UNIT 3 EXPLORING MATHEMATICS

Structure **Page Nos.**

3.1 INTRODUCTION

In the previous unit we elaborated on important processes involved in mathematical thinking, namely, working in a world of abstract objects, specialising and generalising. In this unit we focus on the use of these processes and other processes involved in 'doing mathematics'.

We start with a section in which we discuss the different thought processes involved in exploring mathematical problems. In the next two sections, we carefully observe these processes through exploring some mathematical problems in geometry. Finally, we look at the use of mathematical puzzles for developing these processes.

While you are studying Sec. 3.3 and Sec. 3.4, we expect you to focus on the thought processes involved because these are the processes that your learners need to develop. Therefore, while studying this unit, keep thinking about how you can foster these processes in your learners' minds.

Objectives

After reading this unit, you should be able to

- explain the mathematical thinking involved in problem-solving, conjecturing and other mathematical explorations;
- suggest ways of generating mathematical thinking in your learners;
- design and carry out activities to help your learners investigate the polyhedra and tilings;
- create mathematical puzzles that challenge, but not over-challenge, your learners.

3.2 THE PROCESSES INVOLVED

Let us start this section with a brief look at what 'doing mathematics' means to most children and teachers. The common view is that mathematics is 'done' only in the 'maths class'. During this class, the children are expected to learn a concept by being given the definition and doing a few direct questions based on it. Then they are expected to solve word problems related to the concept. The procedure involved is that first the teacher explains 'the way' to solve a particular kind of problem. Following this, she gives the children many problems of the same type to solve on exactly the same lines. So, solving a problem is reduced to listening to the teacher, memorising certain solutions and mathematical facts and reproducing them appropriately. Any algorithm is treated in the same manner.

Where is the mathematical thinking being developed in the whole process outlined above? It is certainly important to have basic computational skills and definitions, but not by rote. It is important for a learner to understand the mathematics involved, even in an algorithm.

To understand any mathematical concept or process, a child needs to be introduced to it through familiar situations and experiences. In order to improve her understanding of the concept, it is important to use the concept on different occasions in as many ways as possible. The exposure to a variety of problems related to the concept helps her to deepen her understanding of the concept. For the child (or for us) this helps to interlink the different aspects of a concept, which, again, helps to strengthen conceptual understanding. When the child has to think about what to do and how to do it, she is forced to examine the concept seriously, and hence extend its meaning for her. For those whose concepts are half-formed or are erroneous, solving different problems gives an opportunity to discover the errors and to reach a better understanding of the concept. For example, when helping a child to develop the idea of a function, we need to give her an opportunity to identify functions from non-functions, use functions in various ways, allow her a chance to use a variety of functions, etc. In short, **concept formation is linked to the opportunities available to the learner to think, apply her understanding and use her conceptual structures in various ways, finding relationships with other concepts.**

While you're thinking about the point just made, try the following exercises.

E1) Give an example of a mathematics question given to children which does not require them to think mathematically. Also give your reasons for your choice.

E2) Explain, with examples from your own learning of mathematics, how solving problems helps in concept formation.

While trying these exercises, you must have focussed on the essential characteristics of doing mathematics — it must be an opportunity for the learner to think mathematically, choosing which step to take, based on what she knows and where she wants to reach. If she is expected to solve a problem, it should not require her to merely reproduce information or mechanically apply algorithms. She needs to, gradually, be exposed to more and more complex problems built around the concept. A major part of this process is the ability to build one or more representations of the problem that is being dealt with. We shall consider this, and other aspects in the next section.

3.3 SOLVING AND POSING PROBLEMS

Try and recall the last time you were required to do some mathematics — solve a problem based on data given to you, or asked to generalise some mathematical ideas. Did you start by making a representation of the data or of the concepts involved? It may have been a mental or a visual picture of 'skew lines', or of a large number. Once you had this picture, what did you do next? Did you try to relate it to the knowledge you already had and search for relevant pieces that could help in solving the problem? How different is this from the steps your students go through? You may be able to answer this while observing the stages you, and your students, go through when solving the following problem :

How many different ways are there for seating 8 persons at a round table?

See if your steps are similar to the steps I went through, which are:

1) I first drew a circle and made 8 points on its circumference.

2) Looking at this, I searched through my memory to think of what I knew related to this problem, for instance, the permutations (1, 2, ..., 8), (2, 3, ..., 8, 1), ..., (8, 1, 2, ..., 7) represent the same seating in this case.

3) Therefore, for solving the problem, I needed to find the number of distinct permutations, keeping Point (2) in mind.

4) To check my understanding, I tried it for 3 people, instead of 8.

5) I solved it, the answer being $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$.

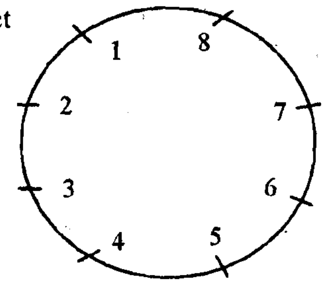


Fig. 1

Breaking up the stages of solving the problem is not very easy because many of these steps get merged and many stages are repeated again and again. In fact, the stages given above are exactly those given by the educationists **Davis and Mayer**. According to them, in order to solve any mathematical problem, we need to go through some or all of the following steps sequentially (perhaps many times).

1. **Build a representation** for the known mathematical information.
2. Use this representation, and **search in our memory** for knowledge that we consider relevant for solving the problem.
3. Apply the retrieved knowledge to the input data and **construct connections** between them.
4. **Check** these constructions to see if they seem to be reasonable and correct
5. **Use technical devices** such as procedures or algorithms (or other information associated with the knowledge representation in order to solve the problem.

As you can see, our ability to represent a problem situation is essential for solving it. In order to solve a problem, Step 2 would require "building a representation of what we consider relevant knowledge". This may be sometimes so quick that we fail to notice it. Children, however, are seen to do it much more often. For instance, consider the problem: "How many integers between 100 and 999 consist of distinct even digits?" You may see this almost immediately as "a problem of counting". But, when a child tries to solve it, there are two separate steps involved. Firstly, she needs to build a representation of the input data. Then she needs a representation of "possibly relevant knowledge", which would require her to put in some mental effort. She may try listing some of the numbers, and then pick out the required ones to construct a numerical representation. Then she may use her earlier knowledge for the single-digit case and the 2-digit case to find a pattern. If she finds a pattern, she may generalise it to find the answer.

As your learner does more problems in different contexts, these procedures become a part of her thinking and turn into an instantaneous strategy. However, we must remember that the learner needs opportunities and encouragement to **tackle open-ended problems and problems with many solutions related to the concept.** The problems can steadily become more challenging. At each stage, **she should also be encouraged to talk about what she is doing** and explain her line of reasoning.

Let's see the abilities developed in the process, by asking your students to do some problem-solving.

E3) Give your students problems like the following one to do.

A company makes 100 computers every month. Its employee union accused the company of discriminating against its female employees. The union said that women were not being given the promotions due to them. The following table gives the data about the promotions in the company.

Year	*No. of women promoted*	*No. of men promoted*
1996	*5*	*15*
1997	*6*	*16*
1998	*10*	*8*
1999	*8*	*10*
2000	*8*	*10*

If an employee who is promoted during these five years is selected at random, what is the probability that the employee is a woman? Is this data enough for deciding whether female employees are discriminated against?

While they are working on the problems, talk to them to try and separate out the various thought processes they are using in the process. Also note down the stages you went through while solving the problem.

When I tried the problem given above, I first tried to understand the situation — **what I knew**, and **what I needed to find out**. Then I needed to think of **the path to use** to move from what I knew to what I needed to find out.

I also needed to know **which information, if any, was extra and not required**. For instance, what the company produced is irrelevant to the problem.

The next step was to write down the mathematical equivalent of the given problem :

Total number of women promoted from 1996 to 2000 = 37
Total number of people promoted in this period = 37 + 59 = 96
To find P(A), where A is the event that a woman was promoted.

Then I solved this problem using the definition of probability of an event, that is,

$$P(A) = \frac{37}{96}.$$

So, I concluded that approximately 1 in 3 promotions is likely to be that of a female worker. However, this probability gives us no indication of whether women workers are discriminated against. This is because we need some more information. For instance, we need to know how many men and women were eligible for promotion in this period.

Should we note down the steps involved in solving this problem?

1. Read the problem carefully to understand what it says — the information and assumptions in it, and what is to be found out, proved or examined.

2. Represent it mathematically, clearly filtering out the irrelevant data in the problem.

3. Gather other relevant information, axioms and earlier proved (or known) results.

4. Look for a path for solving the mathematical equivalent of the problem.

5. Interpret the solution in the problem situation.

These steps may appear to be different from the stages given by Davis and Mayer. But, if you look carefully you'll find some of those stages clubbed in the broader stages we have just listed.

Why don't you do an exercise related to this?

E4) Give some children a problem to do. After they have solved it, talk to them to find out which of the stages above they went through. Note down what they articulate. If you can get them to discuss the stages, note down what comes out in their discussion.

Problem-solving is one important part of doing mathematics. An equally important part is what further questions open up in our minds while solving a problem, that is, **posing a problem**. This requires us to use our abilities to generalise in many ways. For example, if I have proved that there are infinitely many primes, I may wonder if there are infinitely many primes of the kind 4m+3, where $m \in \mathbf{N}$. This process can continue for as long as our mathematical maturity and intuition allows us to. And, each time we pose a problem and try and solve it, we grow mathematically.

Our level of problem-solving and problem-posing reflects our level of mathematical thinking. So does our ability to use a variety of representations while dealing with problems.

Being flexible in moving across representations is a sign of competent mathematical thinking. Each type of representation brings out specific aspects of a concept. Flexibility could mean moving within one type of representation, for example, using one diagram with many different parts that highlight different aspects of a problem. Flexibility also involves moving between quite different representations, for example, between an equation and a graph. Solving multi-stage problems may need the use of several representations.

In fact, we need to help our learners develop such a flexibility. They can have many different ways to represent the abstract concepts which they are in the process of learning. The representations can be in terms of known symbols, icons or concrete objects. Think about the various ways your students use to represent problems while trying the following exercise.

E5) While your students were doing the problem in E4 above, what were the various ways of representation they used?

Let us now gather **the implications** of what we have just discussed in this section **for anyone teaching mathematics**. A learner uses a variety of representations while trying to solve problems, particularly to relate it to the knowledge in her mind. The availability of these representations allows her to refer back and forth to her knowledge system. Further, if a child is given many different kinds of problems around a concept or process, she would be able to develop better ways of constructing representations and relating the concept to the knowledge base in her mind.

Throughout the process of problem-solving, the teacher needs to give **the student several opportunities to explain the process she has followed.** This would help her consolidate the strategies she has used. Given sufficient opportunity to deal with different kinds of problems and to articulate the strategies developed **without fear of ridicule** would help the child to develop more sophisticated problem representations. If the child has to learn to solve problems, this development is of great importance.

You may have noted that the use of procedures, algorithms and shortcuts are **only one step in solving a mathematical problem.** The first four steps where the problem is comprehended, represented suitably, related to the knowledge available and checked as being reasonable are extremely necessary before choosing an algorithm or procedure and applying it. Therefore, we need to give the student many tasks requiring her to build her ability to move flexibly across using various modes of representation. And, **we must not just give her one particular procedure for solving a type of problem.**

While a child is solving problems, she also needs to be encouraged to explore further generalisations. Here the teacher could be a facilitator, suggesting certain conjectures, to start her off. The child should be given many opportunities, maybe prodded several times too, to think about and articulate more problems—some could be of the same kind, and some could be of the kind 'What if ...?'.

Now for an exercise!

E6) Give a detailed account of the teaching strategy you would use to develop the ability of children of Class 11 for using various representations for dealing with sets.

To get more of an insight into the processes we have discussed in this section, here is an opportunity for you to investigate some mathematical areas.

3.4 INVESTIGATING PLATONIC SOLIDS

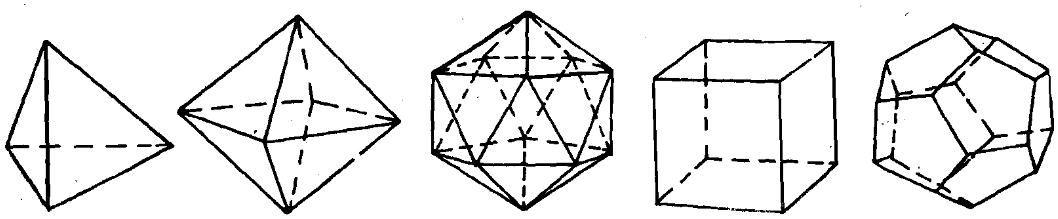
In this section we invite you to explore the processes involved in working mathematically, through a study of polygons and polyhedra. So, let's start with an exercise.

E7) How many different regular polygons are there? How are these polygons related to a circle? Note down other questions that come to your mind while you are working on these questions.

While you were doing E7, what did you notice about the way you deal with mathematical problems? Once you have solved it, do you think your understanding of 'polygon' has improved? In what way? Did you think about other related mathematics questions that could be explored? One problem that you may have thought of exploring could be : Can what I have found true for 2D be generalised to 3D? **(problem-posing)**

When we go from 2-dimensional figures to 3-dimensional objects, the concept of regular polygons generalises to regular polyhedra (the plural of polyhedron). **Regular polyhedra** are solids in which all angles and all sides are equal, for example a cube.

Now, while doing E7 you must have found that there are infinitely many regular polygons because there is no limit to the number of sides they can have. So, you may expect the same about the regular polyhedra. However, there are **only five different regular polyhedra possible**. These are the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron (see Fig.2). These are also known as the **Platonic solids**, after the Greek philosopher Plato (see Fig. 3). They have fascinated mathematicians from the time of the ancient Greeks. The faces of the tetrahedron (4 faces, from the Greek word 'tetra', meaning four), the octahedron (8 faces, from 'okto' meaning eight) and the icosahedron (20 faces, from 'eikosi' meaning twenty) are all equilateral triangles. As you know, the cube has 6 faces, all of which are squares. The 12 faces of the dodecahedron ('dodeka' meaning twelve) are regular pentagons. It is worth noticing that the faces of all the regular polyhedra are regular polygons. (Why?)



Tetrahedron **Octahedron** **Icosahedron** **Cube** **Dodecahedron**

Fig. 2: The five regular polyhedra (the hidden edges are indicated by dashed lines)

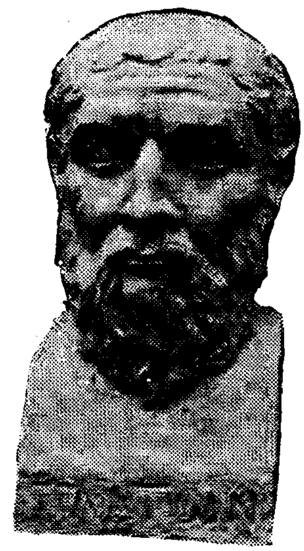


Fig.3 : Bust of Plato (423-347 BC)

What we have just discussed is easy for your students to get interested in. To give them a feel for what the five Platonic solids actually look like, there is nothing better than having models of these solids for them to play around with. As solid models are not easy to come by, it is a good idea to get your students to make models of these solids from paper. With this in mind, in Fig.4 we have given flat diagrams of the five regular polyhedra.

These could be copied on to some stiff paper and cut out along the outer edges. The cut-outs can then be folded along the inner lines and the sides pasted with thin strips of paper to make three-dimensional models. Those corners of the regular polygons that make up the faces of the models and which meet at a common vertex of the polyhedron are labelled with the same letter in our figure.

Now, getting back to exploring mathematics, here is an exercise for you.

E8) Prove that there can only be 5 regular polyhedra. Also ask your students to prove it. The models may come in useful for this.

How did you go about answering E8? Of course, you know that there are **at least** 5 regular polyhedra, the ones made in Fig.2. But, how do you know that **these are the only ones**? That is, how did you go about proving that any regular polyhedron is forced to be one of the five you know? Did you try to use anything you already know or have observed? For instance, did you notice that **at any vertex of a polyhedron there cannot be less than three faces**? One face is clearly not enough and two would only give rise to an edge.

Next, what do you know about the sum of the angles of all the faces at each vertex? Remember that each face has to be a regular polygon. Also, if you 'open up' the polyhedron, place all the adjacent faces in a plane, there need to be some gaps between the edges. So, shouldn't the sum of the angles be less than 360°? If the sum were exactly 360°, then all the faces would lie in one plane and there would be no corner of a 3D solid.

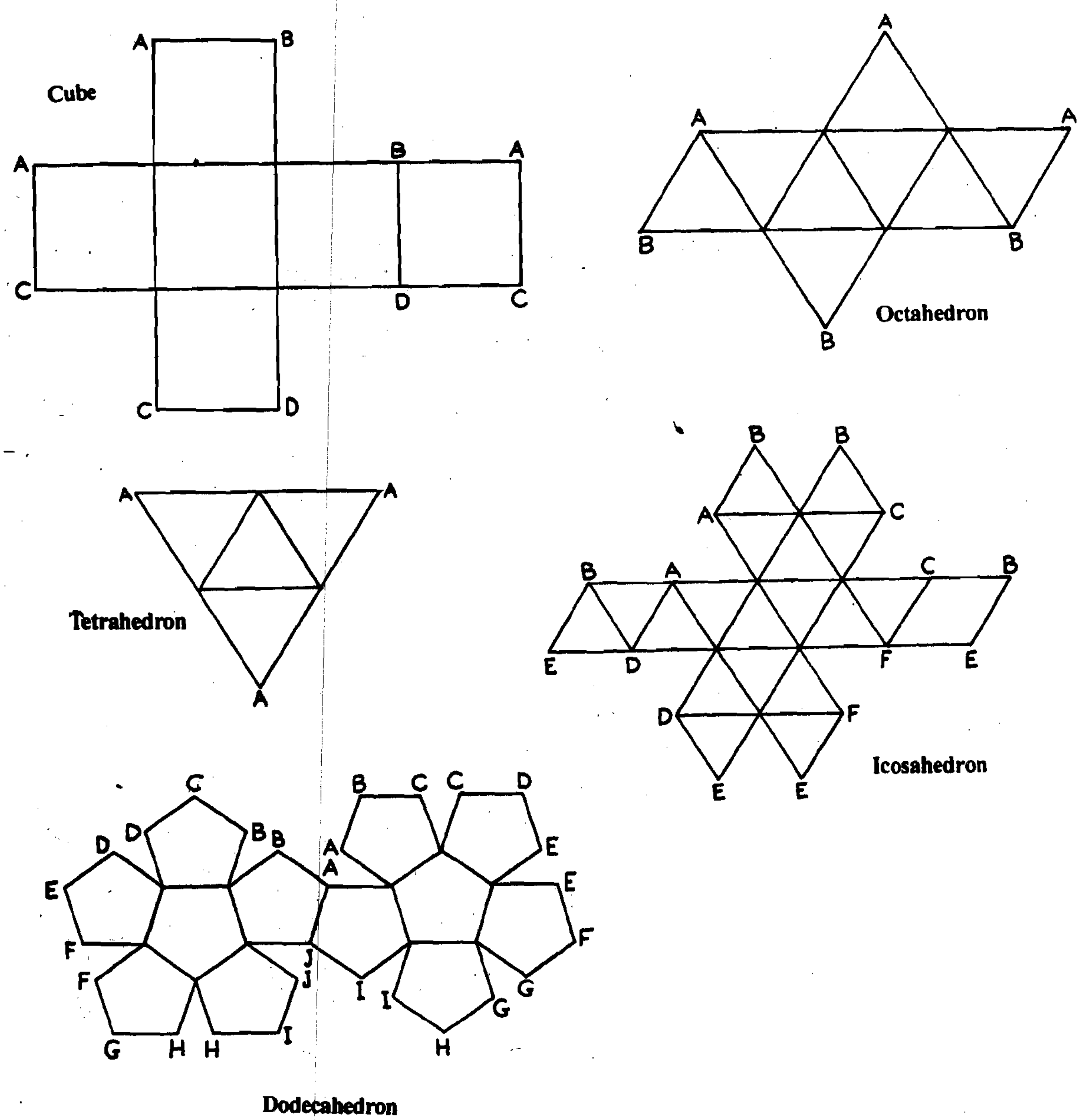


Fig. 4: Cut-outs to make models of the five Platonic solids

Now, given the two facts we have just noted, can **any** regular polygon be a face of a polyhedron? Since the sum of the angles of the faces at each vertex of the polyhedron has to be less than 360°, and since there must be at least three faces at each vertex, the angle of a face at the vertex must be less than 120°. This immediately restricts the regular polygons that can form faces of the regular polyhedra to be equilateral triangles, squares or regular pentagons (see E9 in Unit 2). Therefore, a regular polyhedron can only have these 3 types of polygons as faces.

Once you reached this stage, you probably thought about the various possibilities for the regular polyhedra. The simplest case is that of a regular polyhedron whose faces are equilateral triangles. We have already used the fact that the number of faces at each vertex must be more than two. They must also be less than 6, since each angle of the face is equal to 60°. The number of faces at each vertex of a regular polyhedron whose faces are equilateral triangles can therefore only be 3, 4 or 5. These correspond to the regular tetrahedron, octahedron and icosahedron, respectively.

Now consider the case of regular polyhedra whose faces are squares. The number of faces at each vertex can only be 3. (Why?) The corresponding solid is, of course, the cube.

By exactly the same arguments the number of faces at each vertex of a regular polyhedron whose faces are regular pentagons can only be 3. The corresponding solid is the regular dodecahedron.

You should check your proof to see that you have not made any logical errors, and there are no other possibilities. Once this is done, you have proved that there can only be 5 kinds of regular polyhedra.

Now try these exercises.

E9) Go back to the discussion on ' proof ' in Unit 2 (following E10). Then, note down the mathematical thought processes and the kinds of statements used in the proof above. Under which of the four stages of a proof listed in Unit 2 do they come? Are there any other stages or categories in the proof above that are not mentioned in Unit 2?

E10) We list some of the properties of the five regular solids in the table below. Ask your students to use the paper models to verify the entries in the table for each of the five regular solids.

Table 1: Properties of the regular polyhedra

Type of polyhedron	Faces are n-gons n	Number of faces F	Number of vertices V	Number of edges E	Number of faces at each vertex
Tetrahedron	3	4	4	6	3
Cube	4	6	8	12	3
Octahedron	3	8	6	12	4
Dodecahedron	5	12	20	30	3
Icosahedron	3	20	12	30	5

Then ask them if they see a relationship between F, V and E, and if so, to find it.

Let us now explore another area of spatial mathematics. While you are investigating it, keep thinking about the same broad questions that you kept in mind in the previous section.

3.5 STUDYING TILINGS

'Tiling' is the study of shapes that can be placed alongside each other to fill space completely **without leaving any gaps**, like the tiles covering your floor. If you look around you, you will see a variety of tilings — on floors, on walls, decoration pieces, etc. An example is given in Fig.5.

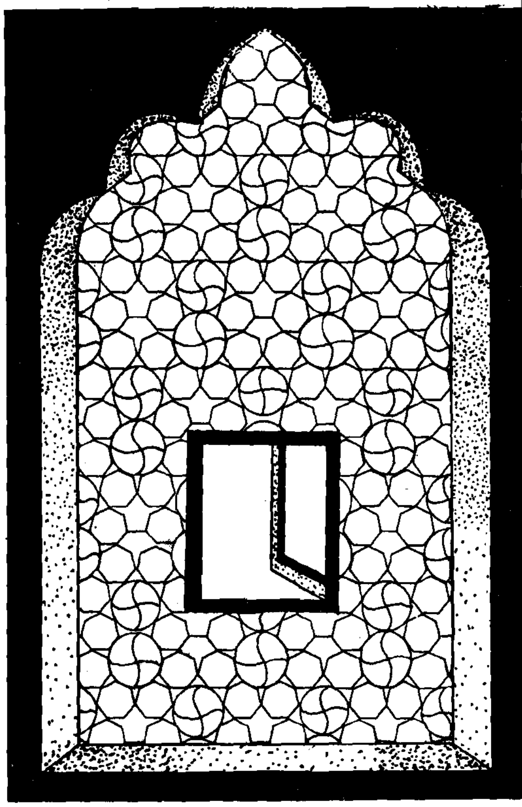


Fig. 5

What are the shapes that are usually used as tiles to fill the tilings? In two dimensions, we usually find squares or rectangles used as tiles. If a tiling is done by one kind of regular polygon of the same shape and size, it is called a **regular tiling**. Do you see regular tilings around you? The most common kind is the one made by squares.

What are the other kinds possible? Here is an exercise about this now.

E11) Prove that the only regular tilings are those made up of the equilateral triangle, the square and the regular hexagon. Further, note down the points you reflect on, the questions you ask yourself and the different routes you may follow while finding the proof.

How did you go about trying the exercise above? Did you physically take several equilateral triangles, say, and try to cover a surface with them? While doing so, did you notice that at any intersection in a regular tiling there must always be more than two tiles meeting? This concrete example may have also helped you realise that the sum of the angles of all the vertices meeting at an edge must be 180°. The sum of the vertices of the regular polygons meeting at other points will be 360°. This means that we can only have three equilateral triangles (or two squares) meeting at an edge. Also, we can have 6 equilateral triangles, 4 squares or 3 regular hexagons meeting vertex to vertex. This exhausts all possibilities for regular tilings. Therefore, there are only three regular tilings, all of which are shown in Fig.6.

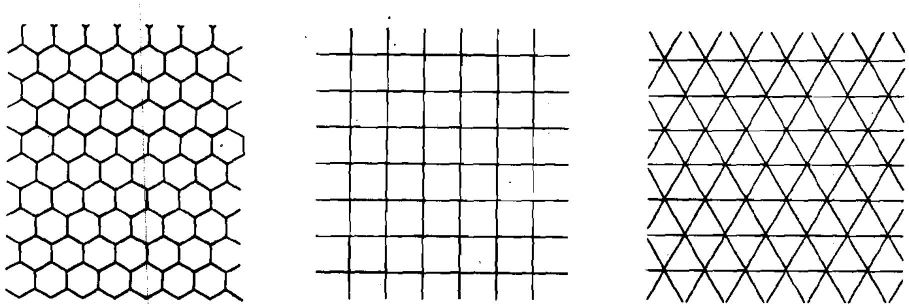


Fig. 6 : The regular tilings

If a mix of non-regular polygons are used in any tiling, then of course the possibilities are endless. The same is true if a mix of regular polygons of different sizes are used. In particular, the plane can be tiled completely by triangles or quadrilaterals of arbitrary shape.

A **tessellation** is another name for a tiling, which is used by artists more than mathematicians. Tessellations use either a single shape which may or may not be regular, or at most a few shapes, to cover the plane. The emphasis is on using shapes which look natural like birds, fish, horses, people, etc.. Through the following activity, you and your students can pick up some basic principles involved in creating tessellations, and make some of your own tilings.

Activity 1 (Making tessellations) : You need to start by establishing a regular grid on the plane. You can use triangles, squares, rectangles, parallelograms, hexagons, etc., to create a grid which covers the whole area you wish to work with. Suppose you start with a grid of squares. You can choose as your unit a 3×3 square. We know that periodic repetitions of this unit will tile the plane. (Why?)

Now, the secret of a tessellation is to remove parts of this square from one side and add it in a corresponding position on the opposite side of the square. In this way, although the shape of the unit changes, its total area remains the same. In the process you create cuts and wedges that fit into each other. (Why does this happen?)

So, suppose you remove a small square from the top left-hand corner of the unit figure (see Fig. 7(a)) and add it to the top right-hand corner. Similarly, remove another small square from the middle of the bottom of the figure and add it to the middle of the top. This, then, produces your basic motif shown on the left-hand side of Fig. 7(b).

Consider your original grid to be tiled by a set of the basic 3×3 squares and replace each such square by the motif you have just created. This will produce the pattern shown on the right of Fig. 7(b). Stretch your imagination a little, and you can consider this to be a tessellation of a horse and rider!

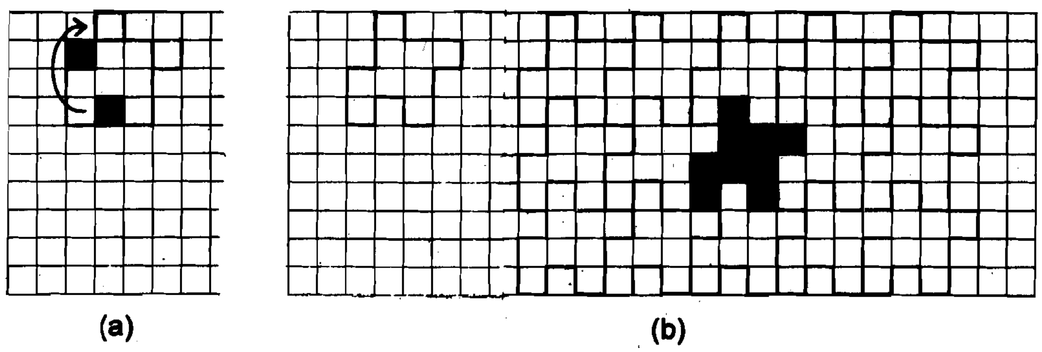


Fig. 7 : (a) Creating the motif.
(b) Tesselleting the plane with the motif to get a tessellation of a horse and rider.

To make a tessellation, we can add and remove any shape from the basic unit we choose. For example, starting with the same basic 3×3 tile, we can add/remove shapes as shown in Fig. 8(a). Then we get a basic motif that gives us the tessellation in Fig. 8(b).

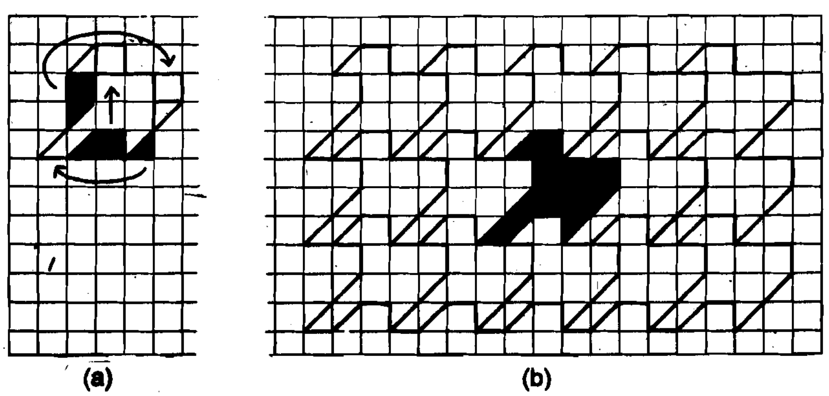


Fig. 8 : (a) A basic motif for a tessellation, (b) The tessellation of horses

The important thing to remember about creating a motif is

i) decide on the grid and the basic unit,

ii) you can remove any shape from the basic unit **provided** you add it back to the unit at the corresponding place on the opposite side to give rise to the new shape.

This can be done as many times as you please. The skill lies in creating a natural looking shape at the end. For example, in Fig.9 we show how, starting from a grid of parallelograms, you can proceed step by step to create a tessellation of roosting birds

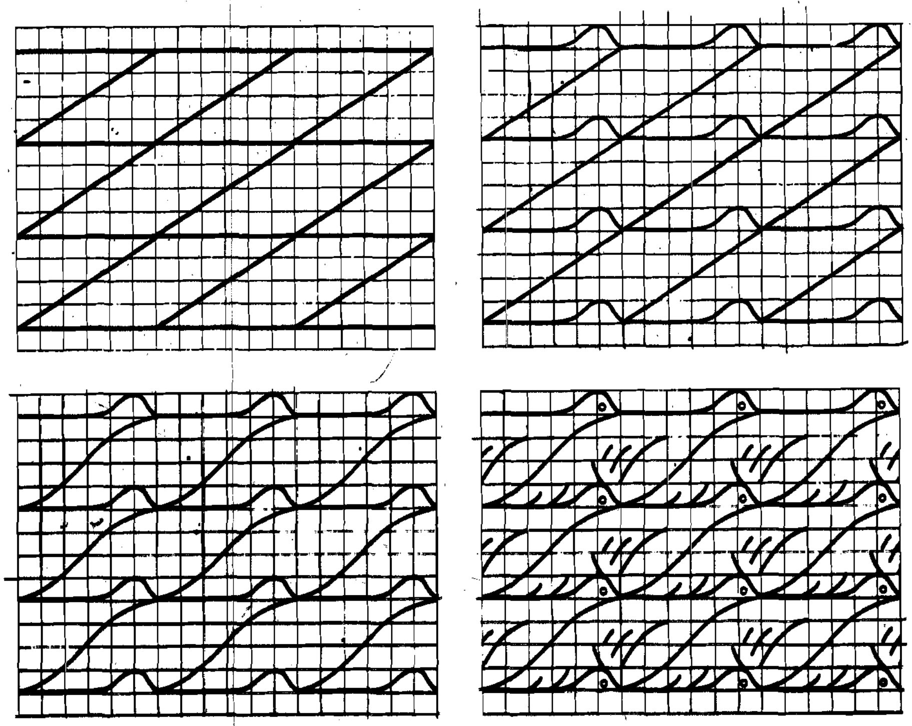


Fig. 9 : A tessellation of roosting birds

Why don't you try some exercises now?

E12) Create at least two tessellations using the steps we have just discussed. Also try out what we have said in this section with your students. What were their reactions?

E13) Ask children of Classes 9 or 10 to tile a plane with squares and regular pentagons, respectively. Note down the discussions that take place amongst them during this activity. What understanding does this give you of their mathematical thought processes?

The tessellations that we have considered so far make repeated use of just one basic form. This need not be so. We can always take our repeating motif and divide it into two parts such that each part looks like a separate natural shape. Escher, who was an acknowledged master, has used many basic shapes to give all kinds of tessellations of the plane (see Fig.10).

While creating tessellations, there is a notion of symmetry that is involved. We shall study this notion in detail in the next unit.

So far, in this unit we have carefully looked at the thought processes we go through while learning mathematics. Both the mathematical areas we have looked at in detail in this unit have been spatial, that is, space related. Now, let us look at the processes involved in solving algebraic and logical puzzles.

REGULAR DIVISION OF THE PLANE I (DETAIL), 1957; WOODCUT, 9½ x 7⅛ IN.

Fig. 10 : A tessellation by M.C.Escher

3.6 WORKING OUT PUZZLES

If you are given a problem like 12345 – 3249, you are likely to do it in a flash. This is because you have acquired the ability to apply a subtraction algorithm. Now, see how long you take to do the following subtraction :

In the problem below, each letter has been assigned one digit from 0 to 9. Find the numbers involved in the subtraction

$$\begin{array}{r} ABCD \\ -\ \underline{EBEB} \\ EDEB \end{array}$$

How have you gone about finding the digits involved? For instance, you may start with the possibility that D = 6, B = 3. Then D–B = B. But then, D and B are occurring again in the 'hundreds' column. And $3 - 3 \neq 6$. So, you would need to try another possibility for D and B. In this way, using logical arguments, what solution did you get? Note that there may be more than one solution to this problem.

Try these exercises now.

E14) While solving the problem above, what were the different aspects of mathematical thinking you applied?

E15) Find the operations K and K′ and the K digits represented by the letters in

ABKAB = ACC and FG K′FH = DE

E16) Try the following problems. Also give your students these problems to do. What problem-solving abilities were the children using in the process, and how did you find out? How different were they from the processes you used for solving them?

i) *There is a sequence of 16 numbers which reads the same from left to right as well as from right to left. Also, the sum of any 7 consecutive numbers in the sequence is –1, and the sum of any 11 consecutive terms is +1. Find the numbers.*

ii) *Ashrafi was convinced that her key had been hidden by one of her friends —Aarti, Birla, Kalyan or Megha. Each of these friends made a statement about this matter. But only one of these four statements was true.*

Aarti said, "I didn't take it."
Birla said, "Aarti is lying."
Kalyan said, "Birla is lying."
Megha said, "Birla has taken it."

Who told the truth?

E17) Ask your learners to think of more problems like the ones mentioned above. What were the puzzles/problems they came out with?

The purpose of asking you to engage with the problems above was two-fold. Firstly, we wanted you to have fun. We also wanted to help you focus on the processes that are used for solving them. If you are aware of these abilities being used, then you would agree that these are the abilities to be fostered in your learners. One way is to give them problems that they would enjoy and that would challenge them a bit. We end this unit with leaving you to think of various ways in which this can be done.

But first, let us see what we have covered in this unit.

3.7 SUMMARY

In this unit we have focussed on the thought processes involved in learning and doing mathematics, particularly while solving and creating problems. More specifically, we covered the following points.

1. Exploring any mathematical concept involves considering it in different ways. Solving a variety of problems related to this concept helps to build and consolidate one's understanding of the concept.

2. We stressed the importance of using a variety of ways for representing a problem situation. Developing mathematical maturity requires the learner to, among other things, move comfortably from one form of representation to another.

3. We looked at why there are only 5 types of regular polyhedra though there are infinitely many regular polygons.

4. We discussed what a tiling is, how many regular tilings there can be and how to create tessellations.

5. The point of studying polyhedra and tilings was to consider the processes involved in investigating mathematics.

6. We looked at interesting non-routine mathematics problems that entertain us and keep the brain ticking. The idea was to focus on the directions in which the thought processes were moving.

7. We asked you to work with your learners on the same lines and analyse their reactions. Through this, you would be able to gauge their use and understanding of mathematical thought processes.

3.8 COMMENTS ON EXERCISES

E1) While thinking of an example, remember the essence of a problem — it should force the learner to think mathematically. This does not mean merely retrieving learnt facts or applying algorithms unthinkingly.

E2) Pick an example of a concept or process you learnt, for example, the concept of limit. Note down what kind of exercises, activities and problems helped you to, develop your understanding of the concept, and in what way. Does 'limit' mean more to you than merely applying the algorithm for finding it? What visual aid did you need to understand when a limit exists, or otherwise? Which practical situations require you to use it? Did finding the answers to these questions help you understand this concept better? In what way?

Similarly, how would solving real-life and other problems related to a concept help your students to learn the concept?

E3) See material following the exercise.

E4) Pick up a problem that requires plenty of thinking. Do not hassle the children when you ask them how they have done the problem. They may not be very clear in remembering or explaining how they solved it. You could also have them sit in a group and try to work out a solution together. Their conversation while they think about the question would help you to understand the processes that they are going through.

If you discuss the stages with them, be sure to use simple language and small logical steps so that they can understand what you are talking about.

E5) Was their ability to use a variety of representations linked with their ability to articulate their thought processes? If so, in what way? What was the relationship, if any, between the child's ability to build representations and being comfortable with mathematics? Note down the other points that you find important.

E6) Here the focus is on helping children develop their ability to use a variety of representations. Accordingly they would need several opportunities to use the concept in different situations. Note down the kind of learning opportunities you can think of for these children.

E7) For each $n \geq 3$, we can define a regular n-gon. Draw them and see what happens as n becomes larger and larger. As $n \to \infty$, the n-gon tends to a circle.

What questions regarding relationships between different mathematical objects, pattern finding and generalisation did you think of in the process?

E8) Think of what a proof involves — first gather together what is known and what is assumed. Then see how you can use this to prove your result. The discussion following E8 will, of course, lead you there.

It may be useful, while you think, to try to actually make these polyhedra and see the implications of this concrete activity.

Ask your students to study these models and try and prove the statement. Were they able to do it? What did they say about this exercise, while discussing amongst themselves as well as with you? Which aspects of mathematical thinking were coming out through their remarks?

E9) For instance, there have to be 3 or more faces at any vertex of a polyhedron. Is this statement an axiom? Or is it based on assumptions? This statement follows logically from the definition.

'Each face has to be a regular polygon' also follows from the definition.

In this way, consider all the other steps in the proof.

E10) The number of faces F, the number of vertices V and the number of edges E of a regular polyhedron are connected by Euler's famous formula $F + V - E = 2$.

E11) One route is given in the discussion following E11. Think of other routes. Compare the thought processes and steps in the different solutions. In fact, think of all the regular tilings. Is the list very long? Try covering a book (any surface) with all these tilings one by one. Did you find a problem with some of them? While doing so, remember that you cannot change a shape of the tile in between.

Is the statement true for tilings which are not regular?

E13) Divide the children into groups of 6-8, depending upon the space available and the number of children. Explain to them what tiling means and let them proceed with the tiling exercises. Do not interfere in their thinking. Observe the discussions among them as they do this activity.

Analyse the discussions for their notions regarding symmetry, angles, vertex, etc. What other mathematical thought process can you study in this exercise?

E14) You probably first assembled various single-digit subtraction facts. Then, from them you chose the ones that may fit. Then, moving step by step, you would eliminate the non-possibilities, based on contradictions you got.

One solution is (A, B, C, D, E) = (2, 5, 3, 0, 1).

E15) Consider the first problem. Note down why the operation can't be subtraction. If it is addition, what value of A would give you A in the hundreds place in the answer?

If the operation is multiplication, what could A, B and C be? One solution is A = 1, B = 2, C = 4. Think of others.

You can try the second problem similarly.

E16) i) How are you going about this one? I started by trying out the sequence

$$1, -1, 1, -1, \ldots$$

This met the second condition, but not the first or the third. In this way, I tried a few more sequences till I decided to use algebra for dealing with this problem. So, using the first condition, my sequence became

$$a, b, c, d, e, f, g, h, h, g, f, e, d, c, b, a.$$

Then I used the second and third conditions to reduce the sequence to

$$a, a, c, a, a, a, c, a, a, c, a, a, a, c, a, a$$

Now, can you guess how I got a and c? Why don't you try and find the sequence? Maybe your solution agrees with mine. An answer I got was $a = 5$, $b = -13$. Are there any other possibilities?

What were the mental processes the children went through while reaching a solution?

ii) This problem can be solved in various ways. Of course, each way requires the use of mathematical logic.

So, let me begin by assuming that Aarti is telling the truth. Then Birla's statement is false, so that Kalyan's statement is true. But we have assumed that both Aarti and Kalyan can't give true statements. So, Aarti must be lying.

Now, let me assume that Birla is telling the truth. See if you find any contradictions with this assumption.

In this way, checking the various possibilities, moving logically step by step, I arrived at the solution. Can you see the mathematical thinking involved in this problem?

E17) Did your learners come out with other kinds of conditions to determine a sequence? Did they come out with minor generalisations, or radically different conditions? What kind of other problems like E16(ii) did they create? Did you ask other students to solve them to see if the newly posed problems made sense? What was the general reaction in the classroom to this exercise?