UNIT 6 TORSION

Structure

6.1 INTRODUCTION

The results obtained during the study of shear enable us to pass over to the study of strength under torsion. Members in torsion are encountered in many engineering applications. The most common application is provided by transmission shafts, which are used to transmit power from one point to another, as from a steam turbine to an electric generator, or from a motor to a machine tool, or from the engine to the rear axle of an automobile. These shafts may either be solid or they may be hollow. In practice, we come across torsion very often; a turning force is always applied to transmit energy by rotation. This turning force is applied either to rim of a pulley, keyed to the shaft, or to any other suitable point at some distance from the axis of the shaft. The product of this turning force, and the distance between the point of application of the force and the axis of the shaft is known as torque, turning moment or twisting moment.

This unit is devoted to the treatment of members with circular, cross-sectional areas. In practice, members that transmit torque, such as shafts of motors, torque tubes of power equipment, etc. are predominantly circular or tabular in cross-section.

Objectives

After studying this unit, you should be able to

- conceptualize the theory of torsion,
- calculate the strength of the solid and hollow circular shaft, deformations and stresses developed in the shafts,
- determine the power transmitted by the shafts in MKS and SI units,
- design the shaft for the required torque as per the strength criteria and as per the stiffness criteria, and
- determine the torque transmitted by the new shaft (should be equal to the torque transmitted by the replaced shaft) in replacing the shaft.

6.2 TORSION OF CIRCULAR SHAFTS

A shaft of circular section is said to be in pure torsion when it is subjected to equal and opposite end couples whose axes coincide with the axis of the shaft. In other words, if the moment is applied in a vertical plane perpendicular to the longitudinal axis of the beam or a shaft, it will be subjected to a torque causing twist or torsion in the member. As the beam bends due to bending moment, the shaft twists due to twisting moment. Figure 6.1 shows a pulley of radius *R* subjected to a system of couple, i.e. equal and opposite force *W*. The couple attached to the shafts cause the turning effect on the pulley.

$\therefore$ Twisting moment or torque, T = Force $\times$ Lever arm

$$T = W \times R$$

Figure 6.1

In order that the body should remain in static equilibrium, it must exert an equal amount of resisting moment.

At any point in the section of the shaft, a shear stress is induced or more exactly, the state of stress at any point in the cross-section of the shaft is one of pure shear, the direction of which is tangential at any point in the shaft. By the principle of complementary shear stresses, we know that in a state of simple shear there are two planes carrying the shear stress of the same intensity. These planes must be perpendicular to each other.

In the case of the shaft in torsion, the planes of shear at a point are

(a) the cross-section itself, and

(b) the plane containing the point and the axis of the shaft.

Figure 6.2

To find internal torque or resisting moment, in statically determinate members, only one equations of statics, $\Sigma M_z = 0$, is required, where Z axis is directed along the member. As in the case of determination of twisting moment at any point along the length of the member, pass a plane at the desired section perpendicular to the longitudinal axis of the member and remove everything to either side of the cut. The internal or resisting torque necessary to maintain equilibrium of the isolated part is determined.

Consider, for example, the system consisting of the turbine A and the generator B connected by the transmission shaft AB (Figure 6.2), and breaking the system into its three component parts (Figure 6.3), we note that the turbine exerts a twisting couple or torque T on the shaft, and that the shaft exerts an equal torque on the generator.

The generator reacts by exerting the equal and opposite torque T' on the shaft, and the shaft by exerting the torque T' on the turbine.

Assumptions

Following assumptions are made, while finding out shear stresses and deformations in a circular shaft subjected to torsion.

(a) The material of the shaft is homogeneous and isotropic.

(b) The twist along the shaft is uniform throughout, i.e. all normal cross-sections which are at the same axial distance suffer equal relative rotation.

(c) Normal cross-sections of the shaft, which were plane and circular before twist, remain plane and circular after twist, i.e. no warping or distortion of parallel planes normal to the axis of the member takes place.

(d) All diameters of the normal cross-section which were straight before twist, remain straight with their magnitude unchanged, after twist.

(e) Stress is proportional to strain, i.e. all the stresses are within the elastic limit.

(f) Intensity of stress varies uniformly from zero at the centre to a maximum at the outside surface and hence the stress is proportional to the distance of that point from the centre.

A little consideration will show that the above assumptions are justified, if the torque applied is small and the angle of twist is also small.

6.2.1 Theory of Torsion

For the purpose of developing the expressions for the torsional stress and strain, we shall assume that one end of the shaft is fixed and a moment is applied at the other end, the plane of application of moment being perpendicular to the longitudinal axis of the beam. This assumption is valid because whether it rotates at uniform speed to transmit the power or is at rest; the stress and strain due to equal and opposite couples at its ends will remain the same.

Consider a shaft fixed at one end, and subjected to a torque (T) at the other end as shown in Figure 6.3.

Let T = Torque in kg cm,

l = Length of the shaft, and

R = Radius of the shaft.

A balancing torque of equal magnitude and opposite in direction will be induced at the fixed end.

Figure 6.3

Let the line CA on the surface of the shaft be deformed to CA′ and OA to OA′ as shown in Figure 6.3.

Let $\angle ACA' = \phi$ and $\angle AOA' = \theta$.

As a result of the torque applied, every cross-section of the shaft will be subjected to shear stresses.

Let f_s = shear stress induced at the outermost surface, and

C = modulus of the rigidity of the shaft material.

We know that

Shear strain = Deformation per unit length

$$= \frac{AA'}{l}$$

$$= \tan \phi$$

$$= \phi \text{ (}\phi\text{ being very small)}$$

We also know that the length of the arc AA′ = $R\theta$

$$\therefore \quad \phi = \frac{AA'}{l} = \frac{R\theta}{l} \qquad \ldots (6.1)$$

Moreover, deformation = $\dfrac{\text{Shear Stress}}{\text{Modulus of Rigidity}}$

$$\phi = \frac{f_s}{C} \qquad \ldots (6.2)$$

Now from Eqs. (6.1) and (6.2), we find that

$$\frac{f_s}{C} = \frac{R\theta}{l}$$

or,

$$\frac{f_s}{R} = \frac{C\theta}{l}$$

The shaft may be taken to consist of an infinite number of elemental hollow shafts, one surrounding the other.

If the deformation of a line on the surface of any such interior cylinder, at a radius r be considered, the shear stress intensity 'q' at the radius 'r' is given by the relation,

$$\frac{q}{r} = \frac{C\theta}{l}$$

$$\therefore \qquad \frac{f_s}{r} = \frac{q}{r} = \frac{C\theta}{l}$$

Thus, it can be stated that the intensity of shear stress at any point in the cross-section of a shaft subjected to pure torsion is proportional to its distance from the centre.

This means that the shear stress is maximum on the outside surface and variation of shear stress with radius is linear.

6.2.2 Resisting Torque

From conditions of equilibrium the external torque T must be balanced by resisting torque, i.e. by moments of tangential shearing stresses acting on any transverse section. Consider a solid circular shaft subjected to some torque.

Let R = Radius of the shaft, and

f_s = Maximum shear stress developed in the outermost layer of the shaft material.

Now, consider an elementary ring of thickness dx at a distance x from the centre as shown in Figure 6.4.

Figure 6.4

We know that the area of the ring, $da = 2\pi x dx$

Shear stress at this section, (f_x) would be as follows:

$$\frac{f_x}{x} = \frac{f_s}{R}$$

$$\therefore \qquad f_x = \frac{f_s}{R} \times x$$

$\therefore$ Turning Force = Stress × Area

$$= f_x \times da$$

$$= f_s \frac{x}{R} da$$

$$= f_s \frac{x}{R} 2\pi x \, dx$$

$$= \frac{2\pi f_s}{R} x^2 \, dx$$

We know that turning moment of this element, or moment of resistance offered by the elemental ring,

dT = Turning force × Distance of the element from the axis of the shaft

$$dT = \frac{2\pi f_s}{R} x^2 \, dx \, x$$

$$= \frac{2\,\pi\, f_s}{R}\, x^3\, dx$$

$\therefore$ Total moment of resistance offered by the whole shaft is

$$T = \int_0^R \frac{2\,\pi\, f_s}{R}\, x^3\, dx$$

$$= \int_0^R \frac{f_s}{R}\, 2\,\pi\, x\, x^2\, dx$$

$$= \frac{f_s}{R} \int_0^R 2\,\pi\, x\, x^2\, dx$$

$$= \frac{f_s}{R}\, J \qquad \ldots (6.3)$$

where, $$J = \int_0^R 2\,\pi\, x\, x^2 dx = \int_0^R da\, x^2$$

Here, J represents the moment of inertia of the shaft section about the axis of the shaft. The moment of inertia of a plane area, with respect to an axis perpendicular to the plane of the figure is called polar moment of inertia with respect to the point, where the axis intersects the plane. In a circular plane, the point is always the centre of the circle. Therefore, J is known as polar moment of inertia, i.e. moment of inertia about ZZ axis.

As per perpendicular axis theorem,

$$I_{zz} = I_{xx} + I_{yy}$$

From a circular section, $I_{xx} = I_{yy} = \dfrac{\pi}{64} D^4$

where D is diameter of the circular shaft.

$\therefore$ $$I_{zz} = J = \frac{\pi D^4}{64} + \frac{\pi D^4}{64}$$

$$J = \frac{\pi D^4}{32}$$

The term $\dfrac{J}{R}$ is known as torsional section modulus or polar modulus, denoted by Z_p. It is similar to section modulus, Z, which is equal to $\dfrac{M}{I}$.

Thus, polar modulus for a solid shaft

$$Z_p = \frac{\pi D^4}{32} \times \frac{2}{D} = \frac{\pi D^3}{16}$$

The resisting torque or torsional moment of resistance is given by

$$T = \frac{f_s}{R} J = \frac{f_s}{R} \frac{\pi D^4}{32}$$

$$T = f_s \frac{2}{D} \frac{\pi D^4}{32} = \frac{\pi}{16} f_s D^3$$

This resisting torque is also known as strength of the shaft. Strength of the shaft is defined as the maximum torque or power the shaft can transmit from one pulley to another.

Connecting Eqs. (6.1), (6.2) and (6.3), we get

$$\frac{q}{r} = \frac{f_s}{R} = \frac{T}{J} = \frac{C\theta}{l} \qquad \ldots (6.4)$$

which is called the *torsion equation.*

Eq. (6.4) can be compared with equation of bending $\frac{M}{I} = \frac{f}{y} = \frac{E}{R}$.

The expression $\frac{q}{r}$ corresponds to $\frac{f}{y}$, $\frac{T}{J}$ corresponds to $\frac{M}{I}$ and $\frac{C\theta}{l}$ corresponds to $\frac{E}{R}$. The expression CJ corresponds to expression EI.

The term CJ is called torsional rigidity and the term EI is called flexural rigidity.

It may be noted that :

(a) $\frac{T}{\theta}$, i.e. torque required for unit twist, is called the torsional stiffness of the shaft.

(b) $\frac{T}{\theta/L}$, i.e. torque divided by the angle of twist per unit length, is called the torsional rigidity CJ.

6.2.3 Deformations in a Circular Shaft

Consider a circular shaft, which is attached to a fixed support at one end. If a torque T is applied to the other end, the shaft will twist with its free end rotating through an angle θ called the angle of twist. Observation shows that within a certain range of values of T, the angle of twist θ is proportional to T. It also shows that θ is proportional to the length of the shaft. In other words, the angle of twist for a shaft of the same material and same cross-section, but twice as long, will be twice as large under the same torque T.

When a circular shaft is subjected to torsion, every cross-section remains plane and undistorted. In other words, while the various cross-sections along the shaft rotate through different amounts, each cross-section rotates as a solid rigid slab. When a bar of square cross-section is subjected to torsion, its various cross-sections are warped and do not remain plane.

The fact that the cross-section of a circular shaft remains plane and undistorted is due to the fact that the circular shaft is axisymmetric, i.e. its appearance remains the same when it is viewed from a fixed position and rotated about its axis through an arbitrary angle. Square bar, on the other hand, retains the same appearance only if they are rotated through 90° or 180°.

If all sections of the shaft, from one end to the other, are to remain plane and undistorted, we must make sure that the couples are applied in such a way that the ends of the shaft themselves remain plane and undistorted. This may be accomplished by applying the couples T and T' to rigid plates, which are solidly attached to the ends of the shaft. We may then be sure that all sections will remain plane and undistorted when the loading is applied, and that the resulting deformation will occur in a uniform fashion throughout the entire length of the shaft.

We shall now determine the distribution of shearing strains in a circular shaft of length L and radius R, which has been twisted through an angle 'θ'. Detaching from the shaft a cylinder of radius 'h', we consider a small square element formed by two adjacent circles and two adjacent straight lines traced on the surface of the cylinder before any load is applied. As the shaft is subjected to a torsional load the element deformed into a rhombus as shown in Figure 6.5.

Figure 6.5

The shearing strain 'ϕ' in a given element is measured by the change in the angles formed by the sides of that element. Since the circles defining two of the sides of the element remain unchanged, the shearing strain 'ϕ' must be equal to the angle between line AB and A′B.

As we discussed in the preceding section,

$$\phi = \frac{r\theta}{L}$$

The above equation shows that the shearing strain 'ϕ' at a given point in a shaft subjected to torsion is proportional to the angle of twist θ. It also shows that ϕ is proportional to the distance 'r' from the axis of the shaft to the point under consideration. In other words, the shearing strain in a circular shaft varies linearly with the distance from the axis of the shaft.

The shearing strain is maximum on the surface of the shaft, where $r = R$.

Thus, we have, $$\phi_{max} = \frac{R\theta}{L}$$

Eliminating θ from above two equations, we may express the shearing strain ϕ at a distance r from the axis of the shaft as

$$Q = \frac{r}{R}\phi_{\max}$$

6.2.4 Stresses in a Circular Shaft

No particular stress-strain relationship has been assumed so far in the discussion of circular shafts in torsion. We shall now consider the case when the torque T is such that all shearing stresses in the shaft remain below the yield strength f_{sy}. For all practical purposes, this means that the stresses in the shaft will remain below the proportional limit and below the elastic limit as well. Thus, Hooke's law will apply and there will be no permanent deformation.

As per Hooke's law for shearing stress and strain,

$$f_r = C\,\phi$$

where,

C = Modulus of rigidity or shear modulus of the material, and

f_r = Shear stress at a radius r from the axis of the shaft.

We know that

$$\phi = \frac{r}{R}\,\phi_{\max}$$

Multiplying by C on both sides, we get

$$C\phi = \frac{r}{R}\,C\phi_{\max}$$

or, $$f_r = \frac{r}{R} f_s$$

where, f_s = Shear stress at a radius R from the axis of the shaft.

The equation obtained shows that as long as the yield strength (or proportional limit) is not exceeded in any part of a circular shaft, the shearing stress in the shaft varies linearly with the distance r from the axis of the shaft. Figure 6.6 shows the stress distribution in a solid circular shaft of radius R.

Figure 6.6

From the discussion we had in the theory of torsion, we have, $T = \dfrac{f_s\,J}{R}$

or $$f_s = \frac{T}{J}R$$

and $$f_r = \frac{T}{J} r$$

If T is expressed in Nm, R or r in meters and J in m^4, the resulting shearing stress will be expressed in N/m^2, that is Pascal (Pa).

Up to this point, our analysis of stresses in a shaft has been limited to shearing stresses. This is due to the fact that the element we had selected was oriented in such a way that its faces were either parallel or perpendicular to the axis of the shaft. We know that the normal stresses, shearing stresses or a combination of both may be found under the same loading condition, depending upon the orientation of the element which has been chosen. Consider the two elements *a* and *b* located on the surface of a circular shaft subjected to torsion, as shown in Figure 6.7. Since the faces of element *a* are parallel and perpendicular to the axis of the shaft, the only stresses on the element will be the shearing stresses given by $f_s = \frac{T}{J} R$, i.e. the element *a* is in pure shear. On the other hand, the faces of element *b*, which form arbitrary angles with the axis of the shaft, will be subjected to a combination of normal and shearing stresses.

We also note that all the stresses involved have the same magnitude, $\frac{TR}{J}$.

Figure 6.7

Ductile materials generally fail in shear. Therefore, when subjected to torsion, a specimen made of a ductile material breaks along a plane perpendicular to its longitudinal axis. On the other hand, brittle materials are weaker in tension than in shear. Thus, when subjected to torsion, a specimen made of brittle material tends to break along surfaces which are perpendicular to the direction in which tension is maximum, i.e. along surfaces forming a 45° angle with the longitudinal axis of the specimen.

Example 6.1

Find the torque which a shaft of 25 cm diameter can safely transmit, if the shear is not to exceed 460 kg/cm^2.

Solution

Diameter of shaft, D = 25 cm

Maximum shear stress, f_s = 460 kg/cm^2

Let T is the torque transmitted by the shaft.

$$T = \frac{\pi}{16} f_s D^3$$

$$= \frac{\pi}{16} \times 460 \times 25^3 = 1411262.33 \text{ kg cm} = 14112.62 \text{ kg m.}$$

Example 6.2

A bar of magnesium alloy 28 mm in diameter was tested on gauge length of 25 cm in tension and in torsion. A tensile load of 5 tonnes produced an extension of 0.4 mm and a torque of 1250 kg cm produced a twist of 1.51°.

Determine

(a) the Young's modulus,

(b) the modulus of rigidity,

(c) the bulk modulus, and

(d) the Poisson's ratio for the material under test.

Solution

Diameter of bar, D = 28 mm = 2.8 cm

Area of bar, $A = \frac{\pi}{4}(2.8)^2 = 6.1575 \text{ cm}^2$

Length of bar, l = 25 cm

Load on the bar, P = 5 t = 5000 kg

Extension of the bar, δl = 0.4 mm = 0.04 cm

Torque, T = 1250 kg cm

Angle of twist, θ = 1.51° = 0.02635 radian

$$\delta l = \frac{Pl}{AE}$$

$$0.04 = \frac{5000 \times 25}{6.1575 \times \text{E}}$$

$$E = 0.5075 \times 10^6 \text{ kg/cm}^2$$

Young's modulus for the alloy, $E = 0.5075 \times 10^6$ kg/cm^2.

$$J = \frac{\pi \text{D}^4}{32} = \frac{\pi}{32}(2.8)^4 = 6.034 \text{ cm}^4$$

$$\frac{T}{J} = \frac{C\theta}{l}$$

$$\frac{1250}{6.034} = \frac{C \times 0.02635}{25}$$

$$C = 0.1965 \times 10^6 \text{ kg/cm}^2$$

Modulus of rigidity for the alloy, $C = 0.1965 \times 10^6$ kg/cm^2

$$C = \frac{ME}{2(m+1)}$$

$$0.1965 \times 10^6 = \frac{m \times 0.5075 \times 10^6}{2(m+1)}$$

$$2 \times 0.1965 \times 10^6 \times (m + 1) = m \times 0.5075 \times 10^6$$

$$0.1145\, m = 0.393$$

$$m = 3.43$$

$$\frac{1}{m} = 0.29$$

Poisson's ratio for the alloy, $\mu = 0.29$

$$K = \frac{ME}{3(m-2)}$$

$$= \frac{3.43 \times 0.5075 \times 10^6}{3(3.43-2)}$$

$$= 0.4058 \times 10^6 \text{ kg/cm}^2.$$

Bulk modulus for the alloy, $K = 0.4058 \times 10^6$ kg/cm^2.

SAQ 1

Find the maximum torque, which can be applied safely to a shaft of 300 mm diameter. The permissible angle of twist is 1.5° in a length of 7.5 m and the shear stress is not to exceed 42 N/mm^2. Take $C = 84.4$ kN/mm^2.

6.3 POWER TRANSMISSION BY SHAFTS

We have already discussed that the main purpose of a shaft is to transmit power from the shaft to another in factories and workshops. Power is the time rate of doing work.

Consider a shaft of radius R subjected to end couples which cause turning effect.

$\therefore$ Work done in revolution = Work done by each force

$$= P \times 2\pi\, R + P \times 2\pi\, R$$

$$= 2\pi \times 2PR$$

$$= 2\pi\, T \qquad (\therefore\ 2PR = T\,)$$

If the shaft rotates at N-rpm, the work done per minute = $2\pi\, nT$,

where, N = Number of revolutions per minute (rpm), and

T = Average torque in kg m.

MKS Unit

1 hp = 75 kg m/sec = 4500 kg m/min

Work done per minute = $2\pi\, NT$

where, N = Number of revolutions per minute, and

T = Average torque in kg m.

Since there are 4500 kg m per minute in one horsepower,

$$\text{Power, } P = \frac{\text{Work done in kg m/min}}{4500} \text{ hp}$$

$$P = \frac{2\pi NT}{4500} \text{ hp}$$

Angular displacement in radians = $2\pi N$

SI Unit

In SI system, power (P) is measured in watts (W).

1 W = 1 Joule/sec = 1 N m/sec = 60 N m/min

Work done per minute = $2\pi\ NT$

where, N = Number of revolutions per minute, and

T = Average torque in N m.

Since there are 60 N m/min in one watt,

$$\text{Power, } P = \frac{\text{Work done in N m/min}}{60} \text{ watts}$$

$$P = \frac{2\pi NT}{60} \text{ watts}$$

$$= T\omega \text{ watts}$$

where, ω = Angular displacement in radians/sec = $\left(\frac{2\pi N}{60}\right)$

Design of Shafts

The principal specifications to be met in the design of transmission shaft are the power to be transmitted and the speed of rotation of the shaft. The role of the designer is to select the material and the dimensions of the cross-section of the shaft, so that the maximum shearing stress allowable in the material will not be exceeded when the shaft is transmitting the required power at the specific speed.

We know that,

$$\text{Power, } P = T\omega$$

$$= T\left(\frac{2\pi N}{60}\right) = T \times 2\pi f$$

where, f = Frequency of the rotation, i.e. number of revolutions per second.

The unit of frequency is thus 1 s^{-1} and is called a Hertz (Hz).

$$\therefore \quad P = 2\pi f T$$

$$P = \frac{2\pi NT}{60} \text{ watts (in SI units)}$$

$$P = \frac{2\pi NT}{4500} \text{ hp (in MKS units)}$$

After having determined the torque T from the above equations, and having selected the material to be used, the designer will carry the values of T and of the maximum allowable stress into the elastic torsion formula.

$$T = f_s \frac{J}{R}$$

$$\frac{J}{R} = \frac{T}{f_s}$$

The torsional section modulus $\frac{J}{R}$ can be calculated from the above equation.

Knowing $\frac{J}{R}$, the diameter of the shaft can be easily calculated

Thus, we get

$$T = \frac{\pi}{16} f_s D^3$$

Example 6.3

Calculate the diameter of a solid shaft transmitting 150 kW at 25 rpm, if the maximum shear stress in the shaft is not to exceed 70 MPa. Compare this with the shaft delivering same power at 25000 rpm.

Solution

Power transmitted, P = 150 kW = 150×10^3 watts

Number of revolutions, N = 25 rpm

Since, 1 Pa = 1 N/m^2 and 1 Mega Pascal = 10^6 N/m^2 = N/mm^2

Then, maximum shear stress, f_s = 70 MPa = 70 N/mm^2

Let T be the torque transmitted in N m.

$$P = \frac{2\pi NT}{60}$$

$$150 \times 10^3 = \frac{2\pi \times 25}{60} \times T$$

$$T = 57.28 \times 10^3 \text{ N m}$$

$$T = 57.28 \times 10^6 \text{ N mm}$$

$$\frac{T}{J} = \frac{f_s}{R}$$

$$\frac{57.28 \times 10^6}{\frac{\pi D^4}{32}} = \frac{70}{\left(\frac{D}{2}\right)}$$

$$D = 160.9 \text{ mm}$$

If N = 25000 rpm, then $P = \frac{2\pi NT}{60}$

$$150 \times 10^3 = \frac{2\pi \times 25000 \times T}{60}$$

$$T = 57.28 \text{ N m} = 57.28 \times 10^3 \text{ N mm}$$

$$\frac{T}{J} = \frac{f_s}{R}$$

$$\frac{57.28 \times 10^3}{\frac{\pi}{32} D^4} = \frac{70}{\left(\frac{D}{2}\right)}$$

$$D = 16.09 \text{ mm}$$

It is seen from this example that the size of the shaft is reduced very much if the power is transmitted at high speed. That is the reason for the modern tendency to use high speed machines, which results in considerable saving in the material cost.

Example 6.4

A steel shaft transmits 105 kW at 160 rpm. If the shaft is 100 mm diameter, find the torque on the shaft and the maximum shear stress induced. Find also the twist of the shaft in a length of 6 m.

Take $C = 8 \times 10^4$ N/mm^2.

Solution

$P = 105 \text{ kW} = 105 \times 10^3$ W

$N = 160$ rpm

$D = 100$ mm

$l = 6 \text{ m} = 6000$ mm

$C = 8 \times 10^4$ N/mm^2

We know, $$P = \frac{2\pi NT}{60}$$

$$105 \times 10^3 = \frac{2\pi \times 160}{60} \times T$$

$$T = 6266 \text{ N m} = 6.266 \times 10^6 \text{ N mm}$$

$$T = \frac{\pi}{16} f_s\, D^3$$

$$6.266 \times 10^6 = \frac{\pi}{16} f_s\, (100)^3$$

$$f_s = 31.19 \text{ N/mm}^2$$

$$\frac{T}{J} = \frac{C\theta}{l}$$

$$\frac{6.266 \times 10^6}{\frac{\pi}{32} \times (100)^4} = \frac{8 \times 10^4}{6000}\theta$$

$$\theta = 0.04786 \text{ radian} = 2° 45'$$

Example 6.5

Find the diameter of the shaft required to transmit 60 kW at 150 r.p.m., if the maximum torque is likely to exceed the mean torque by 25% for a maximum permissible shear stress of 60 N/mm^2. Find also the angle of twist for a length of 2.5 metres.

Take $C = 8 \times 10^4$ N/mm^2.

Solution

Here,
$$P = 60 \text{ kW} = 60 \times 10^3 \text{ W}$$
$$N = 150 \text{ rpm}$$
$$T_{max} = 1.25\, T_{mean}$$
$$f_s = 60 \text{ N/mm}^2$$
$$l = 2.5 \text{ m}$$
$$C = 8 \times 10^4 \text{ N/mm}^2$$

We know,
$$P = \frac{2\pi NT}{60}$$
$$60 \times 10^3 = \frac{2\pi \times 150 \times T}{60}$$
$$T = 3819.7 \text{ N m} = 3.8197 \times 10^6 \text{ N mm}$$
$$T_{max} = 1.25\, T_{mean}$$
$$= 1.25 \times 3.8197 \times 10^6 = 4.746 \times 10^6 \text{ N mm}$$
$$T_{max} = \frac{\pi}{16} f_s\, D^3$$
$$4.7746 \times 10^6 = \frac{\pi}{16} \times 60 \times D^3$$
$$D^3 = \frac{4.7746 \times 10^6 \times 16}{\pi \times 60}$$
$$\therefore \quad D = 74 \text{ mm}$$
$$\frac{T}{J} = \frac{C\theta}{l}$$
$$\frac{4.7746 \times 10^6}{\frac{\pi}{32} \times (74)^4} = \frac{8 \times 10^4}{2500} \times \theta$$
$$\theta = 0.0507 \text{ radians} = 2^\circ\, 54'$$

Example 6.6

Show that for a given maximum shear stress the minimum diameter required for a solid circular shaft to transmit P kW at N rpm can be expressed as

$$d = \text{Constant} \times \left(\frac{P}{N}\right)^{1/3}$$

What value of the maximum shear stress has been used if the constant equals 84.71, being in millimeters?

Solution

We know, $P = \dfrac{2\pi NT}{60}$ watts

$$P \times 10^3 = \frac{2\pi NT}{60}$$

$$T = \frac{60 \times 10^3 P}{2\pi N} \text{ N m}$$

$$= \frac{60 \times 10^3 P}{2\pi N} \times 1000 \text{ N mm}$$

$$= 3 \times 10^7 \left(\frac{P}{\pi N} \right) \text{ N mm}$$

$$T = \frac{\pi}{16} f_s \, D^3$$

$$D^3 = \frac{16T}{\pi f_s}$$

$$D^3 = \frac{16}{\pi f_s} \times 3 \times 10^7 \times \frac{P}{\pi N}$$

$$D^3 = \frac{16 \times 3 \times 10^7}{f_s \times \pi^2} \times \frac{P}{N}$$

$\therefore$ $$D = K \times \left(\frac{P}{N} \right)^{1/3}$$

where, $$K = \sqrt[3]{\frac{16 \times 3 \times 10^7}{f_s \, \pi^2}}$$

when $K = 84.71$, we obtain,

$$84.71 = \sqrt[3]{\frac{16 \times 3 \times 10^7}{f_s \, \pi^2}}$$

$$(84.71)^3 = \frac{16 \times 3 \times 10^7}{f_s \, \pi^2}$$

$$f_s = \frac{16 \times 3 \times 10^7}{(84.71)^3 \times \pi^2} = 80 \text{ N/mm}^2.$$

SAQ 2

(a) A solid shaft made of steel and of 2 m length is to transmit 50kW at 150 rpm. If the shear stress in the shaft material is not to exceed 50 MPa and maximum allowable twist in the shaft is 1°, calculate the shaft diameter. Take C = 80 GPa.

(b) What must be the length of a 5 mm diameter aluminium wire be so that it can be twisted through one complete revolution without exceeding a shearing stress of 42 N/mm^2. Take C = 2.7×10^4 N/mm^2.

(c) Find the power that can be transmitted by a shaft 60 mm diameter at 180 rpm, if the permissible shear stress is 85 N/mm^2.

6.4 SUMMARY

- A shaft of circular section is said to be in pure torsion when it is subjected to equal and opposite end couples whose axes coincide with the axis of the shaft.
- The product of the turning force and the distance between the point of application of the force and the axis of the shaft is known as torque, turning moment or twisting moment.
- When a shaft is subjected to a torque, then

$$\frac{q}{r} = \frac{f_s}{R} = \frac{C\theta}{l}$$

where, q = Intensity of shear stress on a layer, at a distance r from the centre of the shaft,

f_s = Intensity of shear stress on the outermost layer of the shaft, i.e. a distance R from the centre of the shaft,

C = Modulus of rigidity of the shaft material,

θ = Angle (in radius) through which the cross-section of the shaft has been twisted as a result of the torque, and

l = Length of the shaft.

- Polar moment of inertia, denoted by J, is the moment of inertia of a plane area with respect to an axis perpendicular to the plane of the figure, i.e. moment of inertia about ZZ axis. For shaft section, J is moment of inertia about the axis of the shaft.

$$J = \frac{\pi D^4}{32} \qquad \text{(For solid shaft)}$$

- The term $\frac{J}{R}$ is known as torsional section modulus or polar modulus, Z_p.

Thus, polar modulus for a solid shaft, $Z_p = \frac{\pi D^3}{16}$.

- Strength of the shaft is defined as the maximum torque or power the shaft can transmit from one pulley to another. It is also known as resisting torque or torsional moment of resistance.

For solid shaft, $T = \frac{\pi}{16} f_s D^3$

where, f_s = Maximum shear stress at the outermost layer.

- The torsion equation is

$$\frac{q}{r} = \frac{f_s}{R} = \frac{C\theta}{l} = \frac{T}{J}$$

- The torque required for unit twist $\left(\frac{T}{\theta}\right)$ is called the torsional stiffness of the shaft.
- Torque divided by the angle of twist per unit length $\frac{T}{\theta/L}$ is called the torsional rigidity. It is also equal to $C \times J$.
- The shearing strain is maximum (ϕ_{max}) on the surface of the shaft where $r = R$.

$$\phi_{max} = \frac{R\theta}{L}$$

- The shearing strain ϕ at any distance r from the axis of the shaft as

$$\phi = \frac{r}{R}\phi_{max}$$

- Power transmitted by the shaft is as follows :

$$P = \frac{2\pi NT}{4500} \text{ hp} \qquad \text{(in MKS Units)}$$

$$P = \frac{2\pi NT}{60} \text{ watts} \quad \text{(in SI Units)}$$

$$= T\,\omega \text{ watts}$$

where, $\omega = \frac{2\pi N}{60} =$ Angular displacement in radians/sec.

$N =$ Number of revolution per unit.

6.5 ANSWERS TO SAQs

SAQ 1

Torque based on shear stress,

$$T = \frac{\pi}{16} \times 42 \times (300)^3 = 222.7 \times 10^6 \text{ N mm}$$

Torque based on angle of twist,

$$T = \frac{795.2 \times 10^6 \times 84.4 \times 10^3 \times \frac{1.5\pi}{180}}{7.5 \times 10^3}$$

$$T = 234.6 \times 10^6 \text{ N mm}$$

The maximum torque that can be applied safely to the shaft is smaller of the above two values, i.e. 222.7×10^6 N mm.

SAQ 2

(a) We know, $P = \frac{2\pi NT}{60}$

$$50 \times 10^3 = \frac{2\pi \times 150 \times T}{60}$$

$T = 3.183 \times 10^3$ N m = 3.183×10^6 N mm

Diameter of the shaft based on its strength, i.e. stress,

$$\frac{3.183 \times 10^6}{\frac{\pi}{32} D^4} = \frac{50}{\left(\frac{D}{2}\right)}$$

D = 68.7 mm

Diameter of the shaft based on its stiffness, i.e. angle of twist,

$$\frac{3.183 \times 10^6}{\frac{\pi}{32} D^4} = \frac{80 \times 10^3 \times 1 \times \frac{\pi}{80}}{2000}$$

D = 82.55 mm

The required shaft diameter will be larger of the above two values, i.e. 82.55 mm.

(b) We have, $\frac{f_s}{R} = \frac{C\theta}{l}$

$$\frac{42}{2.5} = \frac{2.7 \times 10^4 \times 2\pi}{l}$$

$$l = \frac{2.7 \times 10^4 \times 2\pi}{42}$$

l = 10098 mm = 10.098 m

(c) We have, $T = \frac{\pi}{16} f_s D^3$ = 3605 N m

$$P = \frac{2\pi NT}{60} = \frac{2\pi \times 180 \times 3605}{60} = 67950 \text{ watts}$$

$P = 67.95$ kW