deformations are introduced. The angular strain or change in angle produced is called shear strain denoted by $\gamma$ (Greek gamma) and its magnitude is expressed in radians. The ratio of the applied shear stress and the shear strain produced by it is defined as Rigidity Modulus or Shear Modulus of the material, i.e.

$$\text{Rigidity Modulus, } G = \frac{\text{Shear Stress } (\tau)}{\text{Shear Strain } (\gamma)} \qquad \dots (1.5)$$

We have so far defined four elastic constants, namely, Young's Modulus, (*E*), Poisson's Ratio (υ), Bulk Modulus (*K*), and Shear Modulus (*G*). Later on we shall learn that these four are not independent constants, but are related to each other. We can show that there are only two independent elastic constants and the other two are dependent on them.

**Figure 1.14 Shear Stress and Shear Strain**

## 1.4  DEFORMATION OF BARS

### 1.4.1  Bars of Uniform Section

With the knowledge gained already, you will now be able to learn calculation of deformations in solids due to applications of simple stresses. Let us start with the example of a prismatic bar of length '*L*', uniform cross section of area *A* carrying an axial load '*P*' as shown in Figure 1.15.

**Figure 1.15**

Stress in the solids is in the longitudinal direction and its magnitude may be calculated as

$$\sigma = \frac{\text{Load}}{\text{Area}} = \frac{P}{A}$$

If the Young's Modulus of the material, *E*, is known, the strain induced may be calculated as

$$\varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

As strain is only deformation per unit length, the total elongation of the bar is calculated as

$$\delta = \varepsilon . L = \frac{P}{AE} L$$

Thus,      Elongation $\delta = \dfrac{PL}{AE}$ $\qquad \dots (1.6)$

You may please note that the strain, $\varepsilon$, calculated here is in the longitudinal direction and is accompanied by strains in the lateral directions also whose magnitude is given by $-\upsilon\varepsilon$. Let us now consider a few numerical examples.

## Example 1.1

If the bar shown in Figure 1.15 is 2 m long with rectangular cross section of 300 mm deep and 400 mm wide, calculate the change in volume of the solid due to a longitudinal compressive force of 720 kN/mm², if the elastic constants $E$ and $\upsilon$ for the material are known as 120 kN/mm$^2$ and 0.2 respectively.

**Solution**

Area of cross section of the member $= 300 \times 400 = 120000$ mm²

Longitudinal strain $\varepsilon = \dfrac{P}{AE} = \dfrac{-720 \times 1000}{120000 \times 120 \times 10^3} = -0.00005$

(Note that all the values have to be converted to consistent units; here, it is $N$ for forces and mm for length.)

$\therefore$      Total change in length $\delta = 1000 \times (-0.00005) = -0.05$ mm.

Lateral strain $\varepsilon_l = -\upsilon\varepsilon = -0.2 \times (-0.00005) = 0.00001$

Change in depth $= 0.00001 \times 300 = 0.003$ mm

Change   in width $= 0.00001 \times 400 = 0.004$ mm

$\therefore$      Change in volume of the solid,

$= (1000 - 0.05)(300 + 0.003)(400 + 0.004) - (1000 \times 400 \times 300)$

$= 999.95 \times 300.003 \times 400.004 - (1000 \times 400 \times 300)$

$= -3600.109$ mm³

Let us consider an alternate approximate method also.

Change in volume, $dV = V + dV - V$

$$= (l + \Delta l)(b + \Delta b)(d + \Delta d) - l \cdot b \cdot d$$

where $\Delta l$, $\Delta b$, and $\Delta d$ are changes in length, breadth and depth of the solid.

i.e.      $dV = l(1 + \varepsilon_1) \times b(1 + \varepsilon_2) \times d(1 + \varepsilon_3) - l \cdot b \cdot d$

where $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ are the strains in the three mutually perpendicular directions.

$\therefore$      $dV = l\,bd \times (1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3) - l\,bd$

$= l\,bd \times (1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 + \varepsilon_1\varepsilon_2\varepsilon_3) - l\,bd$

$= l\,bd \times (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 + \varepsilon_1\varepsilon_2\varepsilon_3)$

Neglecting the second order products,

$dV = V \times (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$      $\ldots$ (1.7)

Now let us calculate the change in volume of the given solid using Eq. (1.7).

Change in volume, $dV = V \times (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$

19

$$= 1000 \times 300 \times 400 \, (- 0.00005 + 0.00001 + 0.00001)$$

$$= 3600 \text{ mm}^3$$

Though there is a small error, the approximation is quite satisfactory. (As an exercise you may calculate the percentage error in the value.) If you are very particular about accuracy, you use the following formulation:

$$dV = V \times (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_3 + \varepsilon_3 \varepsilon_1 + \varepsilon_1 \varepsilon_2 \varepsilon_3)$$

## SAQ 1

(a) Taking the dimensions of the solid in Example 1.1 as same, calculate the change in volume of the solid if additional tensile forces of 1200 kN, and 2000 kN are applied in the lateral directions namely horizontal and vertical respectively.

(b) A concrete cylinder of height 300 mm and diameter 150 mm is tested for compression in a Universal Testing Machine. Within the elastic limit, the cylinder was found to be shortened by 0.12 mm and its diameter was found to be increased by 0.01 mm under an axial load of 90 kN. Calculate the Young's Modulus, $E$, and Poisson's Ratio, $\upsilon$, for the specimen.

### 1.4.2 Bars of Varying Cross Section

Let us now consider a few cases of a little more complexity. To give you a feeling of solid reality all solids have so far been represented by pictorial views. Figure 1.16 represents the projected view of a bar whose cross sectional area differs in different segments. Taking the elastic modulus of the material as 60 kN/mm², let us evaluate the total elongation of the bar.

**Figure 1.16**

Although the bar has no uniform cross-section, it consists of segments of uniform cross section. Hence, the total elongation of the bar can be calculated as a sum of the elongations of individual segments.

Thus, $$\delta = \Sigma \delta_i = \frac{P_1 L_1}{A_1 E_1} + \frac{P_2 L_2}{A_2 E_2} + \dots \frac{P_i L_i}{A_i E_i} + \dots$$

In the numerical example under consideration

$$\delta = \frac{750 \times 300}{\frac{\pi}{4} \times 200^2 \times 60} + \frac{750 \times 400}{\frac{\pi}{4} \times 135^2 \times 60} + \frac{750 \times 200}{\frac{\pi}{4} \times 60^2 \times 60} + \frac{750 \times 500}{\frac{\pi}{4} \times 100^2 \times 60}$$

$$= \frac{750}{\frac{\pi}{4} \times 60}\left(\frac{300}{200^2} + \frac{400}{135^2} + \frac{200}{60^2} + \frac{500}{100^2}\right) = 2.14865 \text{ mm}$$

In the bar, shown in Figure 1.16, the axial pull is applied at the ends and hence the axial force in all the members is same. In the bar, shown in Figure 1.17(a), the external forces are applied at intermediate sections also. In such cases, the axial force in each member should be evaluated first, before any deformation calculations. This can be accomplished by considering equilibrium of each of the sections where external loads are applied. Though no external load has been prescribed at the LHS end, the support reaction has to be calculated and taken as the external load. Equilibrium analysis can be easily carried out by treating each segment as a free body as shown in Figure 1.17(b).

For example, consider the equilibrium of the segment 4 in Figure 1.17(b). At the RHS end of the member a point load of 60 kN is applied. Hence, for the member to be in equilibrium a force of – 60 kN should be applied at the RHS end of the member. Hence, the member is subjected to a tensile force of 60 kN which is represented by the internal arrows in accordance with the sign conventions you have already learnt. Member 3 is pulled with a tensile force of 60 kN exerted by member 4 and in addition the external force of 80 kN also pulls the member in the same direction, resulting in the member carrying a total tensile force of 140 kN. Proceeding thus, the axial forces in all the members can be calculated. To simplify the graphical representation we may show the member forces along with external forces as shown in Figure 1.17(c).

Now let us calculate the total elongation of the bar, taking the elastic modulus, *E*, as 200 kN/mm².

**(a)**

**(b)**

**Figure 1.17**

**(c)**

**Figure 1.17**

21

$$\delta = \sum \delta_i = \sum \frac{P_i L_i}{A_i E_i}$$

$$= \frac{160 \times 600}{800 \times 200} + \frac{100 \times 800}{200 \times 200} + \frac{140 \times 800}{600 \times 200} + \frac{60 \times 500}{150 \times 200}$$

$$= 0.6 + 2.0 + 0.9333 + 1.0$$

$$= 4.53333 \text{ mm.}$$

Now, let us consider the case of a bar whose cross section is varying continuously as shown in Figure 1.18. The bar of (truncated) conical shape is subjected to an axial force of $P$. As in the case of stepped bars, here also, we may apply the equation $\delta = \sum \delta_i = \sum \frac{P_i L_i}{A_i E_i}$, if we recognize that $L_i$ is sufficiently small and $A_i$ may be taken as uniform over the length of the small segment. That is, the expression $\sum \frac{P_i L_i}{A_i E_i}$ now becomes $\int_0^L \frac{P_i L_i}{A_i E_i}$.

**Figure 1.18**

**Figure 1.19**

However, to simplify the expression for $A$, the origin may be conveniently be shifted to the apex of the produced cone as shown in Figure 1.21. Thus,

$$\delta = \int_{H_0}^{H} \frac{P dx}{\frac{\pi}{4} \left( \frac{D_2 x}{H} \right)^2 E}$$

The integral in above equation may be evaluated for solids having stepped as well as continuous variation in cross section. Stepped bars require no special treatment except that the equilibrium and deformations of each segment may be analysed separately and added wherever necessary. Even a prismatic segment of the bar may have to be considered as two or more members, if external loads are applied at interior points of the segment.

# SAQ 2

(a) Taking the Elastic modulus, $E = 80\text{kN/mm}^2$ and Poisson's ratio of the bar as 0.24, calculate the change in volume of the bar shown in Figure 1.16.

(b) Calculate the total elongation (or shortening) of the non-uniform bars (with loads) shown in Figures 1.20 and 1.21.

**Figure 1.20** ($E = 180$ GPa)

**Figure 1.21** ($E = 200$ GPa)

(c) Taking the Young's Modulus, $E$, as 120 kN/mm², find the total elongation of the bar shown in Figure 1.22.

**Figure 1.22**

## 1.4.3 Bars of Uniform Strength

If we calculate the stresses induced in the various segments of the bar, shown in Figure 1.17(a), we get the following results:

$$\sigma_1 = \frac{110 \times 1000}{800} = 137.5 \text{ N/mm}^2$$

$$\sigma_2 = \frac{50 \times 1000}{200} = 250 \text{ N/mm}^2$$

$$\sigma_3 = \frac{120 \times 1000}{600} = 200 \text{ N/mm}^2$$

$$\sigma_4 = \frac{30 \times 1000}{150} = 200 \text{ N/mm}^2$$

The results show that the bar should be made of a material with permissible stress not less than 250 N/mm². Such a material (should be specially manufactured, if required) is wasted in segments where the stress induced is much less. For instance, in segment 1, the stress induced is only 137.5 N/mm². If we reduce the cross sectional area of segment 1, 3 and 4 to 440 mm², 480 mm² and 120 mm² respectively, the stress in all segments will be uniformly 250 N/mm², the bar would be safe and we would have effected considerably economy of material. Such a bar is called a bar of uniform strength. A designer will strive to achieve a bar of uniform strength, whenever possible. However, due to other constraints and variations in loadings, it may not always be possible to provide a bar of uniform strength. Thus, the concept of bar of uniform strength is a design ideal aiming to effect maximum saving in the material.

## 1.5 COMPOSITE BARS

So far we have considered the cases of solids made of a single material. However, in a number of engineering applications we use components made of two or more materials. Bars made of two or more materials are called **composite bars.** This section will introduce to you basic principles applied in analyzing such bars.

### 1.5.1 Modular Ratio

Consider the composite bar shown in Figure 1.23 in which an aluminum bar is encased just snug in a Copper tube which in itself is encased just snug in a steel tube. Let us analyze how the three different components of the composite bar share the load applied axially on the composite bar.

**Figure 1.23**

The basic requirement in such a case is that all the three components of the bar should undergo the same amount of deformation while carrying the load. Otherwise, the bar which shortens the least alone will be in contact with the load which is physically impossible, since a component which looses its contact will not be carrying any load and hence there can be no shortening of that component which contradicts the assumption that it shortens more. Hence, it is logically necessary that all the three bars should undergo the same amount of deformation. This condition is called the compatibility condition or condition of consistent deformation. Satisfaction of this condition is the basis on which the problem of composite bars is analysed.

Let the total axial force applied on the composite member $P$ be shared by the different members as $P_1$, $P_2$ and $P_3$. If the deformations produced are $\delta_1$, $\delta_2$ and $\delta_3$, then we get,

$$\delta_1 = \frac{P_1 L_1}{A_1 E_1}$$

$$\delta_2 = \frac{P_2 L_2}{A_2 E_2}$$

$$\delta_3 = \frac{P_3 L_3}{A_3 E_3}$$

The compatibility condition requires that

$$\frac{P_1 L_1}{A_1 E_1} = \frac{P_2 L_2}{A_2 E_2} = \frac{P_3 L_3}{A_3 E_3} \qquad \ldots (1.8)$$

Let the elastic modulii of the materials be redefined as $E_1$, $m_2 E_1$ and $m_3 E_1$, where

$$m_2 = \frac{E_2}{E_1} \quad \text{and} \quad m_3 = \frac{E_3}{E_1}$$

Eq. (1.8) may be rewritten as follows

$$\frac{L_1}{E_1} \times \frac{P_1}{A_1} = \frac{L_2}{m_2 E_1} \times \frac{P_2}{A_2} = \frac{L_3}{m_3 E_1} \times \frac{P_3}{A_3} \qquad \ldots (1.9)$$

Recognizing that $L_1 = L_2 = L_3$, we may rewrite Eq. (1.9) as

$$\frac{P_1}{A_1} = \frac{P_2}{m_2 A_2} = \frac{P_3}{m_3 A_3} \qquad \ldots (1.10)$$

From which, we get $P_2 = \dfrac{P_1}{A_1} \, m_2 A_2$ and

$$P_3 = \frac{P_1}{A_1} \, m_3 A_3 \qquad \ldots (1.11)$$

The equilibrium condition required to be satisfied is

$$P_1 + P_2 + P_3 = P$$

or $\qquad P_1 + \dfrac{P_1}{A_1} \, m_2 A_2 + \dfrac{P_1}{A_1} \, m_3 A_3 = P$

or $\qquad P_1 \left( 1 + \dfrac{m_2 A_2}{A_1} + \dfrac{m_3 A_3}{A_1} \right) = P$

$$P_1 = \frac{P}{\left( 1 + \dfrac{m_2 A_2}{A_1} + \dfrac{m_3 A_3}{A_1} \right)} \qquad \ldots (1.12)$$

After calculating $P_1$ using Eq. (1.12), the loads shared by other members $P_2$ and $P_3$ may be calculated using the Eq. (1.11).

In the process of the above solution, we have introduced two terms, namely, $m_2$ and $m_3$ whose values are the ratios $\dfrac{E_2}{E_1}$ and $\dfrac{E_3}{E_1}$ respectively. These ratios are the ratios of Elastic modulii of two different materials and, hence, are called **Modular Ratios.**

Let us now solve the problem of load sharing in the composite member shown in Figure 1.23.

Here, we have

$$P = 60 \text{ kN}$$

$$L = 400 \text{ mm}$$

$$A_1 = \frac{\pi}{4} \times 50^2 = 1963.5 \text{ mm}^2$$

$$A_2 = \frac{\pi}{4} \times (90^2 - 50^2) = 4398.23 \text{ mm}^2$$

$$A_3 = \frac{\pi}{4} \times (120^2 - 90^2) = 4948 \text{ mm}^2$$

Elastic modulii for the materials, namely aluminium, copper and steel may be taken as 80 kN/mm², 120 kN/mm² and 200 kN/mm² respectively.

$$m_1 = \frac{E_2}{E_1} = \frac{120}{80} = 1.5$$

$$m_2 = \frac{E_3}{E_1} = \frac{200}{80} = 2.5$$

Substituting these values in Eq. (1.12),

$$P_1 = \frac{60}{\left(1 + 1.5 \times \dfrac{4398.23}{1963.5} + 2.5 \times \dfrac{4948}{1963.5}\right)}$$

$$= 5.6285 \text{ kN}$$

Now, using Eq. (1.11),

$$P_2 = \frac{5.62852}{1963.5} \times 1.5 \times 4398.23 = 18.912 \text{ kN}$$

$$P_3 = \frac{5.62852}{1963.5} \times 2.5 \times 4948 = 35.4595 \text{ kN}$$

## 1.5.2 Equivalent Area of a Composite Section

On reconsideration, we may rewrite Eq. (1.12) as follows :

$$P_1 = \frac{PA_1}{[A_1 + m_2 A_2 + m_3 A_3 + ...]}$$

or $\qquad \dfrac{P_1}{A_1} = \dfrac{P}{[A_1 + m_2 A_2 + m_3 A_3 + ...]}$ $\qquad \qquad$ . . . (1.13)

In Eq. (1.13), we recognize that $\dfrac{P_1}{A_1}$ is the stress induced in material 1 and hence,

$$\sigma_1 = \frac{P}{[A_1 + m_2 A_2 + m_3 A_3 + ...]} \qquad \qquad ...\,(1.14)$$

The denominator in Eq. (1.14) has Area unit and is called the equivalent area of the composite bar, as if composite bar entirely made of material 1 alone. In other words, if the composite bar in Figure 1.23 is entirely made of aluminum, then the equivalent area is given as

$$A_{eq} = A_1 + m_2 A_2 + m_3 A_3$$
$$= 1963.5 + (1.5 \times 4398.23) + (2.5 \times 4948)$$
$$= 20930.86 \text{ mm}^2$$

That is, the composite bar, shown in Figure 1.23, is equivalent to an aluminium bar of cross sectional area 20930.86 mm², in resisting axial forces. (An aluminium bar of 163.25 mm diameter will give this area.)

One of the practical examples of composite bars is the R.C.C. column, a typical cross section of which is shown in Figure 1.24. Here, a 400 mm × 400 mm square cross section of a R.C.C. column is reinforced with 8 numbers of 32 mm diameter mild steel bars. Taking the Elastic modulii of steel and concrete as $E_s$ and $E_c$ and the modular ratio, $m = \dfrac{E_S}{E_C}$ as 18, let us find the loads shared by steel and concrete when an axial load of 600 kN is applied on the column.

Area of steel bars, $A_s = 8 \times \dfrac{\pi}{4} \times 32^2 = 6434 \text{ mm}^2$

Area of concrete, $A_c = (400 \times 400 - 6434) = 153566 \text{ mm}^2$

Modular Ratio, $\quad m = 18$

Equivalent Area of Concrete $= (153566 + 18 \times 6434) = 269378 \text{ mm}^2$

Total axial load $= 600$ kN.

Load shared by concrete $= \dfrac{600}{269378} \times 153566 = 342.046$ kN

Load shared by steel $= \dfrac{600}{269378} \times 18 \times 6434 = 257.954$ kN

**Figure 1.24**

**SAQ 3**

    (a)    Show that the composite bar in Figure 1.23 is equivalent to a steel bar of cross sectional area 8372.344 mm².

    (b)    If the composite bar shown in Figure 1.23 is to be replaced by a copper bar, calculate the equivalent area of cross section of the copper bar.

### 1.5.3 Stresses in Composite Bars and Load Carrying Capacity of Composite Bars

You have learnt how a total load applied on a composite bar is shared by different components of a composite. Though this is certainly enlightenment on the behaviour of composite bars, we need to learn more in order to apply the concept in practically more useful way. That is, we need to learn

    (i)    what will be the total load that a composite bar can carry, and

    (ii)    to carry a given load, how should a composite bar be proportioned.

For this purpose, we should know the strength of each material and also we should be able to calculate the stresses induced in the various components of a composite bar.

We have already learnt the compatibility condition that the axial deformations undergone by all the components of a composite bar should be equal.

As the lengths of components are also equal, the strain in each component should also be equal. Taking the strain of the composite bar as $\varepsilon$, stress in any component may be expressed as

$$\sigma_i = E_i \times \varepsilon \qquad \qquad \dots (1.15)$$

or $\qquad \qquad \sigma_i = \dfrac{E_i}{E_1} \times E_1 \times \varepsilon$

or $\qquad \qquad = m_i \times E_1 \times \varepsilon$

Since, $\qquad \qquad E_1 \times \varepsilon = \sigma_1$, then we can write

$$\sigma_i = m_i \times \sigma_1 \qquad \qquad \dots (1.16)$$

Eq. (1.16) expresses the relationship that the stress induced in any component of a composite bar should be proportional to its elastic modulus or modular ratio. For example, in the R.C.C. column shown in Figure 1.24, the stress in steel, $\sigma_s$, will be 18 times the stress in concrete. You could have seen this by yourself, if you had divided the load shared by each member by its area of cross section. Use of Eq. (1.16) saves some computational effort.

We can get stress in concrete $= \dfrac{342.046 \times 1000}{153566} = 2.227355$ N/mm².

Thus,       Stress in steel = $18 \times 2.227355$

$$= 40.0924 \text{ N/mm}^2.$$

In order to calculate the load carrying capacity, we need to know the strength (allowable stress) of all the components. For example, let the allowable stress in concrete be 4 N/mm² and the allowable stress in steel be 120 N/mm². There is another restriction, you can notice, that even though the strength of steel is 120 N/mm², we cannot stress it to the full, because if we stress steel to its full strength, i.e. 120 N/mm², then concrete will be stressed to 6.667 N/mm², i.e. 120/18, which is not permissible. Hence, the maximum stress that may be induced in steel is only 72 N/mm², i.e. $4 \times 18$.

Now, we can calculate the maximum load the column can support or the load carrying capacity of the column as follows :

$$P = \sigma_s \times A_s + \sigma_c \times A_c$$

$$= 72 \times 6434 + 4 \times 153566$$

$$= 1077512 \text{ N or } 1077.512 \text{ kN.}$$

Another type of problem that a designer faces is to decide the amount of steel required if the column is to support a magnitude of load. For instance, if the column is required to carry an axial load of 960 kN, let us calculate the steel requirement.

Maximum stress in steel = 72 N/mm²

Maximum stress in concrete = 4 N/mm²

Let the area of steel be $A_s$.

Area of concrete = $160000 - A_s$.

∴       $72 \times A_s + 4 (160000 - A_s) = 960 \times 1000$

$(72 - 4) A_s + 640000 = 960000$

∴       $A_s = \dfrac{(960000 - 640000)}{72 - 4} = 4706 \text{ mm}^2$

This may be provided suitably (say, 8 nos of 28 mm dia bars, which will be a little more than sufficient).

## SAQ 4

(a)   Evaluate the stress induced in each component of the composite bar shown in Figure 1.23.

(b)   In the case of composite bar shown in Figure 1.23, if the permissible stresses in aluminium, copper and steel respectively are 50 N/mm², 60 N/mm² and 120 N/mm², calculate the load carrying capacity of the composite bar.

(c)   If the column shown in Figure 1.24 is required to carry an axial load of 880 kN, find the requirement of steel, with mechanical properties as given in worked example.

(d)   Provide a graphical chart to relate the strength of column shown in Figure 1.24 to the percentage ratio of $\dfrac{A_S}{A}$ where $A + A_S + A_C$. Take $\sigma_C = 4$ MPa, $\sigma_S = 140$ MPa and $m = 18$.

(e)   If the column, shown in Figure 1.24, is cast with higher quality of concrete with a permissible stress of 5 N/mm$^2$ and $E_C = 0.06\ E_S$, what is its load carrying capacity and what is the quantity of steel required if the column has to carry an axial load of 1200 kN?

# 1.6  THERMAL STRESSES AND STRAINS

## 1.6.1  Effect of Temperature on Bodies

It is a common observation and experience that the size of a body increases as the temperature increases while it decreases as the temperature decreases. Thus, temperature rise results in expansion, i.e. increase in linear dimensions of the body, and temperature fall results in contraction, i.e. reduction in the linear dimensions of the body. The magnitude of the changes brought in linear dimensions of the body by temperature is defined through an intrinsic property of the material, of which the body is made of, called coefficient of linear thermal expansions or contraction.

The coefficient of linear thermal expansion or contraction of a material is defined as the change in length per unit length per degree change in temperature of the body which is made of the material. The values of this coefficient, $\alpha$ of some of the common engineering materials are listed in Table 2.1.

From the definition, $\alpha$ can be written as,

$$\alpha = \frac{\Delta L}{L} \times \frac{1}{\Delta T}$$

where, $\Delta L$ is the change in length over a total length $L$ when the temperature is changed by $\Delta T$. Thus,

$$\Delta L = \alpha \times L \times \Delta T$$

**Table 2.1**

| Material | $\alpha$, Coefficient of Linear Thermal Expansion (k$^{-1}$) |
|---|---|
| Low carbon (mild) steel | $11.7 \times 10^{-6}$ |
| Cast Iron : Gray Malleable | $12 \times 10^{-6}$ |
| Nickel-chrome steel | $11.7 \times 10^{-6}$ |
| Aluminium | $23 \times 10^{-6}$ |
| Brass | $18.9 \times 10^{-6}$ |
| Bronze | $18 \times 10^{-6}$ |
| Magnesium | $28.8 \times 10^{-6}$ |

## 1.6.2 Thermal Stresses in Bodies

In the earlier section, we had seen that if the temperature of a body is increased by $\Delta T$, there will be an expansion $\Delta L$ over a length $L$. However, this expansion occurs only if the body is free to expand. During this free expansion, the body is stressed. But, if this free expansion is restrained either fully or partially, forces of restraint are generated and the body develops internal strain resulting in internal stresses. These stresses are called **thermal stresses**. A similar situation arises if the body is restrained against contraction and the temperature is reduced. When expansion is restrained, compressive stresses will be generated because the restraint is equivalent to applying a compressive force. Similarly, when contraction is restrained, tensile stresses will be generated.

## 1.6.3 Thermal Stresses and Strains in Uniform Bars

In this section, we shall discuss the problem of thermal stresses in bars of uniform cross-section when they are partially or totally restrained against change in length due to temperature changes. Since the area of cross-section is the same over the entire length, the stress will be the same over each cross-section.

**Fully Restrained Bar**

Let us consider a bar of uniform cross-sectional area $A$ and length $L$ and constrained at both ends as shown in Figure 1.25(a) and (b).

**(a)**                                           **(b)**

**Figure 1.25**

Let us increase the temperature of the bar by $\Delta T$. If the bar were free to expand, it would have increased to a length $L + \Delta L$ as in Figure 1.25 (b). However, this increase in length, $\Delta L$, is completely restrained. Hence, the restraining force, $P$, will be equivalent to a force which would have produced the same change in length. Following these arguments

$$\Delta L = \alpha L \, \Delta T = \frac{PL}{AE} \quad \text{or} \quad P = AE \, \alpha \, \Delta T$$

Therefore, Thermal Strain $= \dfrac{\Delta L}{L} = \alpha \, \Delta T$

Thermal Stress in the bar = Strain $\times E = E \, \alpha \, \Delta T$

Which is the same as $\dfrac{P}{A}$. Since the force $P$ is compressive, the stress developed is also compressive in nature.

(b)   **Partially Restrained Bar**

Now, let us assume that one of the supports of the bar in Figure 1.25 (a) can move by a distance $\Delta L'$ ($\Delta L' < \Delta L$) while the bar is expanding. In this case, the bar is free to expand by an amount $\Delta L'$ but is restrained in expanding

further by rest of the amount $(\Delta L - \Delta L')$. Thus, the force of restraint, $P$, is now a force which would produce a change in length $(\Delta L - \Delta L')$ in the bar.

Thus,

$$P = \frac{(\Delta L - \Delta L')\, AE}{L} = \frac{(\alpha L \Delta T - \Delta L')\, AE}{L}$$

$$\text{Strain} = \frac{(\alpha L\, \Delta T - \Delta L')}{L} = \alpha\, \Delta T - \left(\frac{\Delta L'}{L}\right)$$

$$\text{Thermal Stress} = \text{Strain} \times E = E\left(\alpha\, \Delta T - \frac{\Delta L'}{L}\right)$$

---

### Example 1.2

Two parallel walls are stayed together by a steel rod of 5 cm diameter passing through metal plates and nuts at both ends. The nuts are tightened, when the rod is at 150°C, to keep the walls 10 m apart. Determine the stresses in the rod when the temperature falls down to 50° C, if

(a)  the ends do not yield, and

(b)  the ends yield by 1 cm.

Take $E = 2 \times 10^5\,\text{N/mm}^2$ and $\alpha = 12 \times 10^{-6}\,\text{K}^{-1}$.

**Solution**

Given Length of the rod, $L = 10\text{ m} = 10^4\text{ mm}$

Diameter of the rod, $d = 5\text{ cm} = 50\text{ mm}$

Change in temperature, $\Delta T = 150 - 50 = 100°C$

$$E = 2 \times 10^5\,\text{N/mm}^2$$

$$\alpha = 12 \times 10^{-6}\,\text{K}^{-1}$$

(a)  When the ends do not yield (let the stress be $\sigma_1$)

Thermal stress, $\sigma_1$, in the rod $= E\, \alpha\, \Delta T$

$$= 2 \times 10^5 \times 12 \times 10^{-6} \times 100 = 240\,\text{N/mm}^2$$

(b)  When the ends yield by 1 cm (let the stress be $\sigma_2$)

Thermal Stress, $\sigma_2 = \left(\alpha\, \Delta T - \frac{\Delta L'}{L}\right) \times E$

$$= \left(12 \times 10^{-6} \times 100 - \frac{10}{10^4}\right) \times 2 \times 10^5$$

$$= (0.0012 - 0.001) \times 2 \times 10^5 = 40\,\text{N/mm}^2$$

---

### SAQ 5

Two parallel walls 6 m apart are stayed together by a steel rod 20 mm diameter, passing through metal plates and nuts at each end. The nuts are

tightened, when the rod is at a temperature of 100°C. Determine the stress in the rod, when the temperature falls down to 20°C, if

(a)   the ends do not yield, and

(b)   the ends yield by 1 mm.

Take $E = 2 \times 10^5 \text{ N/mm}^2$ and $\alpha = 12 \times 10^{-6} \text{ K}^{-1}$.

## 1.6.4  Thermal Stresses and Strains in Stepped Bars

In this section, we shall discuss the determination of thermal stresses in bars whose cross-sectional area changes in steps over the length.

**Fully Restrained Stepped Bar**

Consider a bar of length $L$ which has a uniform cross-sectional area $A_1$ over length $L_1$ while in the rest of the length $L_2$ the cross-sectional area is $A_2$. Let the Young's Modulus and the coefficient of linear expansion for the two parts be $E_1$, $\alpha_1$ and $E_2$, $\alpha_2$ respectively. The ends of the bar are fully restrained. The bar is shown in Figure 1.26. Let the temperature of the bar be raised by $\Delta T$.

**Figure 1.26**

If the bar were free to expand, it would have extended by a length $\Delta L$ given by

$$\Delta L = L_1 \alpha_1 \Delta T + L_2 \alpha_2 \Delta T$$

$$= (L_1 \alpha_1 + L_2 \alpha_2) \Delta T$$

Hence,     $\sigma_1 = \dfrac{\Delta T E_1 E_2 A_2 (L_1 \alpha_1 + L_2 \alpha_2)}{(A_1 E_1 L_2 + A_2 E_2 L_1)}$ , and

$\therefore$     $\sigma_2 = \dfrac{\Delta T E_1 E_2 A_1 (L_1 \alpha_1 + L_2 \alpha_2)}{(A_1 E_1 L_2 + A_2 E_2 L_1)}$

Herein, we have assumed that the two parts of the bar are made of two different materials. Instead, if the entire bar is of a single material of Young's Modulus $E$ and coefficient of linear expansion $\alpha$,

$$E_1 = E_2 = E \text{ and } \alpha_1 = \alpha_2 = \alpha$$

Then $\alpha_1$ and $\alpha_2$ reduce to

$$\sigma_1 = \frac{\Delta T E A_2 \alpha L}{A_1 L_2 + A_2 L_1} \text{ and}$$

$$\sigma_2 = \frac{\Delta T E A_1 \alpha L}{A_1 L_2 + A_2 L_1} \text{ and}$$

**Partially Restrained Stepped Bar**

Let us now consider the case of a stepped bar as in Figure 1.26, but which is free to extend by an amount $\Delta L'$ ($\Delta L' < \Delta L$) but restrained thereafter. In this case, during the free expansion of $\Delta L'$ the bar remains unstressed but thereafter develops the stresses $\sigma_1$ and $\sigma_2$ whose values will now be different than those of fully restrained stepped bar above.

From the above two expressions, the thermal stresses $\sigma_1$ and $\sigma_2$ can now be written as

$$\sigma_1 = \frac{E_1 E_2 A_2 [\Delta T (L_1 \alpha_1 + L_2 \alpha_2) - \Delta L']}{(A_1 E_1 L_2 + A_2 E_2 L_1)}$$

$$\sigma_2 = \frac{E_1 E_2 A_1 [\Delta T (L_1 \alpha_1 + L_2 \alpha_2) - \Delta L']}{(A_1 E_1 L_2 + A_2 E_2 L_1)}$$

In the case of a single material bar

$$\sigma_1 = \frac{E A_2 [\Delta T \alpha L - \Delta L']}{A_1 L_2 + A_2 L_1} \text{, and}$$

$$\sigma_2 = \frac{E A_1 [\Delta T \alpha L - \Delta L']}{A_1 L_2 + A_2 L_1}$$

**Example 1.3**

A stepped bar made of aluminum is held between two supports. The bar is 900 mm in length at 38ºC, 600 mm of which is having a diameter of 50 mm, while the rest is of 25 mm diameter. Determine the thermal stress in the bar at a temperature of 21ºC, if

(a)    the supports are unyielding, and

(b)    when the supports move towards each other by 0.1 mm.

Given    $E$ for aluminum $= 74$ kN/mm$^2$

$\alpha$ for aluminum $= 23.4 \times 10^{-6}$ K$^{-1}$

**Solution**

*When the Supports are Unyielding*

We have, $L_1 = 600$ mm; diameter $d_1 = 50$ mm

Thus,    $A_1 = \dfrac{\pi(50)^2}{4}$ mm$^2$

Also,    $L_2 = 300$ mm; diameter $d_2 = 25$ mm

Therefore, $A_1 = \dfrac{\pi(25)^2}{4}$ mm$^2$

and    $\Delta T = 38 - 21 = 17$ºC

Therefore, free contraction, $\Delta T = L \, \alpha \, \Delta T = 900 \times 23.4 \times 10^{-6} \times 17$
= 0.358 mm.

Let $\sigma_1$ be the stress in 50 mm $\phi$ part and $\sigma_2$ be the stress in 25 mm $\phi$ part.

Then,

$$\sigma_1 = \frac{\Delta T E A_2 \alpha L}{A_1 L_2 + A_2 L_1} = \frac{900 \times 23.4 \times 10^{-6} \times 17 \times 74 \times 10^3 \times \pi \times (25)^2}{4\left(\dfrac{\pi \times (50)^2}{4} \times 300 + \dfrac{\pi \times (25)^2}{4} \times 600\right)}$$

$$= 14.72 \text{ N/mm}^2$$

$$\sigma_2 = \sigma_1 \frac{A_1}{A_2} = 14.72 \times \frac{(50)^2}{(25)^2} = 58.88 \text{ N/mm}^2$$

These stresses are tensile in nature.

*When the Supports Move towards Each Other by 0.1 mm*

Here, $\Delta L' = 0.1$ mm.

$$\therefore \qquad \sigma_1 = \frac{E A_2 [\Delta L - \Delta L']}{A_1 L_2 + A_2 L_1} = \frac{74 \times 10^3 \times \pi \times (25)^2 \times (0.358 - 0.1)}{4\left(\dfrac{\pi \times (50)^2}{4} \times 300 + \dfrac{\pi \times (25)^2}{4} \times 600\right)}$$

$$= 10.61 \text{ N/mm}^2$$

$$\therefore \qquad \sigma_2 = 10.61 \times \frac{(50)^2}{(25)^2} = 42.44 \text{ N/mm}^2$$

Both these stresses are tensile.

## SAQ 6

A bi-metallic rod of length 450 mm is mounted horizontally between rigid abutments. The rod has a uniform circular cross-section and is made up of a 150 mm length of steel and a 300 mm length of copper coaxial with each other. If the rod is initially stress-free, determine the stress in the rod caused by a temperature rise of 100 K.

Given　$E$, copper = 105 GNm$^{-2}$

$E$, steel = 210 GNm$^{-2}$

$\alpha$, copper = 18 × 10$^{-6}$ K$^{-1}$

$\alpha$, steel = 12 × 10$^{-6}$ K$^{-1}$

## 1.6.5 Thermal Stresses and Strains in Tapered Bars

In this section, we shall study the problem of thermal stresses in gradually tapering bars. Because the cross-sectional area is varying along the length of the bar, so does the stress. Herein, we shall derive expression for thermal stress in bars with circular cross-section. For other cross-sections, like square etc., expressions for stresses can be derived in a similar fashion.

### Fully Restrained Tapered Bar

Consider the bar shown in Figure 1.27, which is circular in cross-section but tapering from a diameter $d_2$ to $d_1$ over the length $L$. The bar is completely restrained at its ends. Now, let the temperature of the bar increased by $\Delta T$.

**Figure 1.27**

If the bar were not restrained but free to expand it would have extended by an amount $\Delta L$, given by

$$\Delta L = \alpha \, L \, \Delta T$$

Due to the restraint, a compressive force $P$ would have developed in the bar whose effect is to produce a contraction equal to $\Delta L$. Under this force, a cross-section at a distance $x$ from the larger end would have developed a stress, $\sigma_x$, equal to $\dfrac{P}{A_x}$, where $A_x$ is the area of that cross-section.

$$\therefore \qquad \sigma_x = \frac{P}{A_x} = = \frac{4P}{\pi\left(d_1 - \dfrac{x}{L}(d_1 - d_2)\right)^2}$$

The strain at that cross-section, $\varepsilon_x$, can be written as

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{4PL}{\pi E d_1 \, d_2}$$

$$P = \frac{\pi E d_1 \, d_2 \, \alpha \, \Delta T}{4}$$

Hence, the thermal stress, $\sigma_x$, at a cross-section with an area of $A_x$ is given as

$$\sigma_{\max} = \frac{E d_1 \, \alpha \, \Delta T}{d_2}$$

where $x$ is measured from the end with diameter $d_1$ which is the larger end.

The maximum stress, $\sigma_{max}$, in the bar occurs at the smaller end with diameter $d_2$.

## Partially Restrained Tapered Bar

Let us consider the bar of Figure 1.27, but now let it be free to extend by an amount $\Delta L'(\Delta L' < \Delta L)$. By following similar arguments as in the previous section, now the compressive load $P$ developed in the bar will be given by

$$P = \frac{\pi E d_1 d_2 (\Delta L - \Delta L')}{4L}$$

$$= \frac{\pi E d_1 d_2 (L \alpha \Delta T - \Delta L')}{4L}$$

and hence,

$$\sigma_x = \frac{E d_1 d_2 (L \alpha \Delta T - \Delta L')}{L \left( d_1 - \frac{(d_1 - d_2)}{L} x \right)^2}$$

$$\sigma_{max} = \frac{E d_1 (L \alpha \Delta T - \Delta L')}{L d_2}$$

## Example 1.4

A circular bar rigidly fixed at both ends is 1 m long and tapers uniformly from 20 cm diameter at one end to 10 cm diameter at the other end. Find the maximum stress in the bar, if its temperature is raised through 50°C. $E = 2 \times 10^5 \text{ N/mm}^2$ and $\alpha = 12 \times 10^{-6} \text{ K}^{-1}$.

## Solution

Here,

$d_1 = 20 \text{ cm} = 200 \text{ mm}$

$d_2 = 10 \text{ cm} = 100 \text{ mm}$

$L = 1 \text{ m} = 1000 \text{ mm}$

$\Delta L = 50°C$

$\therefore \quad \sigma_{max} = \frac{E d_1 \alpha \Delta T}{d_2} = \frac{2 \times 10^5 \times 200 \times 12 \times 10^{-6} \times 50}{100}$

$= 240 \text{ N/mm}^2.$

## SAQ 7

A straight bar has a circular cross-section, the radius of which varies linearly from 30 mm diameter at one end A to 15 mm at the other end B. The bar is 1 m long and is fixed rigidly at A, but longitudinal movement is possible at B against a spring which opposes movement with a constant stiffness of 20 kN/mm. Initially, there is no longitudinal stress in the bar. The temperature of the bar then falls by 100 K. Determine the change in the bar length if $E = 69 \text{ GN/m}^2$, and $\alpha = 23 \times 10^{-6} \text{ K}^{-1}$.

### 1.6.6 Thermal Stresses and Strains in Composite Bars

In Sub-section 1.6.4 on stepped bars, we have seen one type of composite bar where bars of different materials are joined serially. In this section, we shall consider another type of compound bar, a bar constructed from two different materials rigidly joined together as shown in Figure 1.28(a), or two bars of different materials but of equal length held between two rigid plates as shown in Figure 1.28(b).

<div style="text-align: center">

**(a)**                   **(b)**

</div>

If the individual bars were free to expand (or contract) due to temperature changes, they would do so to different amounts (for the same change in temperature), as shown in Figure 1.28(c) and Figure 1.28(d), due to the difference in the coefficients of linear expansion of the two materials.

<div style="text-align: center">

**(c)**                   **(d)**

</div>

<div style="text-align: center">

**(e)**                   **(f)**

**Figure 1.28**

</div>

However, since the two materials are rigidly joined as a compound bar and subjected to the same temperature rise, each material will attempt to expand to its free length position but each will be affected by the movement of the other. The higher coefficient of expansion material will try to pull the lower expansion material to its free length, but will be held back by the latter to its own free length position. In practice, a compromise will be reached with both extending to a common position in between the individual free length positions. This, in effect, is equivalent to a contraction in bar 2 from its free length position and an expansion of bar 1 from its free position. Thus, the higher coefficient of

expansion material develops tensile stresses, when the temperature of the compound bar increases. It will be vice-versa when the temperature decreases. From Figure 1.28(c) to 1.28(f), it is clear that extension of bar 1 + contraction of bar 2 = difference in free lengths

Let the stresses in bars 1 and 2 be $\sigma_1$ and $\sigma_2$ due to the temperature change. Then the above rule can be written as

$$\frac{\sigma_1 L}{E_1} + \frac{\sigma_2 L}{E_2} = (\alpha_2 - \alpha_1) \, L \Delta T$$

Since there are no external forces acting on the compound bar, for equilibrium, the compressive force in bar 2 should be equal to the tensile force in bar 1. This means that

$$\sigma_1 A_1 = \sigma_2 A_2$$

From the above two expressions, $\sigma_1$ and $\sigma_2$ can be written as

$$\sigma_1 = \frac{A_2 E_1 E_2 \, (\alpha_2 - \alpha_1) \Delta T}{A_1 E_1 + A_2 E_2} \, ,$$

and

$$\sigma_2 = \frac{A_1 E_1 E_2 \, (\alpha_2 - \alpha_1) \Delta T}{A_1 E_1 + A_2 E_2}$$

The extension of the compound bar, i.e. $\Delta L$ is given as

$$\Delta L \quad = \frac{(A_1 E_1 \alpha_1 + A_2 E_2 \alpha_2) L \Delta T}{A_1 E_1 + A_2 E_2}$$

## Example 1.5

A compound bar is constructed from three bars 50 mm wide by 12 mm thick fastened together to form a bar 50 mm wide by 36 mm thick. The middle bar is of aluminum alloy for which $E = 70$ GN/m$^2$ and the outside bars are of brass with $E = 100$ GN/m$^2$. If the bars are initially fastened at 18°C and the temperature of whole assembly is then raised to 50°C, determine the stresses set up in the brass and the aluminium.

For computation purposes, take following values :

$$\alpha_{\text{brass}} = 18 \times 10^{-6} \, \text{K}^{-1} \, , \text{ and } \alpha_{\text{aluminium}} = 22 \times 10^{-6} \, \text{K}^{-1}$$

**Solution**

Let the stress in aluminium bar be $\sigma_a$ and that in each brass bar be $\sigma_b$. Then for equilibrium

Force in brass = Force in aluminium

$$\sigma_b \times 50 \times 12 \times 2 = \sigma_a \times 12 \times 50$$

or $\qquad\qquad\qquad 2 \, \sigma_b = \sigma_a$

and from extension considerations, we get

$$\frac{\sigma_a}{E_a} L + \frac{\sigma_b}{E_b} L = L(\alpha_a - \alpha_b) \, \Delta T$$

$$\therefore \qquad \frac{2\sigma_b}{70\times10^9} + \frac{\sigma_b}{100\times10^9} = (22-18)\times10^{-6}\times(50-18)$$

or $\qquad 27\sigma_b = 4 \times 10^{-6} \times 32 \times 700 \times 10^9$

or $\qquad \sigma_b = 3.32$ N/mm$^2$ (T)

and $\qquad \sigma_a = 6.64$ N/mm$^2$ (C)

## Example 1.6

A steel bolt of diameter 12 mm and length 175 mm is used to clamp a brass sleeve of length 150 mm to a rigid base plate as in Figure 1.29. The sleeve has an internal diameter of 25 mm and a wall thickness of 3 mm. The thickness of the base plate is 25 mm. Initially, the nut is tightened until there is tensile force of 5 kN in the bolt. The temperature is now increased by 100°C. Determine the final stresses in the bolt and the sleeve.

For computation purposes, take following values :

$$E_b = 105 \text{ GNm}^{-2}; \; E_s = 210 \text{ GNm}^{-2}$$

$$\alpha_b = 20 \times 10^{-6} \text{ K}^{-1}; \; \alpha_s = 12 \times 10^{-6} \text{ K}^{-1}$$

**Figure 1.29**

### Solution

Let the free thermal expansions of steel and brass be $\Delta_s$ and $\Delta_b$ and $\Delta$ be the common expansion. Then

$$\Delta_s = (175)\,(12 \times 10^{-6})\,(100) = 0.21 \text{ mm}$$

$$\Delta_b = (150)\,(22 \times 10^{-6})\,(100) = 0.3 \text{ mm}$$

If the initial stresses in steel and brass due to 5 kN load are $\sigma_{s1}$ and $\sigma_{b1}$, then

$$\sigma_{s1} = + \frac{5 \times 10^3 \times 4}{\pi (12)^2} = +44.21 \text{ N/mm}^2 \qquad \text{(Tensile)}$$

$$\sigma_{b1} = + \frac{5 \times 10^3 \times 4}{\pi (31^2 - 25^2)} = -18.95 \text{ N/mm}^2 \qquad \text{(Compressive)}$$

Equilibrium of thermal stresses $\sigma_{s2}$ and $\sigma_{b2}$ requires that

$$\sigma_{s2} \times \frac{\pi (12)^2}{4} + \sigma_{b2} \times \frac{\pi (31^2 - 25^2)}{4} = 0$$

or $\quad 3\,\sigma_{s2} + \sigma_{b2} = 0$

The thermal strains are as follows :

$$\varepsilon_s = \frac{\Delta - \Delta_s}{175} \quad \text{and} \quad \varepsilon_b = \frac{(\Delta - \Delta_b)}{150}$$

Thus, the thermal stresses are as follows :

$$\sigma_{s2} = \frac{210}{175}(\Delta - \Delta_s) \times 10^3 \ \text{N/mm}^2$$

and $\quad \sigma_{b2} = \dfrac{105}{150}(\Delta - \Delta_b) \times 10^3 \ \text{N/mm}^2$

Substituting these in the equilibrium equation,

$$3 \times \frac{210}{175}(\Delta - 0.21) \times 10^3 + 7 \times \frac{105}{150}(\Delta - 0.30) \times 10^3 = 0$$

Hence, $\quad \Delta = 0.262$ mm

Thus,

$$\sigma_{s2} = \frac{210}{175}(0.262 - 0.21) \times 10^3 = +\ 62.26 \ \text{N/mm}^2 \quad \text{(Tensile)}$$

$$\sigma_{b2} = \frac{105}{150}(0.262 - 0.3) \times 10^3 = -\ 26.28 \ \text{N/mm}^2 \quad \text{(Compressive)}$$

Total stresses are therefore as follows :

$$\sigma_s = \sigma_{s1} + \sigma_{s2} = -\ 106.5 \ \text{N/mm}^2 \quad \text{(Tensile)}$$

$$\sigma_b = \sigma_{b1} + \sigma_{b2} = -\ 45.23 \ \text{N/mm}^2 \quad \text{(Compressive)}$$

## SAQ 8

(a) A steel rod of cross-sectional area 600 mm$^2$ and a coaxial copper tube of cross-sectional area 1000 mm$^2$ are firmly attached at their ends to form a compound bar. Determine the stress in the steel and in copper when the temperature of the bar is raised by 80°C and an axial tensile force of 60 kN is applied.

For steel, $E = 200$ GNm$^{-2}$ and $\alpha = 11 \times 10^{-6}$ K$^{-1}$

For copper, $E = 100$ GNm$^{-2}$ and $\alpha = 16.5 \times 10^{-6}$ K$^{-1}$

(b) Steel rod of 20 mm diameter passes centrally through a tight fitting copper tube of external diameter 40 mm. The tube is closed with the help of rigid washers of negligible thickness and nuts threaded on the rod. The nuts are tightened till the compressive load on the tube is 50 kN. Determine the stresses in the rod and the tube, when the temperature of the assembly falls by 50°C.

For steel, $E = 200$ GNm$^{-2}$ and $\alpha = 12 \times 10^{-6}$ K$^{-1}$

For copper, $E = 100$ GNm$^{-2}$ and $\alpha = 18 \times 10^{-6}$ K$^{-1}$

(c) A rigid slab weighing 600 kN is placed upon two bronze rods and one steel rod each of 60 cm$^2$ cross-sectional area at a temperature of 15°C.

The bronze rods are 25 cm while the steel rod is 30 cm long. Before the slab was placed, the top of all the three rods are level. Find the temperature, at which the stress in the steel rod will be zero.

$E$ of steel $= 200$ GNm$^{-2}$   and   $E$ of bronze $= 80$ GNm$^{-2}$

$\alpha$ of steel $= 12 \times 10^{-6}$ K$^{-1}$ and $\alpha$ of bronze $= 18 \times 10^{-6}$ K$^{-1}$

# 1.7 RELATIONSHIP BETWEEN ELASTIC CONSTANTS

In Section 1.3.3, we have defined the four Elastic Constants $E$, $\upsilon$, $K$ and $G$ and also stated that they are not independent. Now, we shall establish the relationship between them.

## 1.7.1 Relationship between $E$ and $K$

Figure 1.13 shows the spherical state of stress in which the normal stress component is the same, $\sigma_0$ in any direction and shear stress components are zero on any plane. Such a state of stress is also called as volumetric stress. If the Young's Modulus and Poisson's Ratio of the solid are known as $E$ and $\upsilon$, the strain components are given by

$$\varepsilon_x = \frac{\sigma_x}{E} - \upsilon \frac{\sigma_y}{E} - \upsilon \frac{\sigma_z}{E} \qquad \ldots (1.17)$$

Since,        $\sigma_x = \sigma_y = \sigma_z = \sigma_0$

$$\varepsilon_x = \frac{\sigma_0}{E}(1 - 2\upsilon) \qquad \ldots (1.18)$$

Similarly,    $\varepsilon_y = \dfrac{\sigma_0}{E}(1 - 2\upsilon)$ and   $\varepsilon_z = \dfrac{\sigma_0}{E}(1 - 2\upsilon)$

From Eq. (1.7), we can express the change in volume of the solid as

$$dV = V(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

Volumetric strain, $\varepsilon_v = \dfrac{dV}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$

Bulk Modulus,    $K = \dfrac{\text{Volumetric stress}}{\text{Volumetric strain}}$

$$= \frac{\sigma_0}{\varepsilon_x + \varepsilon_y + \varepsilon_z} = \frac{\sigma_0}{3\varepsilon_x}$$

$$= \frac{\sigma_0}{\dfrac{\sigma_0}{E} \times 3(1 - 2\upsilon)}$$

$$K = \frac{E}{3(1-2\upsilon)} \qquad \ldots (1.19)$$

## 1.7.2 Relationship between $E$ and $G$

Figure 1.30(a) shows what is known as the state of pure shear.

The deformation of the body due to the shear is shown in Figure 1.30(b) (projected view).

(a)                                              (b)

**Figure 1.30**

Normal stress component, $\sigma = \tau \sin 2\theta$

$$= \tau \sin \frac{2\pi}{4} = -\tau$$

Shear stress component $\qquad = \tau \cos 2\theta$

$$= \tau \cos \frac{2\pi}{4} = 0$$

Normal stress component $\qquad = \tau \sin 2\theta = -\tau$

Shear stress component $\qquad = \tau \cos 2\theta = 0$

or shear strain $\qquad \gamma = \dfrac{2\tau (1-\upsilon)}{E}$

Rigidity Modulus $\qquad G = \dfrac{\text{Shear stress}}{\text{Shear strain}} = \dfrac{\tau}{\dfrac{2\tau}{E}(1-\upsilon)}$

$$G = \frac{E}{2(1-\upsilon)} \qquad \ldots (1.20)$$

## 1.7.3 Significance of the Relationships

We have already mentioned that there are only two independent elastic constants. But this is true only in the case of isotropic solids. The term isotropy means same property in all directions. We have defined Elastic Modulus as ratio between

stress and strain and implied (without stating) that the ratio holds good for all directions, i.e. $\dfrac{\sigma_x}{\varepsilon_x} = \dfrac{\sigma_y}{\varepsilon_y} = \dfrac{\sigma_z}{\varepsilon_z}$ .

Materials for which this is true are called isotropic materials and throughout this course you will be learning only about isotropic solids.

When any two of the elastic constants are known, the other two may be calculated using Eqs. (1.19) and (1.20). Let us have an example.

## Example 1.7

In separate experiments, Young's Modulus and Rigidity Modulus of a material have been determined as 120 GPa and 50 GPa respectively. Calculate the Poisson's Ratio and Bulk Modulus of the material.

**Solution**

Here, Young's Modulus, $E = 120$ GPa.

Let Poisson's ratio be $\upsilon$.

Rigidity Modulus, $G = \dfrac{E}{2(1+\upsilon)} = 50$ GPa

$$(1 + \upsilon) = \frac{E}{2G} = \frac{120}{2 \times 50} = 1.2$$

$\therefore$ Poisson's Ratio, $\upsilon = 1.2 - 1 = 0.2$

Bulk Modulus, $K = \dfrac{E}{3(1-2V)}$

$$= \frac{120}{3(1 - 2 \times 0.2)}$$

$$= 66.667 \text{ GPa.}$$

## SAQ 9

(a) Through separate experiments the elastic constants of a material, namely, $E$, $\upsilon$, $G$ and $K$ were determined as 75 GPa, 0.247, 30 GPa and 49.2 GPa respectively. Taking the values of $E$ and $G$ as correct, find the percentage errors in the values of Poisson's Ratio and Bulk Modulus.

(b) The values of Young's Modulus and Rigidity Modulus of a material are known to be 20.8 GPa and 8 GPa respectively. If a spherical ball of diameter 150 mm made of the material is immersed in water to a depth of 120 meters, find the change in volume of the ball.

(c) By eliminating Poisson's ratio $\upsilon$ from Eqs. (1.19) and (1.20), show that $G = \dfrac{3KE}{9K - E}$ .

# 1.8 SUMMARY

In this unit you have learnt the concepts of stress, strain, elastic modulus and the basic concepts in application to analysis of stresses and deformation in simple solids. Any designer of engineering systems has to have a clear perception of how forces are resisted by solids and the effects produced in solids so as to enable him to produce designs with safety and stability. The study of these aspects will require learning further new concepts and methods, understanding of which will require the knowledge you acquired in this unit. This unit is, therefore, like the first step in a full flight of stairs climbing which will provide you with a powerful tool of engineering.

# 1.9 ANSWERS TO SAQs

**SAQ 1**

   (a)   Change in volume $= -1800$ mm$^3$

   (b)   Young's Modulus $E = 12.7324$ kN/mm$^2$

          Poisson's Ratio, $\upsilon = 0.16667$.

**SAQ 2**

   (a)   $dv_1 = 1462.5$ mm$^2$, $dv_2 = 1950$ mm$^3$, $dv_3 = 975$ mm$^2$,
          $dv_4 = 2437.5$ mm$^3$

          Total change in volume $dv = 6825$ mm$^3$

   (b)   For bar given in Figure 1.20, $\delta = 24.92$ mm

          For bar given in Figure 1.21, $\delta = 0.0268$ mm

**SAQ 3**

   (b)   Equivalent area of copper bar $= 13953.91$ mm$^2$

**SAQ 4**

   (a)   Stress in steel $= 7.16645$ N/mm$^2$

          Stress in copper $= 4.3$ N/mm$^2$

          Stress in aluminium $= 2.8666$ N/mm$^2$

   (b)   Load carrying capacity of bar $= 837.23$ kN.

   (c)   Area of reinforcement steel required $= 3529.4$ mm$^2$

   (d)   The chart will be straight line with the equation

   $$P = 640 \left(1 + 17 \frac{A_S}{A}\right) \text{ kN}$$

   (e)   Load carrying capacity of given column $= 1305$ kN.

          Area of steel required $= 5106.4$ mm$^2$ if $P = 1200$ kN.

**SAQ 5**

   (a)   $9.192$ N/mm$^2$,

   (b)   $158.7$ N/mm$^2$

**SAQ 6**

201.6 MPa

**SAQ 7**

– 1.26 mm

**SAQ 8**

(a)   94.6 N/mm$^2$, 3.3 N/mm$^2$

(b)   $\sigma_s = 123.15$ N/mm$^2$

(c)   188.6°C

$\sigma_c = 41.05$ N/mm$^2$

**SAQ 9**

Error in Poisson's Ratio = 1.2%

Error in Bulk Modulus = 1.6%

Change in volume of Spherical ball = 122341 mm$^3$ or 6.823%.