UNIT 2 THE ESSENCE OF MATHEMATICS

Structure **Page Nos.**

2.1 INTRODUCTION

In the previous unit we have discussed ways in which we can communicate better with our learners. Now, what do we want to communicate to them? I can hear you say, "Obviously, mathematics!" But, the point is what is mathematics? And, what aspects of it do we want our learners to learn?

In this unit we try to understand this. To do so, we focus on the most important part of mathematics, which is the way it is created and it grows. That is, we explain what mathematical reasoning involves, particularly in the context of what abilities, therefore, need to be stressed while it to the children.

What we find in the classrooms nowadays is that mathematics is reduced to calculations and applying algorithms mindlessly. The procedure in the algorithms has become routine without the underlying logic being known or understood by many of us. Therefore, our students see an algorithm as a set of rules to be followed mechanically. Through this unit, we aim to encourage you to think about the situation, and what aspects of mathematics really need to be stressed when teaching any topic.

We start with bringing out the essence of the abstraction of the objects and relationships in the world of mathematics. Then we talk of the most important processes involved in understanding and developing this world, namely, generalising from observing particular instances and particularising from general statements and conditions. Next, we discuss what a proof is and why it is so important in the context of mathematics.

And finally, we bring you a detailed example to show you that there is nothing sacrosanct or final about mathematics, as we know it. This world is based on many axioms, and any growth needs to be consistent with these axioms. So, if we remove or change some axioms, we get a new mathematical theory. The example we discuss is of some non-Euclidean geometries.

This unit is closely linked with the next one, and with your teaching practice, in general. Therefore, please ensure that you have achieved the following objectives before going to the next block.

Objectives

After studying this unit, you should be able to

- explain in what way 'thinking mathematically' requires dealing with abstraction;

- explain how the processes of particularisation and generalisation are essential for doing mathematics;
- describe what a mathematical proof is and its importance for mathematics;
- identify the thought processes that need to be developed in children when teaching them mathematics;
- help your students realise how any mathematical theory depends completely on the axioms that it assumes.

2.2 ABSTRACTION

The other day a teacher trainer, Aruna, was speaking to a group of school teachers who had gathered for a training session. They got talking about their views on mathematics teaching. Many of the teachers were complaining that the children don't bother learning mathematics. After a bit, Aruna asked them what mathematics meant to each of them. Let me ask you the same.

? What do you think mathematics is?

Do you agree with most of the teachers present that day, who said that mathematics was about numbers and calculations? If so, then does geometry and the study of form and space fit this definition? Aruna asked the teachers this too. So, after a bit of collective thinking, out came a 'definition' that 'mathematics is the study of numbers and space'.

Here Aruna changed track a bit, asking them what numbers are. Again, a lot more discussion took place in which comments like, "Numbers are ... um... for example, 5 is 5 people, 5 chairs, etc" were coming out.

Aruna: So, what you are saying is that all these things have a common property of how many of them there are. You call that 5.
A teacher: Yes, and all numbers are like that.
Aruna: How would you then explain –5?
Another teacher: It would be 5 things missing. For example, if we owe someone 5 rupees.

The conversation continued in this manner for a bit, and then Aruna reminded the teachers of her original question. Everyone went back to a major discussion with each other, and with Aruna. Sometimes she caught a word here, and a word there, and wrote it on the board. After 10 minutes, she had 'logic', 'measurement', 'geometry', 'definitions' on the board, and she asked the group as a whole to concentrate on these terms and her question, and give a final answer. At this point some teachers asked her to give the definition, which she demurred from, wanting them to do it.

Finally, with some prodding, thinking and hints, the following understanding of mathematics was accepted by all.

Mathematics consists of theories that have as a basis some undefined terms, and a self-consistent set of unproved statements (called ***axioms*** *or* ***postulates****) about these terms. Once these terms and postulates are laid down the theory develops further by proving any proposition from these postulates according to clearly stated principles of logic. Also, every term of the theory is definable in terms of the given undefined terms.*

Aruna went on to give them an example — showing them how Peano's axiom system is the postulate system that the entire arithmetic of natural numbers is based on.

Let us now go back to an earlier part of the interaction regarding our understanding of number, and look closely at what it tells us about our mathematical thinking.

What is interesting in this part of the conversation is how clear it is that **we abstract the notion of numbers from using them as adjectives**. When we talk of a number, we are essentially referring to a certain physical property of a set. Thus, when we talk of the number 'two', we could be referring to any collection of objects that can be put in one-to-one correspondence with, for example, the number of sleeves in a shirt. Thus, we say that a coin has *two* sides (each side corresponding to one sleeve), most humans have *two* eyes, a line segment has *two* end points, and there are (usually!) *two* sides to an argument. We abstract a common property of these different concrete objects, namely the number of objects in each of them. This is the number that we call 'two'. As in the previous case, having abstracted the property and understanding what 'two' means, we can now think of the number two without referring to the objects from which we derived the concept. It also has completely abstract and formal relationships with other numbers like 6, $\sqrt{2}$, 2i, etc., and with other abstract mathematical objects (e.g., rectangles).

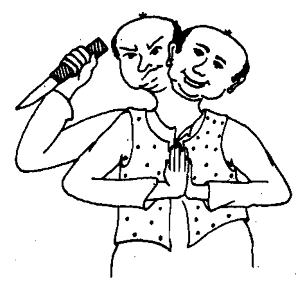


Fig.1 : Do all of us have two faces?

So, **abstracting a concept** is the ability to look at several particular examples of the concept, find what is common to them, separate that common property from the objects and look at the property as something on its own, having an independent existence. This existence is in the world of mathematics. This world is made up of such abstract objects. These objects generate further abstract concepts and relations between such objects.

We acquire our understanding of these abstract objects in two ways. One way is the way we develop our concept of number or of shape. This consists of a process of careful observation and analysis of different objects, noticing a certain property common to these objects and separating the property from the objects from which it was abstracted. This property, then, becomes an object of study as a concept. This is true of several non-mathematical concepts (like colour) too. In the following exercise we ask you to mull over this process.

E1) Identify two other concepts in mathematics and two from non-mathematical areas that arise through a process of abstraction. Explain how this abstraction takes place.

As we have just seen, several mathematical and other concepts are derived by abstracting them from particular instances. Would you be able to abstract the notion of a point or a line by this process? To answer this, let us first consider a point. In school, we are told that a point marks a position in space and that it is dimensionless. How, then, do we represent a point? Even the tiniest dot in space has some dimension. So, we can't abstract the concept from particular concrete instances of the concept, because ideally there cannot be any concrete representation of a point. There is no easy way out of this difficulty. The way out for mathematicians was to adopt the convention that a small dot would represent a point. Thus, on paper we often mark points like the origin O, while in our minds we know that a point cannot exist in reality. It is an abstract entity present only in our minds.

Similar situations arise with many other geometrical concepts as well, such as a line, a segment, or a ray. All these abstract concepts exist because of certain accepted rules and conventions in the world of mathematics. These rules are called **axioms**. And, to be able to deal with such abstract concepts, we choose conventions for representing them symbolically. Once we define one convention, we use it to define conventions for the other objects that exist only in our minds. This is another kind of abstraction. It is by this other form of abstraction that Euclid stated that "a point moves to describe a line". This line, an abstraction itself, moves to generate a surface, and so on.

The essence of mathematics lies in dealing with these forms of abstraction. In the next few sections we shall talk about what we mean by 'dealing'. For now, try this exercise.

E2) Explain what the difference is in the two forms of abstraction we have just discussed, with examples that haven't been given so far.

In this section we have discussed a defining characteristic of mathematical thinking. This thought process moves along a path of generalisation. In fact, generalisation is the way the world of mathematics grows. Let us see how.

2.3 PARTICULARISING AND GENERALISING

One of the most important mathematical thought processes is that of generalisation. We do it in real life all the time. For example, consider the way we formulate the concept of 'tail' in our minds. The process involves observing the tails of some objects, such as a horse or a cow. We also notice that different tails may look different, but all of them are called 'tails'. So, our initial concept of a tail may be that it is that part of an animal that is seen at the back of the rest of the body. Then we extend this concept to the appendage at the rear end of a bird or a fish. We may extend this notion further and modify our image of a tail to include the tails of aeroplanes and kites, thus generalising our notion from living to non-living creatures also. As we examine more objects that have a tail, we continue to generalise this notion. Ultimately, we arrive at an image of a tail that may not include some of the specific features of the tails of the different objects that we are considering, but will include common features of all of them.

We engage in this kind of generalisation all the time in our daily lives in order to formulate a concept. The process is useful in extending our activities — for example, we can generalise our observations about plant growth in order to grow new plants; and, we are able to generalise our experiences of a child's mental development in order to construct learning and teaching methodologies. In the study of mathematics, the process of generalisation assumes a special significance. It helps us to understand the structure of specific mathematical objects and to build further knowledge upon existing structures. But what is even more significant is the fact that often such extension of knowledge may become impossible without going through the process of generalisation.

In mathematics, we find generalisation occurs in different contexts — we generalise to arrive at definitions of new concepts, as in the case of the definition of quadrilaterals. We generalise procedures, for example, the procedure to add two polynomials. We generalise results to new sets of mathematical objects, such as extending the statement 'the sum of the four angles of a square is 360 degrees' to the statement 'the sum of the four angles of a quadrilateral is 360 degrees'. And, of course, algebra is a generalisation of arithmetic, where the use of variables helps us to extend our study and use of numbers in new ways.

In this section, we study generalisation in different mathematical contexts. For instance, think about the way most of us develop the general concept of a polygon. We get to know triangles of various shapes and sizes. We get to know rectangles, squares and other quadrilaterals. We look around us and see patterns having pentagons. We wonder — can we have figures having 20 sides, 50 sides, 77 sides, and so on? If so, what would their properties be? Is anything common to all these figures? In this way we develop our concept of a polygon as a closed figure having three or more sides. This is an example of generalisation. With such generalisation we also generalise related notions like those of area, perimeter and other concepts associated with polygons.

Usually, to understand what the general concept is, we begin learning about it by observing and studying properties of particular cases. For instance, by studying the areas of squares, parallelograms or triangles, we may naturally acquire the general concept of 'area of a polygon'.

For another example, try and recall the way you acquired your understanding of 'place value'. Initially it developed in the context of 'base 10', i.e., in the decimal system. Then you may have heard that computers function with a binary system, i.e., base 2. Did this make you wonder: Given any number, can I write it in other bases, say base 5, base 60, base 12, or for that matter, base n $\forall$ n $\in$ **N**? This process of 'wondering' is also called **'making a conjecture'**. Since the conjecture is about a situation in greater and greater generality, we consider these thought processes as an example of generalisation from several particular cases. However, be **warned** that at present we do not know if our generalisation is mathematically acceptable or not. (This shall be discussed in the next section, and in the last block of this course.)

Before going further, why don't you try some related exercises?

E3) How would you write 'hundred' in base 5, base 50, base 100 and base 101?

E4) If 303 (in base 10) is written as 213 in base n, find n.

Now suppose you have proved your conjecture. Then you know how to write any number in base n, where n $\in$ **N**. You have **a generalised procedure**. Therefore, if you are required to write a number in the hexadecimal (i.e., base 16) system, you apply your procedure for the particular case n=16. We call this process **particularisation** or **specialisation.**

You could do this whole exercise of generalising and particularising for concepts or procedures that your students are learning. Such examples can be used to help your learners understand the processes of generalisation and particularisation while studying these concepts/processes/skills. In this way, they will realise that while understanding or creating mathematics, we are moving **from particular to general and from general to particular** all the time.

In fact, to understand a concept, it helps the learner to gradually construct it in her mind. This is done through experiencing concrete examples, studying several particular cases and gradually grasping the generalised concept. Though many of us accept this fact in theory, how often do we find this happening in our classrooms? Not commonly. In fact, it is more common to find teachers introducing the students to a concept by giving them the definition in all generality, and expecting the children to remember it. Even when examples to illustrate the definition are given, they are not varied enough. Some teachers introduce the children to a concept by giving some particular examples in the textbook or on the board, quickly followed by the general definition.

Neither kind of teaching helps the young minds in acquiring the concept because children require more opportunities to think about and use the concept concerned. They also need to think about examples **and non-examples** of the concept on their own. This gap between teaching and learning is very evident in geometry where, for example, students learn about different polygons without building any links among them. This is one reason why so many people wrongly believe, for instance, that a square is not a rectangle!

The point we are emphasising here is that, in most cases, the move from particular to general cases represents a move towards a higher cognitive plane. The children need to, first, become somewhat familiar with a concept in particular cases by dealing with plenty of concrete examples. They need to build links between these specific cases and the essence that they have abstracted. Only then can they move towards understanding the concept in all its generality. We teachers need to understand this if we don't want concepts to be reduced to mere definitions, which are rote learnt.

A word here about generalising algorithms. For example, the algorithm for adding elements of **Q** is nothing but a **generalised procedure** for adding **any** two fractions. (In fact, we can identify two levels of generalisation in this process. At one level, we have evolved a method that works for **all** rational numbers. At a different level, we are also generalising the idea of addition — we are now adding not only integers, but also parts of integers.) Think of any algorithm in mathematics — may be one for finding the roots of a quadratic equation, or that of finding the solution set of a system of equations. Each of these algorithms is **a generalised step-by-step procedure.** Each such algorithm has an underlying logic. What we mean by generalisation in this case is that the logic of the algorithm is not restricted to just a few particular cases. It works in exactly the same way for any member of the class. You have already seen this in the case of the algorithm for writing a number in a system with any base. **Your learners also need to understand the logic behind the working of the algorithm,** the mathematics of it. Otherwise, the process will reduce to a meaningless mechanical procedure for your students.

Why don't you try some exercises now?

E5) Give an example, with justification, of a generalised procedure in your students' daily lives.

E6) Give an example of movement 'from general to particular' taken from your daily life. Also explain why you chose that example.

E7) Not all generalisations related to mathematical objects are valid. Give an example to show this, taken from the secondary school level mathematics.

We have seen that the world of mathematics grows through the process of generalisation — of concepts and processes, and relations between them. When we are generalising concepts or algorithms, we need to ensure that the generalisation is valid. There are broadly two forms of reasoning we use for this purpose, which we shall discuss next.

2.4 WHAT IS A PROOF?

We will discuss methods of proof in detail in Block 5 of this course.

In the previous section we noted that doing mathematics involves generalising on the basis of observations of particular cases. Once we have noticed patterns in these instances, we make inferences based on these patterns. Thus, you may infer that June is the hottest month of the year (if you live in Punjab, say). Or you may infer what a one-year-old child will look like based on your observations of several children of that age. You may see a cow eating grass, then another one doing the same thing and infer that all cows feed on grass. This form of drawing inferences based on repeated similar experiences is called **inductive logic**. The form of this logic that we use in mathematics is called **mathematical induction**. This principle **uses inductive logic to formulate a conjecture** based on observed patterns. For instance, you may observe that $1^3 + 2^3 = 9 = 3^2$, $1^3 + 2^3 + 3^3 = 36 = 6^2$, and so on. You may also notice that $3 = 1+2$, $6 = 1 + 2 + 3$, and so on. Based on these particular cases, you may conjecture that $1^3 + 2^3 + \ldots + n^3 = (1+2+\ldots+n)^2$.

The other form of reasoning is **deduction**, that is, the use of deductive logic. According to this logic, we use known facts to arrive at a conclusion. For instance, knowing that there is severe water shortage in a given town, you can deduce that the price of drinking water will be high over there. In mathematics we apply deductive logic all the time — when we use known results, definitions, axioms and rules of inference to prove or disprove a statement.

You know that in mathematics when we claim that a statement is true in general, what we really mean is that **it holds true, without exception**, in all cases in which the conditions of the statement are satisfied. This means that mathematically speaking, it is not enough to show that the particular statement is true in several different cases (even if the number of such cases is very large); what we must be able to do is to actually show, through a process of inductive and/or deductive reasoning, that the statement is valid in all the cases where the conditions of the statement are true. This 'showing' constitutes a 'proof'.

Fig.2

It is no exaggeration to say that **the idea of proof is the single most important idea in all of mathematics.** Consider any **mathematical proof** of a statement. It consists of one or more steps, deduced from earlier steps or accepted facts, which make up **mathematically acceptable** evidence to support that statement. Let us look at an example to see how inductive and deductive logic go hand in hand to give a proof in mathematics.

Suppose I ask you to find the sum of the interior angles of any convex polygon. How do you go about trying to answer this question? You may already know that the sum of the interior angles is related to the number of sides of a polygon in some way. You would probably begin by looking at special cases. You already know that this sum is 180° for a triangle and 360° for a quadrilateral. Suppose you also know that for a pentagon this sum is 540° and for a hexagon it is 720°. You could try drawing a chart like the following one to find some pattern:

Making charts is often a good way of looking for patterns.

Number of sides of polygon	3	4	5	6
Sum of the interior angles (in degrees)	180	360	540	720

After a little thought, you may notice that each number in the second row is a multiple of 180. You may then decide to write each number down as a multiple of 180. Thus, you will get: $180 = 1 \times 180$, $360 = 2 \times 180$, $540 = 3 \times 180$, $720 = 4 \times 180$. Are these numbers related to the number of sides in each case? In other words, is there a common rule relating 3 to 1, 4 to 2, 5 to 3, and so on? Some reflection on this question may lead you to infer that the sum is $[(n-2) \times 180]°$, where n is the number of sides of the polygon. But how would you check whether your guess (or **conjecture**) is right? After all, it may happen that this result may not hold if you take a 20-sided polygon, or one with 62537 sides. You would need to find a proof to show that the statement 'the sum of the interior angles of an n-sided polygon is $(n-2) \times 180$ degrees, for any $n \geq 3$' is valid. You would do so **through a series of steps, each of which is deduced logically from the previous ones**. This would constitute the **proof of the statement**. There can be several proofs. Let us consider one of them.

As you may remember, to logically derive a result we must accept certain definitions and/or axioms and/or earlier proven statements. In this case, two statements that we shall assume are
'The sum of the interior angles of a triangle is 180°', and
'The sum of the angles around a point is 360°' (as illustrated for one case in Fig.3):

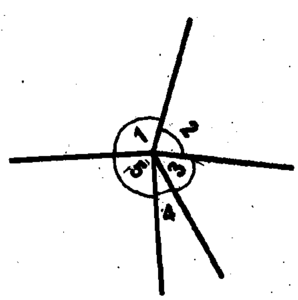


Fig.3

Consider any n-sided polygon and take any point, say O, inside it. Join this point to each of the vertices of the polygon. As there are n vertices, the interior of the polygon gets divided into n triangles. (In order to understand this picture more clearly, we could even draw a polygon and make the necessary construction as in Fig.4. (However, we must **remember that this picture (or any) is only an aid to see the logic of the proof.** Sometimes a picture could give you an incomplete or wrong understanding of the general situation.)

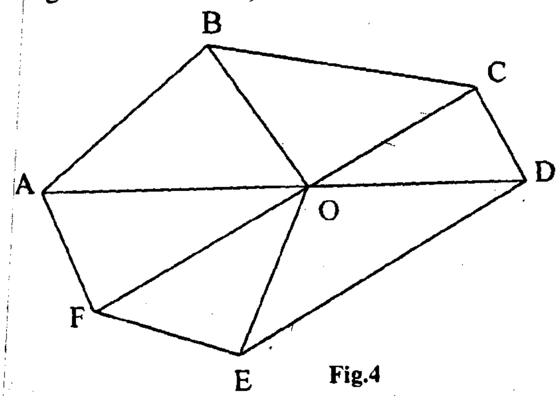


Fig.4

Now, for each of the n triangles, the sum of the angles in it is 180°. Since there are n triangles, the total sum of all the angles inside this polygon is $(n \times 180)^\circ$.
But the total sum of the angles inside the polygon is the sum of the interior angles of the polygon plus the angles around the point O. Since the sum of the angles around O is 360°, the sum of the interior angles of the polygon is $[(n \times 180) - 360]^\circ = [(n-2) \times 180]^\circ$. (Remember, in the picture $n = 6$, but we are actually dealing with any $n \geq 3$.)

The series of statements above constitutes a ' mathematical proof ' for the stated result. In it, each step follows logically from the preceding step and/or one of the results that we assumed before we began this proof. This method of reasoning is what is called 'deductive logic'. Thus, here, by a piece of deductive logic, we have actually shown that what we had **inferred** through inductive logic is indeed true in each and every case.

Here are some exercises for you now.

E8) Go back to the conjecture made earlier, that $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$ for every $n \geq 1$. Give a proof by the **principle of induction.** While doing so, explain which part is using inductive logic and which part is using deductive logic.

E9) Prove that each interior angle of an n-sided regular polygon is $\left[180 - \frac{360}{n}\right]^o$ for $n \geq 3$.

E10) What strategy would you use for inculcating in your students the ability to prove/disprove statements?

Let us, now, take a brief look at what we have just said about proofs, namely, proving a mathematical statement involves the following:

- A general statement about a certain class of objects that satisfy a set of conditions. This statement may be formulated on the basis of observation of patterns found in particular cases, or on the basis of mathematical intuition, or on some other basis.

- The objective is to show, through deductive reasoning, that the given statement is true in all cases where the conditions of the statement are valid.

- What we could use to achieve our objectives are one or more statements, which we call **premises**. These premises can be of four types :

  i) a statement that has been proved earlier;

  ii) a statement that follows logically from the earlier statements given in the proof;

  iii) a mathematical fact that has never been proved, but is universally accepted as true, that is, an axiom;

  iv) the definition of a mathematical term.

- The proof of the statement, then, consists of these premises.

Once we successfully show that the given statement is valid, we say that our statement has been proved.

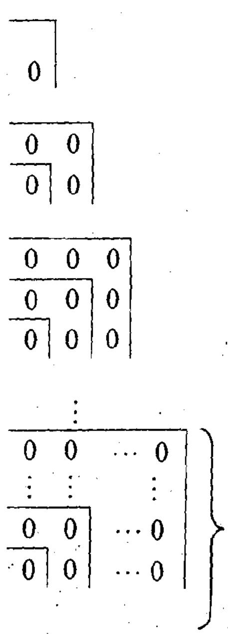
As we see above, proving any statement about a given collection of objects mathematically involves proving it for **each and every object** in the collection. This means that a statement about a collection of objects is false if it does not hold for even one case in the collection. So, one way to **disprove** a mathematical statement (i.e., prove that it is false) is to find one example of an object that satisfies the hypotheses but not the conclusions.

For an example, consider the statement: Every continuous function is derivable. To disprove this statement, we only have to find one example of a function that is continuous but not derivable. You know several such functions.

Try an exercise now.

E11) Give one example each of a true mathematical statement and a false one related to 3D. Also prove that these statements are true and false, respectively.

You may wonder whether the process of proving a statement that we have outlined above is the only way of doing so. How about "visual proofs"? For example, consider the following proof for the statement $1 + 3 + 5 + \ldots + (2n - 1) = n^2$, where n is a natural number.



$1 = 1^2$

$1 + 3 = 2^2$

$1 + 3 + 5 = 3^2$

$\vdots$

$1+3+5+\cdots+(2n-1) = n^2$

The truth is that though such visual evidence can be **useful as an aid** for proving the relevant statement rigorously, mathematicians do not accept it as proof. This is because we have to remember that in mathematics, what we demand of a proof is that **it should be valid in all situations where the conditions of the statement we are proving are valid**. It would often be quite impossible to visually consider all the possible situations in which the conditions of a statement are true. In fact, what is even worse is that we may draw a diagram in which a particular statement is true and not even realise that there are other possible situations where all the conditions of the statement are satisfied and yet the statement is actually false.

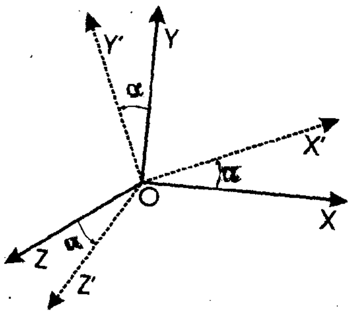


Fig.5

For example, recall what happens when children represent a rotation of three mutually perpendicular axes in three-dimensional space on paper. Very often they show all three axes as having rotated through the same angle α (see Fig.5), something that is not possible (as will be stressed in Unit 7).

If the lines are all in a plane, this is possible, but not otherwise. And this false generalisation has come about entirely because of the examples through which our mathematical intuition was built up.

We can find many other such examples related to functions and other topics. The other point that comes out from these examples is that if we find that we are arriving at a result that appears to be going against our common (and mathematical) sense, then we probably need to pause and re-examine our work carefully. But, sometimes our intuition or common sense may be **wrong** because this may be limited by what one sees in a few particular instances. That is, our generalisation from particular instances may be wrong.

Here's a related question for you.

E12) Give an example of mathematical concepts or processes being understood by your students due to excessive weightage given by them to visual aids.

Let us end this section with a brief overview of what mathematics is. It is a world of abstract objects and relationships between them, based on undefined terms and axioms about them. It is extended further by the processes of generalisation, abstraction and some laid down rules of mathematical logic. What is very important is that everything has to be consistent with what is known earlier.

Therefore, if even one of the axioms is changed, the whole theory that is built around it will change. A very good example of this is given in the following section.

2.5 CHANGE A POSTULATE, AND THE WORLD CHANGES!

"From nothing I have created another new world".
(Janos Bolyai in a letter to his father.)

Most people are unaware that around a century and a half ago a revolution took place in the field of geometry that was as scientifically profound as the Copernican revolution in astronomy and, in its impact, as philosophically important as the Darwinian theory of evolution. "The effect of the discovery of hyperbolic geometry on our ideas of truth and reality has been so profound," writes the great Canadian geometer H.S.M. Coxeter, "that we can hardly imagine how shocking the possibility of a geometry different from Euclid's must have seemed in 1820." Today, however, when it is known that the space-time continuum is closely related to the non-Euclidean geometries, some knowledge of these geometries is an essential prerequisite for a proper understanding of relativistic cosmology.

Mathematics is a deductive system in which one starts from some definitions, some undefined terms and some self-evident truths (which may be based on experience) called **axioms**. Using these as a basis, we move to further results by a process of deductive logic. This is perhaps most evident when we study Euclidean geometry, starting in high school.

Fig.6: Euclid, who produced the definitive treatment of Greek geometry and number theory in his 13-volume 'Elements' around 300BC.

2.5.1 Euclid's Postulates

As in all mathematical theories, Euclid built a theory involving some abstract objects and relationships between them. Some of the objects are undefined, for example, 'point', 'line', etc. Others are defined in terms of these objects. Of course, the effort is to keep the number of undefined terms to a minimum.

Each theorem in Euclid's geometry is proved from some preceding results. Of course, he started with a set of five assumptions about the undefined terms, which are the axioms or **postulates** of the theory. Any set of statements can be laid down as postulates so long as they do not lead to any logical contradictions or inconsistencies. Obviously, the fewer the postulates, the better. Mathematicians try to derive more and more theorems from fewer and fewer postulates.

Euclid's five postulates are:

1. Given two distinct points, there is a unique straight line that passes through them.
2. A line segment can be prolonged indefinitely.
3. For every point O and every point A distinct from O a circle can be constructed with centre O and radius OA.
4. All right angles are congruent to each other.
5. **The Parallel Postulate** (sometimes called **Playfair's axiom**) **:** If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

Remark: The fifth postulate can also be worded as: For every line l and for every point P not lying on l, there exists a unique line m through P that is parallel to l.

In the first of his thirteen books, Euclid made no use of parallel lines, defined as lines in a plane that do not meet. He proved several theorems and propositions without using his fifth postulate. Later mathematicians have added to this number, and together they are now known as **absolute geometry**, a term first used by Janos Bolyai.

It was almost as if Euclid sensed that his fifth postulate was on a different footing from the other four. He may have not been totally sure about whether or not it could be derived from the other four. In fact, Euclid himself proved that if AB is any straight line and P is a point in the plane of AB but not on it, then **at least** one line parallel to AB can be drawn through P. However, he could not prove that there is only one such line. Had he proved this, his fifth postulate would not have been required.

Why don't you try an exercise now?

E13) Show that the two ways of presenting the 5th postulate given above (Pt.5 and the remark) are equivalent.

Several mathematicians, since Euclid, have tried to prove the uniqueness of the parallel line passing through P using only the other four postulates and the theorems derived from them. But no one was successful. However, these efforts led to a great achievement — the creation of several non-Euclidean geometries. This is considered a landmark in the history of thought because till then everyone had believed that Euclid's was the only geometry, and that the world itself was Euclidean. Now the geometry of the universe we live in has been demonstrated to be non-Euclidean.

In the process of creating different geometries, two important ideas have been established. They are

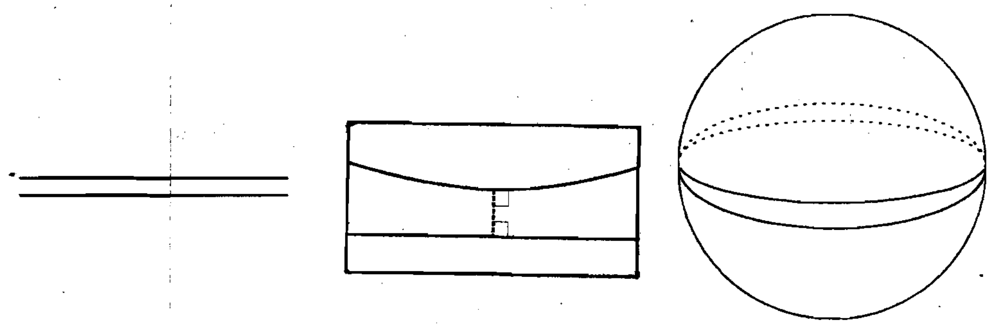
i) there is no single 'correct' geometry; and
ii) mathematical theories are not necessarily real.

Let us take a glimpse of some of these geometries.

2.5.2. Non-Euclidean Geometries

There are several non-Euclidean geometries now. Each of them is built on Euclid's postulates, except for the second one or the fifth one. All these geometries fall into one of two categories: hyperbolic or elliptic. Hyperblic geometry was discovered by Gauss, J.Bolyai and Lobachevsky. Elliptic geometry was discovered by Riemann.

Consider two straight lines drawn perpendicular to another straight line AB at A and B on the same side of AB. In Euclidean geometry, the mutual distance between the two straight lines will remain constant. In hyperbolic geometries, the two straight lines will grow further apart (as in Fig. 7(b)). In elliptic geometries they will come closer together (as in Fig. 7(c)).

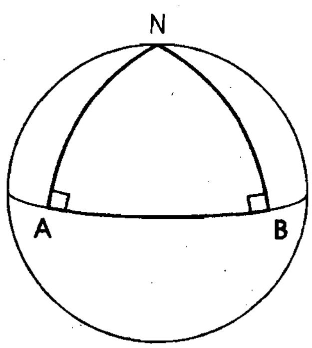


(a) (b) (c)

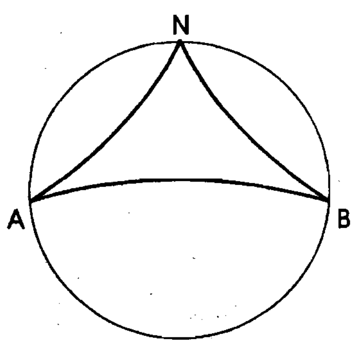
Fig.7: 'Parallel lines' in a)Euclidean geometry, b) hyperbolic geometry, c) elliptic geometry.

In hyperbolic geometries, if the perpendicular at A is replaced by a straight line making a slightly smaller angle with AB, then this new straight line will at first converge towards the straight line perpendicular to AB at B, come to some minimum distance and then diverge. Therefore, Euclid's fifth postulate is no longer valid in this case.

In elliptic geometries, the parallel postulate (in the form stated by Euclid) is satisfied trivially, but his second postulate is violated because now every straight line closes on itself like a circle. Note that **in any elliptic geometry, any two straight lines, both of which are perpendicular to a third straight line, intersect.** This means that these three straight lines form a triangle. Therefore, **in elliptic geometry the sum of the interior angles of a triangle must be greater than two right angles** (see Fig.8).



(a)



(b)

Fig.8: The sum of the interior angles of the triangle NAB
(a) is greater than 180^0 in elliptic geometry,
(b) is less than 180^0 in

On the other hand **in hyperbolic geometry the sum of the angles of the triangle will be less than two right angles.**

Why don't you try an exercise now?

E14) What activities would you give your learners to help them see the differences pointed out between hyperbolic and Euclidean geometry?

As a concrete example of a non-Euclidean geometry, let us now consider the geometry on the surface of a sphere, called **spherical geometry**. This is one example of an elliptic geometry.

2.5.3. Spherical Geometry

Let us imagine that the surface of the earth is a perfect sphere and we are restricted all the time to move only along this surface. If we take as our definition of a straight line that path which has the shortest distance between two points, then a straight line joining any two points on the surface of the sphere is an arc of the great circle passing through these points (see Fig.9). Thus, on the surface of a perfectly spherical earth, the equator and all the lines of longitude are great circles, i.e., straight lines. Since these great circles are of finite circumference, the straight lines cannot be of infinite length. So, **Euclid's second postulate is not valid for spherical geometry.**

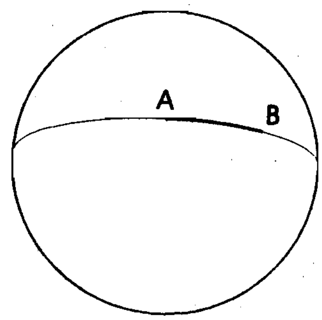


Fig.9

A number of corollaries follow from these statements:

1. Since all the lines of longitude are perpendicular to the line of the equator, they must all be parallel to each other. Yet they all intersect at the North and South poles. Therefore, an infinite number of parallel lines, all distinct from each other, can pass through the poles. Similarly, they can pass through any point on the surface of a sphere.

2. Other than the equator, none of the lines of latitude are great circles. Therefore, none of these lines are straight lines. Consequently, they cannot be considered to be straight lines on the surface of the sphere.

3. Since two lines of longitude intersect at the North and the South poles, between them they enclose a region. This shows that in spherical geometry, unlike in Euclidean geometry, two straight lines can enclose a region between them.

4. Consider two lines of longitude NAS and NBS which are perpendicular to each other, i.e., $\angle ANB = 90°$. Here A and B are the points of intersection of these lines with the equator. Then $\angle NAB = \angle NBA = 90°$. Further each of the sides of the triangle NAB, namely, NA = NB = AB = $1/4^{th}$ the circumference of a great circle. Thus, triangle NAB is an equilateral triangle on the surface of the sphere. So, we immediately have two results that are different from Euclidean geometry.

   i) First, each angle of an equilateral triangle in spherical geometry is 90°, and not 60° as in plane geometry. An immediate consequence of this is that **Pythagoras' theorem is not meaningful in spherical geometry.**

   ii) Second, we have the result that **the sum of the interior angles of a triangle on the surface of a sphere is greater than two right angles,** instead of equal to two right angles, as in plane geometry. Incidentally, this second result establishes that spherical geometry is an example of elliptic geometry since the sum of the angles of a spherical triangle has been shown to be greater than two right angles.

Some more features are given in the following exercises.

E15) Show that the sum of the angles of a spherical triangle can vary between π and 3π, the actual value depending on the triangle.

E16) Show that, in spherical geometry the ratio of the circumference of a circle to its diameter can vary between π and 2.

E17) Imagine you are standing on the Earth, and you walk one mile due South, then one mile due east, then one mile due north and find yourself back at your starting point with a bear staring you in the face. What colour is the bear?

With this we end our short discussion on one of the most exciting mathematical advances of the 19^{th} century. We shall look into other aspects of mathematical thinking in some more detail in the next unit and Block 5. For now, let us summarise what we have done in this unit so far.

2.6 SUMMARY

In this unit we have covered the following points.

1. We have seen that the world of mathematics is made up of abstract objects, and relations between these objects. There are certain rules and conventions that we agree that these objects will follow.

2. The essence of mathematical reasoning is generalising on the basis of patterns observed in particular instances. These generalisations should be valid. Validity means that they should hold true **for every case** that fits the conditions under which the generalisation is made.

3. We have seen what a proof is in mathematics and what disproving a statement involves.

4. We have studied an example of the implications of changing the basis of a mathematical theory, and hence developing new theories which are consistent within themselves. The example presented here is that of Euclidean and non-Euclidean geometries.

2.7 COMMENTS ON EXERCISES

E1) For example, mathematical concepts could be 'closed and open figures'. Examples of non-mathematical ideas could be 'heat' or sharpness.

Think about how each of these concepts develop in our minds. As infants, we don't know what 'heat' is. What kind of experiences gradually make us understand this concept? How we learn to compare and find out which is hotter?

E2) One form of abstraction is abstracting from the concrete. For example, from pens, ball pens, pencils, etc, the idea of things that write is abstracted. Think of more examples of this kind of abstraction.

The second kind of abstraction is abstraction from ideas that are not actually available in reality. For example, abstracting the concept of negative numbers and imaginary numbers.

E3) 100 (base 10) $= 4 \times 25 + 0 \times 5 + 0 \times 1 = 400$ (base 5)
$= 20$ (base 50)
$= 1$ (base 100)

For writing in base 101, you need to first have 101 symbols, say $a_0, a_1, \ldots, a_{100}$. Then
100 (base 10) $= a_{100}$ (base 101).

E4) We know that $2n^2 + n^1 + 3n^0 = 3 \times 10^2 + 0 \times 10^1 + 3 \times 10^0$. Solving this for n, we get $n = 12$.

E5) For example, if one knows how to make one kind of 'daal', the same procedure of using a pressure cooker to cook, can be used for cooking other 'daals' as well. The procedure for cooking a vegetable can also be similarly generally developed. You should think of more examples of procedures that are generalised, including some from mathematics.

E6) We know that, in general, people smile when they are pleased or when they want to be friends. Therefore, in particular, when we meet a new person and he/she smiles at us, we assume that the person wants to be friends. This is an example of particularisation.

E7) There could be many examples in which the generalisation could be wrong. For example, since every linear polynomial has a real root, many children generalise this fact to quadratic and cubic polynomials wrongly. Give other examples of possible generalisations that are erroneous.

E8) The steps are

1) Writing P(n) $\forall\ n \geq 1$.
2) Checking that P(1) is true. (This is called the basis of induction, though its proof is by deductive logic.)
3) For some $m \geq 1$, assuming P(m) is true, and then showing P(m+1) is true. (This is called the inductive step, and again, it is proved by deductive logic.)
4) Hence, P(n) is true $\forall\ n \geq 1$. (Inductive logic)

E9) You know that each interior angle of an equilateral triangle is 60°, which is $[180 - \frac{360}{3}]^\circ$. Now, use what you have proved earlier about the sum of the interior angles of a polygon. Also, use the definition of a regular polygon — all its sides are of the same length, and hence, all its interior angles have the same degree measure. Then see if you can prove the given statement.

E10) You need to give the activities you use for developing the abilities of using inductive and deductive logic. You also need to present the methods you use for helping them develop their mathematical intuition, particularly what kind of conjectures to make, and how to verify if they 'make sense'.
The method that **will not** work is to give a series of proofs to be rote learnt, and coughed up at exam time.

E11) For example, consider the statement: Every planar section of a sphere is a circle. Prove or disprove it.

E12) Several misconceptions regarding skew and parallel lines in 3D are because of this factor. Think of other examples.

E13) You need to show that Playfair's axiom implies the Remark, and vice versa.

E14) Can you think of some results in Euclidean geometry that are based on the parallel postulate? Ask them the results would alter if this postulate does not hold any/more.

You could go further and ask them what would happen if the first postulate didn't hold, and so on.

There are several websites on hyperbolic geometry that you can ask your students to access to visually see the difference triangles, perpendicular lines, etc., in this geometry and Euclidean geometry.

E15) Look again at Fig.8(a). In Δ NAB, what happens to $\angle$ANB if we move the points A and B closer together? As these points approach each other, $\angle$ANB will become smaller and smaller, nearer and nearer to zero. And then the sum of the interior angles of the triangle will become nearer and nearer to two right-angles. If, on the other hand, we move the points A and B further and apart, $\angle$ANB will become larger and larger, with a maximum value of 360°, as the length of the side AB approaches the circumference of the great circle. In such a situation, the sum of the interior angles of ΔNAB will approach six right-angles.

E16) Consider a sphere of radius R. Note that the diameter of any circle on the sphere, being a straight line, has to be an arc of a great circle. Let A be a point on the surface of the sphere, which we shall take as the pole, and let O be the centre of the sphere (see Fig.10). Consider a line of latitude with A as the pole and α as the co-latitude (angle between OA and a line joining O to any point B on the line of latitude). Then the diameter of the circle forming the latitude is $2\alpha R$, since this is an arc of a great circle subtending an angle 2α at the centre O. On the other hand, the circumference of the circle of latitude is $2\pi R\sin\alpha$ (as this is also the circumference of a planar circle of diameter $2R\mathrm{Sin}\ \alpha$).

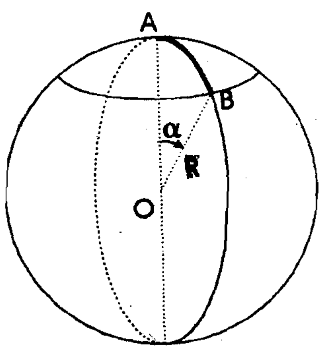


Fig.10

Thus the ratio of the circumference of a spherical circle to its diameter is $\pi_\alpha = 2\pi R\sin\alpha / 2\alpha R = \pi\ \mathrm{Sin}\ \alpha/\alpha$.

The largest circle that one can draw on the surface of the sphere is a great circle (like the equator) for which $\alpha = \pi/2$, so that $\pi_\alpha = 2$.
The smallest circle is one of radius 0, for which $\alpha = 0$, so that $\pi_\alpha = \pi$.
Thus in spherical geometry, π_α varies between 2 and π.

E17) White, because you are near the North Pole, since no bear exists around the South Pole. But if the question had been 'where are we?', there are an infinite number of locations close to the South Pole which fit the bill!

UNIT 3 EXPLORING MATHEMATICS

Structure

3.1 INTRODUCTION

In the previous unit we elaborated on important processes involved in mathematical thinking, namely, working in a world of abstract objects, specialising and generalising. In this unit we focus on the use of these processes and other processes involved in 'doing mathematics'.

We start with a section in which we discuss the different thought processes involved in exploring mathematical problems. In the next two sections, we carefully observe these processes through exploring some mathematical problems in geometry. Finally, we look at the use of mathematical puzzles for developing these processes.

While you are studying Sec. 3.3 and Sec. 3.4, we expect you to focus on the thought processes involved because these are the processes that your learners need to develop. Therefore, while studying this unit, keep thinking about how you can foster these processes in your learners' minds.

Objectives

After reading this unit, you should be able to

- explain the mathematical thinking involved in problem-solving, conjecturing and other mathematical explorations;
- suggest ways of generating mathematical thinking in your learners;
- design and carry out activities to help your learners investigate the polyhedra and tilings;
- create mathematical puzzles that challenge, but not over-challenge, your learners.

3.2 THE PROCESSES INVOLVED

Let us start this section with a brief look at what 'doing mathematics' means to most children and teachers. The common view is that mathematics is 'done' only in the 'maths class'. During this class, the children are expected to learn a concept by being given the definition and doing a few direct questions based on it. Then they are expected to solve word problems related to the concept. The procedure involved is that first the teacher explains 'the way' to solve a particular kind of problem. Following this, she gives the children many problems of the same type to solve on exactly the same lines. So, solving a problem is reduced to listening to the teacher, memorising certain solutions and mathematical facts and reproducing them appropriately. Any algorithm is treated in the same manner.

Where is the mathematical thinking being developed in the whole process outlined above? It is certainly important to have basic computational skills and definitions, but not by rote. It is important for a learner to understand the mathematics involved, even in an algorithm.

To understand any mathematical concept or process, a child needs to be introduced to it through familiar situations and experiences. In order to improve her understanding of the concept, it is important to use the concept on different occasions in as many ways as possible. The exposure to a variety of problems related to the concept helps her to deepen her understanding of the concept. For the child (or for us) this helps to interlink the different aspects of a concept, which, again, helps to strengthen conceptual understanding. When the child has to think about what to do and how to do it, she is forced to examine the concept seriously, and hence extend its meaning for her. For those whose concepts are half-formed or are erroneous, solving different problems gives an opportunity to discover the errors and to reach a better understanding of the concept. For example, when helping a child to develop the idea of a function, we need to give her an opportunity to identify functions from non-functions, use functions in various ways, allow her a chance to use a variety of functions, etc. In short, **concept formation is linked to the opportunities available to the learner to think, apply her understanding and use her conceptual structures in various ways, finding relationships with other concepts.**

While you're thinking about the point just made, try the following exercises.

E1) Give an example of a mathematics question given to children which does not require them to think mathematically. Also give your reasons for your choice.

E2) Explain, with examples from your own learning of mathematics, how solving problems helps in concept formation.

While trying these exercises, you must have focussed on the essential characteristics of doing mathematics — it must be an opportunity for the learner to think mathematically, choosing which step to take, based on what she knows and where she wants to reach. If she is expected to solve a problem, it should not require her to merely reproduce information or mechanically apply algorithms. She needs to, gradually, be exposed to more and more complex problems built around the concept. A major part of this process is the ability to build one or more representations of the problem that is being dealt with. We shall consider this, and other aspects in the next section.

3.3 SOLVING AND POSING PROBLEMS

Try and recall the last time you were required to do some mathematics — solve a problem based on data given to you, or asked to generalise some mathematical ideas. Did you start by making a representation of the data or of the concepts involved? It may have been a mental or a visual picture of 'skew lines', or of a large number. Once you had this picture, what did you do next? Did you try to relate it to the knowledge you already had and search for relevant pieces that could help in solving the problem? How different is this from the steps your students go through? You may be able to answer this while observing the stages you, and your students, go through when solving the following problem :

How many different ways are there for seating 8 persons at a round table?

See if your steps are similar to the steps I went through, which are:

1) I first drew a circle and made 8 points on its circumference.

2) Looking at this, I searched through my memory to think of what I knew related to this problem, for instance, the permutations (1, 2, ..., 8), (2, 3, ..., 8, 1), ..., (8, 1, 2, ..., 7) represent the same seating in this case.

3) Therefore, for solving the problem, I needed to find the number of distinct permutations, keeping Point (2) in mind.

4) To check my understanding, I tried it for 3 people, instead of 8.

5) I solved it, the answer being $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$.

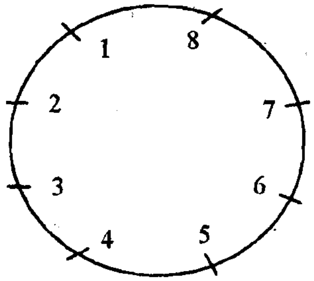


Fig. 1

Breaking up the stages of solving the problem is not very easy because many of these steps get merged and many stages are repeated again and again. In fact, the stages given above are exactly those given by the educationists **Davis and Mayer**. According to them, in order to solve any mathematical problem, we need to go through some or all of the following steps sequentially (perhaps many times).

1. **Build a representation** for the known mathematical information.
2. Use this representation, and **search in our memory** for knowledge that we consider relevant for solving the problem.
3. Apply the retrieved knowledge to the input data and **construct connections** between them.
4. **Check** these constructions to see if they seem to be reasonable and correct
5. **Use technical devices** such as procedures or algorithms (or other information associated with the knowledge representation in order to solve the problem.

As you can see, our ability to represent a problem situation is essential for solving it. In order to solve a problem, Step 2 would require "building a representation of what we consider relevant knowledge". This may be sometimes so quick that we fail to notice it. Children, however, are seen to do it much more often. For instance, consider the problem: "How many integers between 100 and 999 consist of distinct even digits?" You may see this almost immediately as "a problem of counting". But, when a child tries to solve it, there are two separate steps involved. Firstly, she needs to build a representation of the input data. Then she needs a representation of "possibly relevant knowledge", which would require her to put in some mental effort. She may try listing some of the numbers, and then pick out the required ones to construct a numerical representation. Then she may use her earlier knowledge for the single-digit case and the 2-digit case to find a pattern. If she finds a pattern, she may generalise it to find the answer.

As your learner does more problems in different contexts, these procedures become a part of her thinking and turn into an instantaneous strategy. However, we must remember that the learner needs opportunities and encouragement to **tackle open-ended problems and problems with many solutions related to the concept.** The problems can steadily become more challenging. At each stage, **she should also be encouraged to talk about what she is doing** and explain her line of reasoning.

Let's see the abilities developed in the process, by asking your students to do some problem-solving.

E3) Give your students problems like the following one to do.

A company makes 100 computers every month. Its employee union accused the company of discriminating against its female employees. The union said that women were not being given the promotions due to them. The following table gives the data about the promotions in the company.

Year	*No. of women promoted*	*No. of men promoted*
1996	*5*	*15*
1997	*6*	*16*
1998	*10*	*8*
1999	*8*	*10*
2000	*8*	*10*

If an employee who is promoted during these five years is selected at random, what is the probability that the employee is a woman? Is this data enough for deciding whether female employees are discriminated against?

While they are working on the problems, talk to them to try and separate out the various thought processes they are using in the process. Also note down the stages you went through while solving the problem.

When I tried the problem given above, I first tried to understand the situation — **what I knew**, and **what I needed to find out**. Then I needed to think of **the path to use** to move from what I knew to what I needed to find out.

I also needed to know **which information, if any, was extra and not required**. For instance, what the company produced is irrelevant to the problem.

The next step was to write down the mathematical equivalent of the given problem :

Total number of women promoted from 1996 to 2000 = 37
Total number of people promoted in this period = 37 + 59 = 96
To find P(A), where A is the event that a woman was promoted.

Then I solved this problem using the definition of probability of an event, that is,

$$P(A) = \frac{37}{96}.$$

So, I concluded that approximately 1 in 3 promotions is likely to be that of a female worker. However, this probability gives us no indication of whether women workers are discriminated against. This is because we need some more information. For instance, we need to know how many men and women were eligible for promotion in this period.

Should we note down the steps involved in solving this problem?

1. Read the problem carefully to understand what it says — the information and assumptions in it, and what is to be found out, proved or examined.

2. Represent it mathematically, clearly filtering out the irrelevant data in the problem.

3. Gather other relevant information, axioms and earlier proved (or known) results.

4. Look for a path for solving the mathematical equivalent of the problem.

5. Interpret the solution in the problem situation.

These steps may appear to be different from the stages given by Davis and Mayer. But, if you look carefully you'll find some of those stages clubbed in the broader stages we have just listed.

Why don't you do an exercise related to this?

E4) Give some children a problem to do. After they have solved it, talk to them to find out which of the stages above they went through. Note down what they articulate. If you can get them to discuss the stages, note down what comes out in their discussion.

Problem-solving is one important part of doing mathematics. An equally important part is what further questions open up in our minds while solving a problem, that is, **posing a problem**. This requires us to use our abilities to generalise in many ways. For example, if I have proved that there are infinitely many primes, I may wonder if there are infinitely many primes of the kind $4m+3$, where $m \in \mathbf{N}$. This process can continue for as long as our mathematical maturity and intuition allows us to. And, each time we pose a problem and try and solve it, we grow mathematically.

Our level of problem-solving and problem-posing reflects our level of mathematical thinking. So does our ability to use a variety of representations while dealing with problems.

Being flexible in moving across representations is a sign of competent mathematical thinking. Each type of representation brings out specific aspects of a concept. Flexibility could mean moving within one type of representation, for example, using one diagram with many different parts that highlight different aspects of a problem. Flexibility also involves moving between quite different representations, for example, between an equation and a graph. Solving multi-stage problems may need the use of several representations.

In fact, we need to help our learners develop such a flexibility. They can have many different ways to represent the abstract concepts which they are in the process of learning. The representations can be in terms of known symbols, icons or concrete objects. Think about the various ways your students use to represent problems while trying the following exercise.

E5) While your students were doing the problem in E4 above, what were the various ways of representation they used?

Let us now gather **the implications** of what we have just discussed in this section **for anyone teaching mathematics**. A learner uses a variety of representations while trying to solve problems, particularly to relate it to the knowledge in her mind. The availability of these representations allows her to refer back and forth to her knowledge system. Further, if a child is given many different kinds of problems around a concept or process, she would be able to develop better ways of constructing representations and relating the concept to the knowledge base in her mind.

Throughout the process of problem-solving, the teacher needs to give **the student several opportunities to explain the process she has followed.** This would help her consolidate the strategies she has used. Given sufficient opportunity to deal with different kinds of problems and to articulate the strategies developed **without fear of ridicule** would help the child to develop more sophisticated problem representations. If the child has to learn to solve problems, this development is of great importance.

You may have noted that the use of procedures, algorithms and shortcuts are **only one step in solving a mathematical problem.** The first four steps where the problem is comprehended, represented suitably, related to the knowledge available and checked as being reasonable are extremely necessary before choosing an algorithm or procedure and applying it. Therefore, we need to give the student many tasks requiring her to build her ability to move flexibly across using various modes of representation. And, **we must not just give her one particular procedure for solving a type of problem.**

While a child is solving problems, she also needs to be encouraged to explore further generalisations. Here the teacher could be a facilitator, suggesting certain conjectures, to start her off. The child should be given many opportunities, maybe prodded several times too, to think about and articulate more problems—some could be of the same kind, and some could be of the kind 'What if ...?'.

Now for an exercise!

E6) Give a detailed account of the teaching strategy you would use to develop the ability of children of Class 11 for using various representations for dealing with sets.

To get more of an insight into the processes we have discussed in this section, here is an opportunity for you to investigate some mathematical areas.

3.4 INVESTIGATING PLATONIC SOLIDS

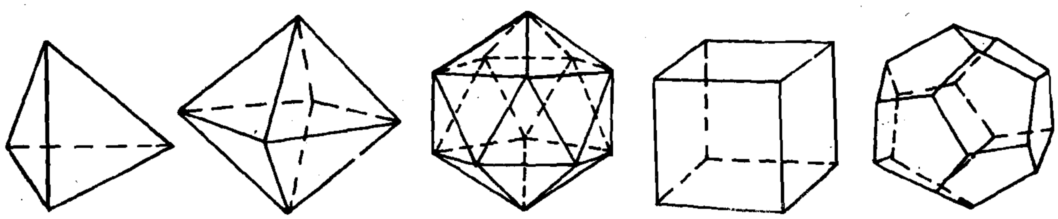
In this section we invite you to explore the processes involved in working mathematically, through a study of polygons and polyhedra. So, let's start with an exercise.

E7) How many different regular polygons are there? How are these polygons related to a circle? Note down other questions that come to your mind while you are working on these questions.

While you were doing E7, what did you notice about the way you deal with mathematical problems? Once you have solved it, do you think your understanding of 'polygon' has improved? In what way? Did you think about other related mathematics questions that could be explored? One problem that you may have thought of exploring could be : Can what I have found true for 2D be generalised to 3D? **(problem-posing)**

When we go from 2-dimensional figures to 3-dimensional objects, the concept of regular polygons generalises to regular polyhedra (the plural of polyhedron). **Regular polyhedra** are solids in which all angles and all sides are equal, for example a cube.

Now, while doing E7 you must have found that there are infinitely many regular polygons because there is no limit to the number of sides they can have. So, you may expect the same about the regular polyhedra. However, there are **only five different regular polyhedra possible**. These are the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron (see Fig.2). These are also known as the **Platonic solids**, after the Greek philosopher Plato (see Fig. 3). They have fascinated mathematicians from the time of the ancient Greeks. The faces of the tetrahedron (4 faces, from the Greek word 'tetra', meaning four), the octahedron (8 faces, from 'okto' meaning eight) and the icosahedron (20 faces, from 'eikosi' meaning twenty) are all equilateral triangles. As you know, the cube has 6 faces, all of which are squares. The 12 faces of the dodecahedron ('dodeka' meaning twelve) are regular pentagons. It is worth noticing that the faces of all the regular polyhedra are regular polygons. (Why?)



Tetrahedron **Octahedron** **Icosahedron** **Cube** **Dodecahedron**

Fig. 2: The five regular polyhedra (the hidden edges are indicated by dashed lines)

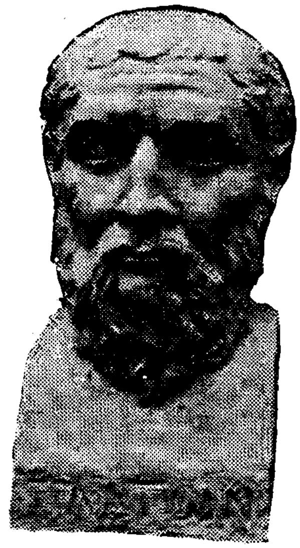


Fig.3 : Bust of Plato (423-347 BC)

What we have just discussed is easy for your students to get interested in. To give them a feel for what the five Platonic solids actually look like, there is nothing better than having models of these solids for them to play around with. As solid models are not easy to come by, it is a good idea to get your students to make models of these solids from paper. With this in mind, in Fig.4 we have given flat diagrams of the five regular polyhedra.

These could be copied on to some stiff paper and cut out along the outer edges. The cut-outs can then be folded along the inner lines and the sides pasted with thin strips of paper to make three-dimensional models. Those corners of the regular polygons that make up the faces of the models and which meet at a common vertex of the polyhedron are labelled with the same letter in our figure.

Now, getting back to exploring mathematics, here is an exercise for you.

E8) Prove that there can only be 5 regular polyhedra. Also ask your students to prove it. The models may come in useful for this.

How did you go about answering E8? Of course, you know that there are **at least** 5 regular polyhedra, the ones made in Fig.2. But, how do you know that **these are the only ones**? That is, how did you go about proving that any regular polyhedron is forced to be one of the five you know? Did you try to use anything you already know or have observed? For instance, did you notice that **at any vertex of a polyhedron there cannot be less than three faces**? One face is clearly not enough and two would only give rise to an edge.

Next, what do you know about the sum of the angles of all the faces at each vertex? Remember that each face has to be a regular polygon. Also, if you 'open up' the polyhedron, place all the adjacent faces in a plane, there need to be some gaps between the edges. So, shouldn't the sum of the angles be less than 360°? If the sum were exactly 360°, then all the faces would lie in one plane and there would be no corner of a 3D solid.

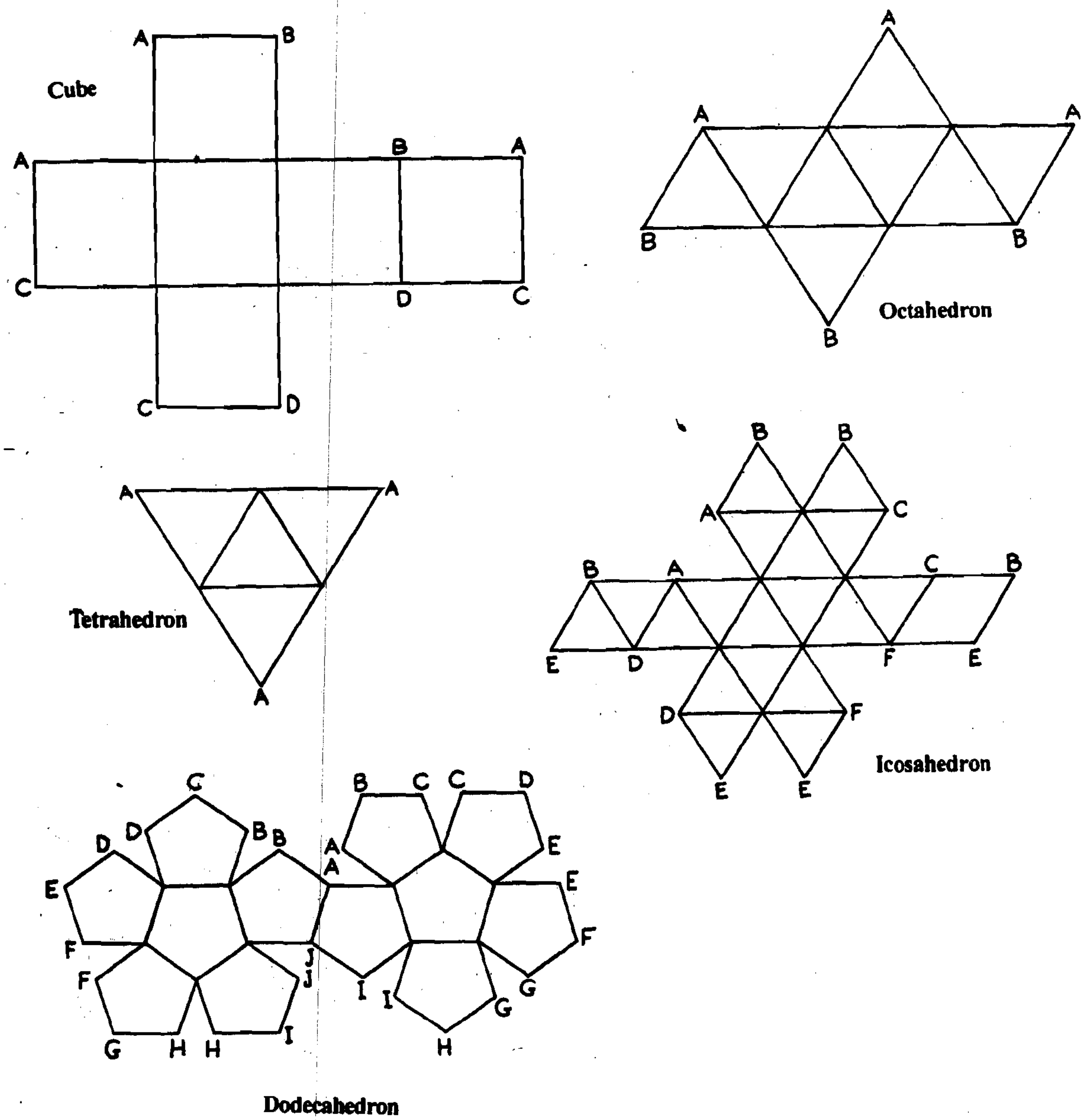


Fig. 4: Cut-outs to make models of the five Platonic solids

Now, given the two facts we have just noted, can **any** regular polygon be a face of a polyhedron? Since the sum of the angles of the faces at each vertex of the polyhedron has to be less than 360°, and since there must be at least three faces at each vertex, the angle of a face at the vertex must be less than 120°. This immediately restricts the regular polygons that can form faces of the regular polyhedra to be equilateral triangles, squares or regular pentagons (see E9 in Unit 2). Therefore, a regular polyhedron can only have these 3 types of polygons as faces.

Once you reached this stage, you probably thought about the various possibilities for the regular polyhedra. The simplest case is that of a regular polyhedron whose faces are equilateral triangles. We have already used the fact that the number of faces at each vertex must be more than two. They must also be less than 6, since each angle of the face is equal to 60°. The number of faces at each vertex of a regular polyhedron whose faces are equilateral triangles can therefore only be 3, 4 or 5. These correspond to the regular tetrahedron, octahedron and icosahedron, respectively.

Now consider the case of regular polyhedra whose faces are squares. The number of faces at each vertex can only be 3. (Why?) The corresponding solid is, of course, the cube.

By exactly the same arguments the number of faces at each vertex of a regular polyhedron whose faces are regular pentagons can only be 3. The corresponding solid is the regular dodecahedron.

You should check your proof to see that you have not made any logical errors, and there are no other possibilities. Once this is done, you have proved that there can only be 5 kinds of regular polyhedra.

Now try these exercises.

E9) Go back to the discussion on ' proof ' in Unit 2 (following E10). Then, note down the mathematical thought processes and the kinds of statements used in the proof above. Under which of the four stages of a proof listed in Unit 2 do they come? Are there any other stages or categories in the proof above that are not mentioned in Unit 2?

E10) We list some of the properties of the five regular solids in the table below. Ask your students to use the paper models to verify the entries in the table for each of the five regular solids.

Table 1: Properties of the regular polyhedra

Type of polyhedron	Faces are n-gons n	Number of faces F	Number of vertices V	Number of edges E	Number of faces at each vertex
Tetrahedron	3	4	4	6	3
Cube	4	6	8	12	3
Octahedron	3	8	6	12	4
Dodecahedron	5	12	20	30	3
Icosahedron	3	20	12	30	5

Then ask them if they see a relationship between F, V and E, and if so, to find it.

Let us now explore another area of spatial mathematics. While you are investigating it, keep thinking about the same broad questions that you kept in mind in the previous section.

3.5 STUDYING TILINGS

'Tiling' is the study of shapes that can be placed alongside each other to fill space completely **without leaving any gaps**, like the tiles covering your floor. If you look around you, you will see a variety of tilings — on floors, on walls, decoration pieces, etc. An example is given in Fig.5.

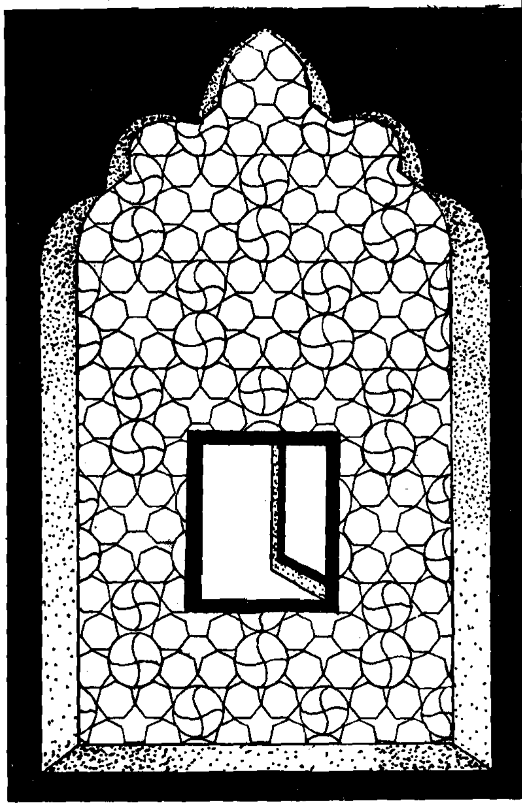


Fig. 5

What are the shapes that are usually used as tiles to fill the tilings? In two dimensions, we usually find squares or rectangles used as tiles. If a tiling is done by one kind of regular polygon of the same shape and size, it is called a **regular tiling**. Do you see regular tilings around you? The most common kind is the one made by squares.

What are the other kinds possible? Here is an exercise about this now.

E11) Prove that the only regular tilings are those made up of the equilateral triangle, the square and the regular hexagon. Further, note down the points you reflect on, the questions you ask yourself and the different routes you may follow while finding the proof.

How did you go about trying the exercise above? Did you physically take several equilateral triangles, say, and try to cover a surface with them? While doing so, did you notice that at any intersection in a regular tiling there must always be more than two tiles meeting? This concrete example may have also helped you realise that the sum of the angles of all the vertices meeting at an edge must be 180°. The sum of the vertices of the regular polygons meeting at other points will be 360°. This means that we can only have three equilateral triangles (or two squares) meeting at an edge. Also, we can have 6 equilateral triangles, 4 squares or 3 regular hexagons meeting vertex to vertex. This exhausts all possibilities for regular tilings. Therefore, there are only three regular tilings, all of which are shown in Fig.6.

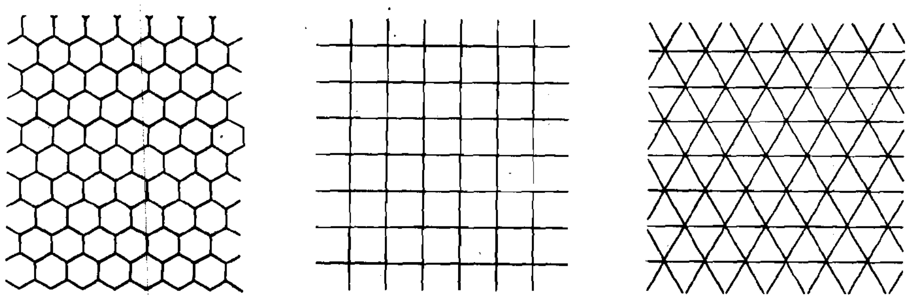


Fig. 6 : The regular tilings

If a mix of non-regular polygons are used in any tiling, then of course the possibilities are endless. The same is true if a mix of regular polygons of different sizes are used. In particular, the plane can be tiled completely by triangles or quadrilaterals of arbitrary shape.

A **tessellation** is another name for a tiling, which is used by artists more than mathematicians. Tessellations use either a single shape which may or may not be regular, or at most a few shapes, to cover the plane. The emphasis is on using shapes which look natural like birds, fish, horses, people, etc.. Through the following activity, you and your students can pick up some basic principles involved in creating tessellations, and make some of your own tilings.

Activity 1 (Making tessellations) : You need to start by establishing a regular grid on the plane. You can use triangles, squares, rectangles, parallelograms, hexagons, etc., to create a grid which covers the whole area you wish to work with. Suppose you start with a grid of squares. You can choose as your unit a 3×3 square. We know that periodic repetitions of this unit will tile the plane. (Why?)

Now, the secret of a tessellation is to remove parts of this square from one side and add it in a corresponding position on the opposite side of the square. In this way, although the shape of the unit changes, its total area remains the same. In the process you create cuts and wedges that fit into each other. (Why does this happen?)

So, suppose you remove a small square from the top left-hand corner of the unit figure (see Fig. 7(a)) and add it to the top right-hand corner. Similarly, remove another small square from the middle of the bottom of the figure and add it to the middle of the top. This, then, produces your basic motif shown on the left-hand side of Fig. 7(b).

Consider your original grid to be tiled by a set of the basic 3×3 squares and replace each such square by the motif you have just created. This will produce the pattern shown on the right of Fig. 7(b). Stretch your imagination a little, and you can consider this to be a tessellation of a horse and rider!

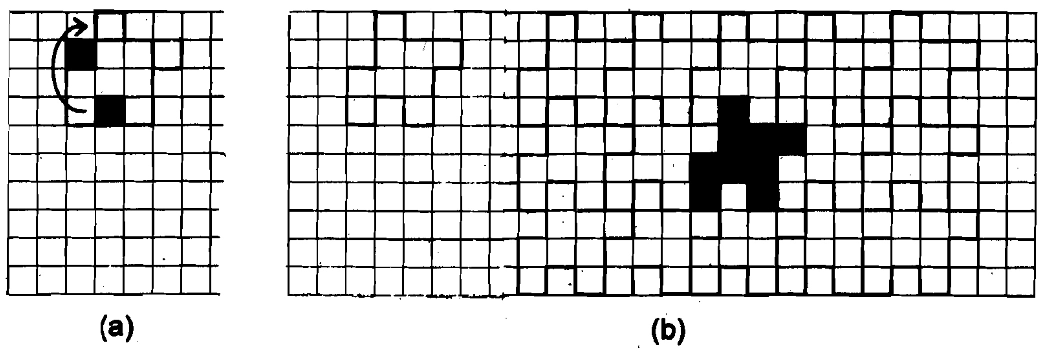


Fig. 7 : (a) Creating the motif.
(b) Tesselleting the plane with the motif to get a tessellation of a horse and rider.

To make a tessellation, we can add and remove any shape from the basic unit we choose. For example, starting with the same basic 3×3 tile, we can add/remove shapes as shown in Fig. 8(a). Then we get a basic motif that gives us the tessellation in Fig. 8(b).

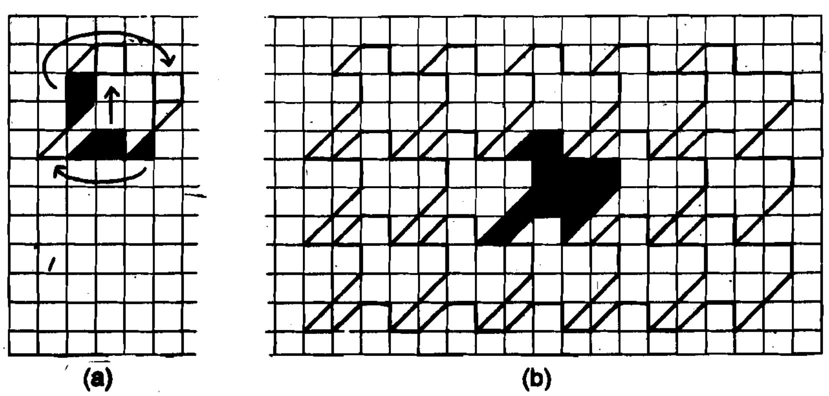


Fig. 8 : (a) A basic motif for a tessellation, (b) The tessellation of horses

The important thing to remember about creating a motif is

i) decide on the grid and the basic unit,

ii) you can remove any shape from the basic unit **provided** you add it back to the unit at the corresponding place on the opposite side to give rise to the new shape.

This can be done as many times as you please. The skill lies in creating a natural looking shape at the end. For example, in Fig.9 we show how, starting from a grid of parallelograms, you can proceed step by step to create a tessellation of roosting birds

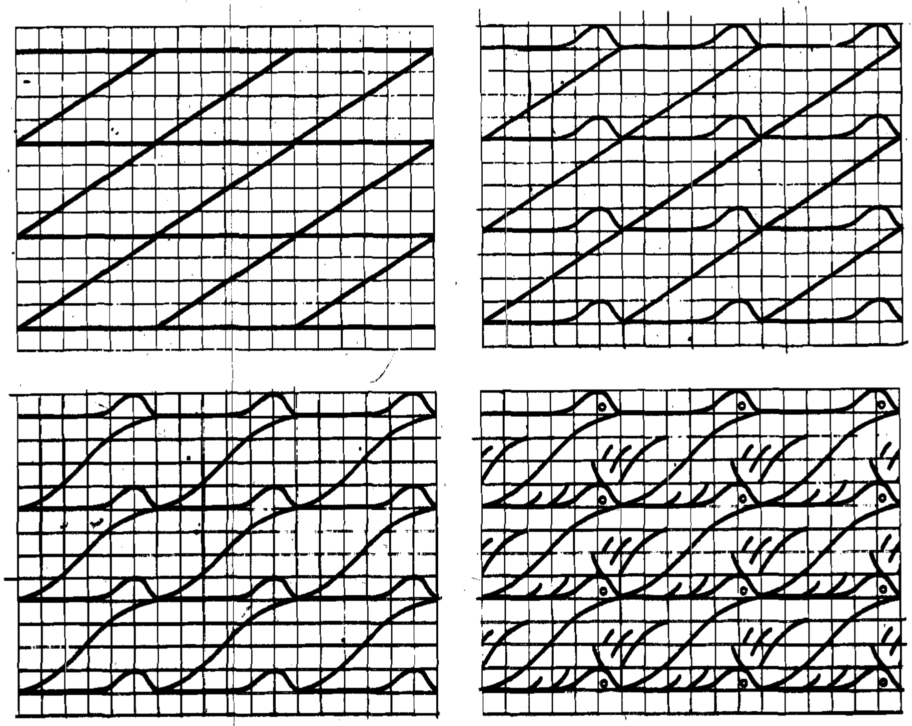


Fig. 9 : A tessellation of roosting birds

Why don't you try some exercises now?

E12) Create at least two tessellations using the steps we have just discussed. Also try out what we have said in this section with your students. What were their reactions?

E13) Ask children of Classes 9 or 10 to tile a plane with squares and regular pentagons, respectively. Note down the discussions that take place amongst them during this activity. What understanding does this give you of their mathematical thought processes?

The tessellations that we have considered so far make repeated use of just one basic form. This need not be so. We can always take our repeating motif and divide it into two parts such that each part looks like a separate natural shape. Escher, who was an acknowledged master, has used many basic shapes to give all kinds of tessellations of the plane (see Fig.10).

While creating tessellations, there is a notion of symmetry that is involved. We shall study this notion in detail in the next unit.

So far, in this unit we have carefully looked at the thought processes we go through while learning mathematics. Both the mathematical areas we have looked at in detail in this unit have been spatial, that is, space related. Now, let us look at the processes involved in solving algebraic and logical puzzles.

Regular Division of the Plane I (detail), 1957; woodcut, 9½ x 7⅛ in.

Fig. 10 : A tessellation by M.C.Escher

3.6 WORKING OUT PUZZLES

If you are given a problem like 12345 – 3249, you are likely to do it in a flash. This is because you have acquired the ability to apply a subtraction algorithm. Now, see how long you take to do the following subtraction :

In the problem below, each letter has been assigned one digit from 0 to 9. Find the numbers involved in the subtraction

$$\begin{array}{r} A\,B\,C\,D \\ -\ \underline{E\,B\,E\,B} \\ E\,D\,E\,B \end{array}$$

How have you gone about finding the digits involved? For instance, you may start with the possibility that D = 6, B = 3. Then D–B = B. But then, D and B are occurring again in the 'hundreds' column. And 3 – 3 ≠ 6. So, you would need to try another possibility for D and B. In this way, using logical arguments, what solution did you get? Note that there may be more than one solution to this problem.

Try these exercises now.

E14) While solving the problem above, what were the different aspects of mathematical thinking you applied?

E15) Find the operations K and K′ and the K digits represented by the letters in

ABKAB = ACC and FG K′FH = DE

E16) Try the following problems. Also give your students these problems to do. What problem-solving abilities were the children using in the process, and how did you find out? How different were they from the processes you used for solving them?

i) *There is a sequence of 16 numbers which reads the same from left to right as well as from right to left. Also, the sum of any 7 consecutive numbers in the sequence is –1, and the sum of any 11 consecutive terms is +1. Find the numbers.*

ii) *Ashrafi was convinced that her key had been hidden by one of her friends —Aarti, Birla, Kalyan or Megha. Each of these friends made a statement about this matter. But only one of these four statements was true.*

Aarti said, "I didn't take it."
Birla said, "Aarti is lying."
Kalyan said, "Birla is lying."
Megha said, "Birla has taken it."

Who told the truth?

E17) Ask your learners to think of more problems like the ones mentioned above. What were the puzzles/problems they came out with?

The purpose of asking you to engage with the problems above was two-fold. Firstly, we wanted you to have fun. We also wanted to help you focus on the processes that are used for solving them. If you are aware of these abilities being used, then you would agree that these are the abilities to be fostered in your learners. One way is to give them problems that they would enjoy and that would challenge them a bit. We end this unit with leaving you to think of various ways in which this can be done.

But first, let us see what we have covered in this unit.

3.7 SUMMARY

In this unit we have focussed on the thought processes involved in learning and doing mathematics, particularly while solving and creating problems. More specifically, we covered the following points.

1. Exploring any mathematical concept involves considering it in different ways. Solving a variety of problems related to this concept helps to build and consolidate one's understanding of the concept.

2. We stressed the importance of using a variety of ways for representing a problem situation. Developing mathematical maturity requires the learner to, among other things, move comfortably from one form of representation to another.

3. We looked at why there are only 5 types of regular polyhedra though there are infinitely many regular polygons.

4. We discussed what a tiling is, how many regular tilings there can be and how to create tessellations.

5. The point of studying polyhedra and tilings was to consider the processes involved in investigating mathematics.

6. We looked at interesting non-routine mathematics problems that entertain us and keep the brain ticking. The idea was to focus on the directions in which the thought processes were moving.

7. We asked you to work with your learners on the same lines and analyse their reactions. Through this, you would be able to gauge their use and understanding of mathematical thought processes.

3.8 COMMENTS ON EXERCISES

E1) While thinking of an example, remember the essence of a problem — it should force the learner to think mathematically. This does not mean merely retrieving learnt facts or applying algorithms unthinkingly.

E2) Pick an example of a concept or process you learnt, for example, the concept of limit. Note down what kind of exercises, activities and problems helped you to, develop your understanding of the concept, and in what way. Does 'limit' mean more to you than merely applying the algorithm for finding it? What visual aid did you need to understand when a limit exists, or otherwise? Which practical situations require you to use it? Did finding the answers to these questions help you understand this concept better? In what way?

Similarly, how would solving real-life and other problems related to a concept help your students to learn the concept?

E3) See material following the exercise.

E4) Pick up a problem that requires plenty of thinking. Do not hassle the children when you ask them how they have done the problem. They may not be very clear in remembering or explaining how they solved it. You could also have them sit in a group and try to work out a solution together. Their conversation while they think about the question would help you to understand the processes that they are going through.

If you discuss the stages with them, be sure to use simple language and small logical steps so that they can understand what you are talking about.

E5) Was their ability to use a variety of representations linked with their ability to articulate their thought processes? If so, in what way? What was the relationship, if any, between the child's ability to build representations and being comfortable with mathematics? Note down the other points that you find important.

E6) Here the focus is on helping children develop their ability to use a variety of representations. Accordingly they would need several opportunities to use the concept in different situations. Note down the kind of learning opportunities you can think of for these children.

E7) For each $n \geq 3$, we can define a regular n-gon. Draw them and see what happens as n becomes larger and larger. As $n \to \infty$, the n-gon tends to a circle.

What questions regarding relationships between different mathematical objects, pattern finding and generalisation did you think of in the process?

E8) Think of what a proof involves — first gather together what is known and what is assumed. Then see how you can use this to prove your result. The discussion following E8 will, of course, lead you there.

It may be useful, while you think, to try to actually make these polyhedra and see the implications of this concrete activity.

Ask your students to study these models and try and prove the statement. Were they able to do it? What did they say about this exercise, while discussing amongst themselves as well as with you? Which aspects of mathematical thinking were coming out through their remarks?

E9) For instance, there have to be 3 or more faces at any vertex of a polyhedron. Is this statement an axiom? Or is it based on assumptions? This statement follows logically from the definition.

'Each face has to be a regular polygon' also follows from the definition.

In this way, consider all the other steps in the proof.

E10) The number of faces F, the number of vertices V and the number of edges E of a regular polyhedron are connected by Euler's famous formula $F + V - E = 2$.

E11) One route is given in the discussion following E11. Think of other routes. Compare the thought processes and steps in the different solutions. In fact, think of all the regular tilings. Is the list very long? Try covering a book (any surface) with all these tilings one by one. Did you find a problem with some of them? While doing so, remember that you cannot change a shape of the tile in between.

Is the statement true for tilings which are not regular?

E13) Divide the children into groups of 6-8, depending upon the space available and the number of children. Explain to them what tiling means and let them proceed with the tiling exercises. Do not interfere in their thinking. Observe the discussions among them as they do this activity.

Analyse the discussions for their notions regarding symmetry, angles, vertex, etc. What other mathematical thought process can you study in this exercise?

E14) You probably first assembled various single-digit subtraction facts. Then, from them you chose the ones that may fit. Then, moving step by step, you would eliminate the non-possibilities, based on contradictions you got.

One solution is (A, B, C, D, E) = (2, 5, 3, 0, 1).

E15) Consider the first problem. Note down why the operation can't be subtraction. If it is addition, what value of A would give you A in the hundreds place in the answer?

If the operation is multiplication, what could A, B and C be? One solution is A = 1, B = 2, C = 4. Think of others.

You can try the second problem similarly.

E16) i) How are you going about this one? I started by trying out the sequence

$$1, -1, 1, -1, \ldots$$

This met the second condition, but not the first or the third. In this way, I tried a few more sequences till I decided to use algebra for dealing with this problem. So, using the first condition, my sequence became

a, b, c, d, e, f, g, h, h, g, f, e, d, c, b, a.

Then I used the second and third conditions to reduce the sequence to

a, a, c, a, a, a, c, a, a, c, a, a, a, c, a, a

Now, can you guess how I got a and c? Why don't you try and find the sequence? Maybe your solution agrees with mine. An answer I got was a = 5, b = −13. Are there any other possibilities?

What were the mental processes the children went through while reaching a solution?

ii) This problem can be solved in various ways. Of course, each way requires the use of mathematical logic.

So, let me begin by assuming that Aarti is telling the truth. Then Birla's statement is false, so that Kalyan's statement is true. But we have assumed that both Aarti and Kalyan can't give true statements. So, Aarti must be lying.

Now, let me assume that Birla is telling the truth. See if you find any contradictions with this assumption.

In this way, checking the various possibilities, moving logically step by step, I arrived at the solution. Can you see the mathematical thinking involved in this problem?

E17) Did your learners come out with other kinds of conditions to determine a sequence? Did they come out with minor generalisations, or radically different conditions? What kind of other problems like E16(ii) did they create? Did you ask other students to solve them to see if the newly posed problems made sense? What was the general reaction in the classroom to this exercise?

Let us now go back to an earlier part of the interaction regarding our understanding of number, and look closely at what it tells us about our mathematical thinking.

What is interesting in this part of the conversation is how clear it is that **we abstract the notion of numbers from using them as adjectives**. When we talk of a number, we are essentially referring to a certain physical property of a set. Thus, when we talk of the number 'two', we could be referring to any collection of objects that can be put in one-to-one correspondence with, for example, the number of sleeves in a shirt. Thus, we say that a coin has *two* sides (each side corresponding to one sleeve), most humans have *two* eyes, a line segment has *two* end points, and there are (usually!) *two* sides to an argument. We abstract a common property of these different concrete objects, namely the number of objects in each of them. This is the number that we call 'two'. As in the previous case, having abstracted the property and understanding what 'two' means, we can now think of the number two without referring to the objects from which we derived the concept. It also has completely abstract and formal relationships with other numbers like 6, $\sqrt{2}$, 2i, etc., and with other abstract mathematical objects (e.g., rectangles).

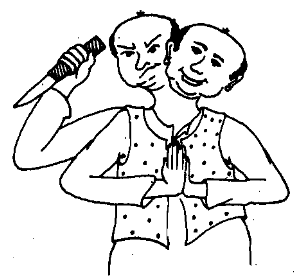


Fig.1 : Do all of us have two faces?

So, **abstracting a concept** is the ability to look at several particular examples of the concept, find what is common to them, separate that common property from the objects and look at the property as something on its own, having an independent existence. This existence is in the world of mathematics. This world is made up of such abstract objects. These objects generate further abstract concepts and relations between such objects.

We acquire our understanding of these abstract objects in two ways. One way is the way we develop our concept of number or of shape. This consists of a process of careful observation and analysis of different objects, noticing a certain property common to these objects and separating the property from the objects from which it was abstracted. This property, then, becomes an object of study as a concept. This is true of several non-mathematical concepts (like colour) too. In the following exercise we ask you to mull over this process.

E1) Identify two other concepts in mathematics and two from non-mathematical areas that arise through a process of abstraction. Explain how this abstraction takes place.

As we have just seen, several mathematical and other concepts are derived by abstracting them from particular instances. Would you be able to abstract the notion of a point or a line by this process? To answer this, let us first consider a point. In school, we are told that a point marks a position in space and that it is dimensionless. How, then, do we represent a point? Even the tiniest dot in space has some dimension. So, we can't abstract the concept from particular concrete instances of the concept, because ideally there cannot be any concrete representation of a point. There is no easy way out of this difficulty. The way out for mathematicians was to adopt the convention that a small dot would represent a point. Thus, on paper we often mark points like the origin O, while in our minds we know that a point cannot exist in reality. It is an abstract entity present only in our minds.

Similar situations arise with many other geometrical concepts as well, such as a line, a segment, or a ray. All these abstract concepts exist because of certain accepted rules and conventions in the world of mathematics. These rules are called **axioms**. And, to be able to deal with such abstract concepts, we choose conventions for representing them symbolically. Once we define one convention, we use it to define conventions for the other objects that exist only in our minds. This is another kind of abstraction. It is by this other form of abstraction that Euclid stated that "a point moves to describe a line". This line, an abstraction itself, moves to generate a surface, and so on.

The essence of mathematics lies in dealing with these forms of abstraction. In the next few sections we shall talk about what we mean by 'dealing'. For now, try this exercise.

E2) Explain what the difference is in the two forms of abstraction we have just discussed, with examples that haven't been given so far.

In this section we have discussed a defining characteristic of mathematical thinking. This thought process moves along a path of generalisation. In fact, generalisation is the way the world of mathematics grows. Let us see how.

2.3 PARTICULARISING AND GENERALISING

One of the most important mathematical thought processes is that of generalisation. We do it in real life all the time. For example, consider the way we formulate the concept of 'tail' in our minds. The process involves observing the tails of some objects, such as a horse or a cow. We also notice that different tails may look different, but all of them are called 'tails'. So, our initial concept of a tail may be that it is that part of an animal that is seen at the back of the rest of the body. Then we extend this concept to the appendage at the rear end of a bird or a fish. We may extend this notion further and modify our image of a tail to include the tails of aeroplanes and kites, thus generalising our notion from living to non-living creatures also. As we examine more objects that have a tail, we continue to generalise this notion. Ultimately, we arrive at an image of a tail that may not include some of the specific features of the tails of the different objects that we are considering, but will include common features of all of them.

We engage in this kind of generalisation all the time in our daily lives in order to formulate a concept. The process is useful in extending our activities — for example, we can generalise our observations about plant growth in order to grow new plants; and, we are able to generalise our experiences of a child's mental development in order to construct learning and teaching methodologies. In the study of mathematics, the process of generalisation assumes a special significance. It helps us to understand the structure of specific mathematical objects and to build further knowledge upon existing structures. But what is even more significant is the fact that often such extension of knowledge may become impossible without going through the process of generalisation.

In mathematics, we find generalisation occurs in different contexts — we generalise to arrive at definitions of new concepts, as in the case of the definition of quadrilaterals. We generalise procedures, for example, the procedure to add two polynomials. We generalise results to new sets of mathematical objects, such as extending the statement 'the sum of the four angles of a square is 360 degrees' to the statement 'the sum of the four angles of a quadrilateral is 360 degrees'. And, of course, algebra is a generalisation of arithmetic, where the use of variables helps us to extend our study and use of numbers in new ways.

In this section, we study generalisation in different mathematical contexts. For instance, think about the way most of us develop the general concept of a polygon. We get to know triangles of various shapes and sizes. We get to know rectangles, squares and other quadrilaterals. We look around us and see patterns having pentagons. We wonder — can we have figures having 20 sides, 50 sides, 77 sides, and so on? If so, what would their properties be? Is anything common to all these figures? In this way we develop our concept of a polygon as a closed figure having three or more sides. This is an example of generalisation. With such generalisation we also generalise related notions like those of area, perimeter and other concepts associated with polygons.

Usually, to understand what the general concept is, we begin learning about it by observing and studying properties of particular cases. For instance, by studying the areas of squares, parallelograms or triangles, we may naturally acquire the general concept of 'area of a polygon'.

For another example, try and recall the way you acquired your understanding of 'place value'. Initially it developed in the context of 'base 10', i.e., in the decimal system. Then you may have heard that computers function with a binary system, i.e., base 2. Did this make you wonder: Given any number, can I write it in other bases, say base 5, base 60, base 12, or for that matter, base n $\forall$ n $\in$ **N**? This process of 'wondering' is also called **'making a conjecture'**. Since the conjecture is about a situation in greater and greater generality, we consider these thought processes as an example of generalisation from several particular cases. However, be **warned** that at present we do not know if our generalisation is mathematically acceptable or not. (This shall be discussed in the next section, and in the last block of this course.)

Before going further, why don't you try some related exercises?

E3) How would you write 'hundred' in base 5, base 50, base 100 and base 101?

E4) If 303 (in base 10) is written as 213 in base n, find n.

Now suppose you have proved your conjecture. Then you know how to write any number in base n, where n $\in$ **N**. You have **a generalised procedure**. Therefore, if you are required to write a number in the hexadecimal (i.e., base 16) system, you apply your procedure for the particular case n=16. We call this process **particularisation** or **specialisation.**

You could do this whole exercise of generalising and particularising for concepts or procedures that your students are learning. Such examples can be used to help your learners understand the processes of generalisation and particularisation while studying these concepts/processes/skills. In this way, they will realise that while understanding or creating mathematics, we are moving **from particular to general and from general to particular** all the time.

In fact, to understand a concept, it helps the learner to gradually construct it in her mind. This is done through experiencing concrete examples, studying several particular cases and gradually grasping the generalised concept. Though many of us accept this fact in theory, how often do we find this happening in our classrooms? Not commonly. In fact, it is more common to find teachers introducing the students to a concept by giving them the definition in all generality, and expecting the children to remember it. Even when examples to illustrate the definition are given, they are not varied enough. Some teachers introduce the children to a concept by giving some particular examples in the textbook or on the board, quickly followed by the general definition.

Neither kind of teaching helps the young minds in acquiring the concept because children require more opportunities to think about and use the concept concerned. They also need to think about examples **and non-examples** of the concept on their own. This gap between teaching and learning is very evident in geometry where, for example, students learn about different polygons without building any links among them. This is one reason why so many people wrongly believe, for instance, that a square is not a rectangle!

The point we are emphasising here is that, in most cases, the move from particular to general cases represents a move towards a higher cognitive plane. The children need to, first, become somewhat familiar with a concept in particular cases by dealing with plenty of concrete examples. They need to build links between these specific cases and the essence that they have abstracted. Only then can they move towards understanding the concept in all its generality. We teachers need to understand this if we don't want concepts to be reduced to mere definitions, which are rote learnt.

A word here about generalising algorithms. For example, the algorithm for adding elements of **Q** is nothing but a **generalised procedure** for adding **any** two fractions. (In fact, we can identify two levels of generalisation in this process. At one level, we have evolved a method that works for **all** rational numbers. At a different level, we are also generalising the idea of addition — we are now adding not only integers, but also parts of integers.) Think of any algorithm in mathematics — may be one for finding the roots of a quadratic equation, or that of finding the solution set of a system of equations. Each of these algorithms is **a generalised step-by-step procedure.** Each such algorithm has an underlying logic. What we mean by generalisation in this case is that the logic of the algorithm is not restricted to just a few particular cases. It works in exactly the same way for any member of the class. You have already seen this in the case of the algorithm for writing a number in a system with any base. **Your learners also need to understand the logic behind the working of the algorithm,** the mathematics of it. Otherwise, the process will reduce to a meaningless mechanical procedure for your students.

Why don't you try some exercises now?

E5) Give an example, with justification, of a generalised procedure in your students' daily lives.

E6) Give an example of movement 'from general to particular' taken from your daily life. Also explain why you chose that example.

E7) Not all generalisations related to mathematical objects are valid. Give an example to show this, taken from the secondary school level mathematics.

We have seen that the world of mathematics grows through the process of generalisation — of concepts and processes, and relations between them. When we are generalising concepts or algorithms, we need to ensure that the generalisation is valid. There are broadly two forms of reasoning we use for this purpose, which we shall discuss next.

2.4 WHAT IS A PROOF?

We will discuss methods of proof in detail in Block 5 of this course.

In the previous section we noted that doing mathematics involves generalising on the basis of observations of particular cases. Once we have noticed patterns in these instances, we make inferences based on these patterns. Thus, you may infer that June is the hottest month of the year (if you live in Punjab, say). Or you may infer what a one-year-old child will look like based on your observations of several children of that age. You may see a cow eating grass, then another one doing the same thing and infer that all cows feed on grass. This form of drawing inferences based on repeated similar experiences is called **inductive logic**. The form of this logic that we use in mathematics is called **mathematical induction**. This principle **uses inductive logic to formulate a conjecture** based on observed patterns. For instance, you may observe that $1^3 + 2^3 = 9 = 3^2$, $1^3 + 2^3 + 3^3 = 36 = 6^2$, and so on. You may also notice that $3 = 1+2$, $6 = 1 + 2 + 3$, and so on. Based on these particular cases, you may conjecture that $1^3 + 2^3 + \ldots + n^3 = (1+2+\ldots+n)^2$.

The other form of reasoning is **deduction**, that is, the use of deductive logic. According to this logic, we use known facts to arrive at a conclusion. For instance, knowing that there is severe water shortage in a given town, you can deduce that the price of drinking water will be high over there. In mathematics we apply deductive logic all the time — when we use known results, definitions, axioms and rules of inference to prove or disprove a statement.

You know that in mathematics when we claim that a statement is true in general, what we really mean is that **it holds true, without exception**, in all cases in which the conditions of the statement are satisfied. This means that mathematically speaking, it is not enough to show that the particular statement is true in several different cases (even if the number of such cases is very large); what we must be able to do is to actually show, through a process of inductive and/or deductive reasoning, that the statement is valid in all the cases where the conditions of the statement are true. This 'showing' constitutes a 'proof '.

Fig.2

It is no exaggeration to say that **the idea of proof is the single most important idea in all of mathematics.** Consider any **mathematical proof** of a statement. It consists of one or more steps, deduced from earlier steps or accepted facts, which make up **mathematically acceptable** evidence to support that statement. Let us look at an example to see how inductive and deductive logic go hand in hand to give a proof in mathematics.

Suppose I ask you to find the sum of the interior angles of any convex polygon. How do you go about trying to answer this question? You may already know that the sum of the interior angles is related to the number of sides of a polygon in some way. You would probably begin by looking at special cases. You already know that this sum is 180° for a triangle and 360° for a quadrilateral. Suppose you also know that for a pentagon this sum is 540° and for a hexagon it is 720°. You could try drawing a chart like the following one to find some pattern:

Number of sides of polygon	3	4	5	6
Sum of the interior angles (in degrees)	180	360	540	720

Making charts is often a good way of looking for patterns.

After a little thought, you may notice that each number in the second row is a multiple of 180. You may then decide to write each number down as a multiple of 180. Thus, you will get: $180 = 1 \times 180$, $360 = 2 \times 180$, $540 = 3 \times 180$, $720 = 4 \times 180$. Are these numbers related to the number of sides in each case? In other words, is there a common rule relating 3 to 1, 4 to 2, 5 to 3, and so on? Some reflection on this question may lead you to infer that the sum is $[(n-2) \times 180]°$, where n is the number of sides of the polygon. But how would you check whether your guess (or **conjecture**) is right? After all, it may happen that this result may not hold if you take a 20-sided polygon, or one with 62537 sides. You would need to find a proof to show that the statement 'the sum of the interior angles of an n-sided polygon is $(n-2) \times 180$ degrees, for any $n \geq 3$' is valid. You would do so **through a series of steps, each of which is deduced logically from the previous ones**. This would constitute the **proof of the statement**. There can be several proofs. Let us consider one of them.

As you may remember, to logically derive a result we must accept certain definitions and/or axioms and/or earlier proven statements. In this case, two statements that we shall assume are
' The sum of the interior angles of a triangle is 180°', and
'The sum of the angles around a point is 360°' (as illustrated for one case in Fig.3):

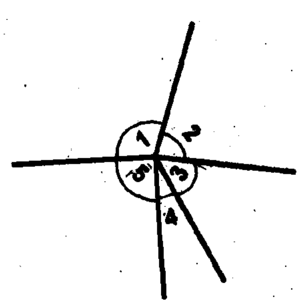


Fig.3

Consider any n-sided polygon and take any point, say O, inside it. Join this point to each of the vertices of the polygon. As there are n vertices, the interior of the polygon gets divided into n triangles. (In order to understand this picture more clearly, we could even draw a polygon and make the necessary construction as in Fig.4. (However, we must **remember that this picture (or any) is only an aid to see the logic of the proof.** Sometimes a picture could give you an incomplete or wrong understanding of the general situation.)

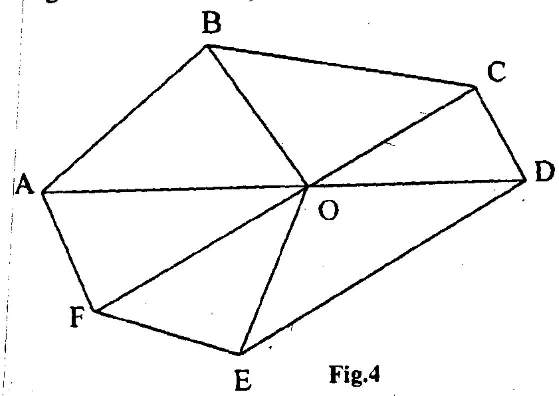


Fig.4

Now, for each of the n triangles, the sum of the angles in it is 180°. Since there are n triangles, the total sum of all the angles inside this polygon is $(n \times 180)^\circ$.
But the total sum of the angles inside the polygon is the sum of the interior angles of the polygon plus the angles around the point O. Since the sum of the angles around O is 360°, the sum of the interior angles of the polygon is $[(n \times 180) - 360]^\circ = [(n-2) \times 180]^\circ$. (Remember, in the picture $n = 6$, but we are actually dealing with any $n \geq 3$.)

The series of statements above constitutes a ' mathematical proof ' for the stated result. In it, each step follows logically from the preceding step and/or one of the results that we assumed before we began this proof. This method of reasoning is what is called 'deductive logic'. Thus, here, by a piece of deductive logic, we have actually shown that what we had **inferred** through inductive logic is indeed true in each and every case.

Here are some exercises for you now.

E8) Go back to the conjecture made earlier, that $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$ for every $n \geq 1$. Give a proof by the **principle of induction.** While doing so, explain which part is using inductive logic and which part is using deductive logic.

E9) Prove that each interior angle of an n-sided regular polygon is $\left[180 - \frac{360}{n}\right]^\circ$ for $n \geq 3$.

E10) What strategy would you use for inculcating in your students the ability to prove/disprove statements?

Let us, now, take a brief look at what we have just said about proofs, namely, proving a mathematical statement involves the following:

- A general statement about a certain class of objects that satisfy a set of conditions. This statement may be formulated on the basis of observation of patterns found in particular cases, or on the basis of mathematical intuition, or on some other basis.

- The objective is to show, through deductive reasoning, that the given statement is true in all cases where the conditions of the statement are valid.

- What we could use to achieve our objectives are one or more statements, which we call **premises**. These premises can be of four types :

  i) a statement that has been proved earlier;

  ii) a statement that follows logically from the earlier statements given in the proof;

  iii) a mathematical fact that has never been proved, but is universally accepted as true, that is, an axiom;

  iv) the definition of a mathematical term.

- The proof of the statement, then, consists of these premises.

Once we successfully show that the given statement is valid, we say that our statement has been proved.

As we see above, proving any statement about a given collection of objects mathematically involves proving it for **each and every object** in the collection. This means that a statement about a collection of objects is false if it does not hold for even one case in the collection. So, one way to **disprove** a mathematical statement (i.e., prove that it is false) is to find one example of an object that satisfies the hypotheses but not the conclusions.

For an example, consider the statement: Every continuous function is derivable. To disprove this statement, we only have to find one example of a function that is continuous but not derivable. You know several such functions.

Try an exercise now.

E11) Give one example each of a true mathematical statement and a false one related to 3D. Also prove that these statements are true and false, respectively.

You may wonder whether the process of proving a statement that we have outlined above is the only way of doing so. How about "visual proofs"? For example, consider the following proof for the statement $1 + 3 + 5 + \ldots + (2n - 1) = n^2$, where n is a natural number.

$$\begin{array}{c|}\hline 0\end{array} \qquad 1 = 1^2$$

$$\begin{array}{cc|}\hline 0 & 0 \\ \cline{1-1} 0 & 0\end{array} \qquad 1 + 3 = 2^2$$

$$\begin{array}{ccc|}\hline 0 & 0 & 0 \\ \cline{1-2} 0 & 0 & 0 \\ \cline{1-1} 0 & 0 & 0\end{array} \qquad 1 + 3 + 5 = 3^2$$

$$\vdots \qquad\qquad \vdots$$

$$\left.\begin{array}{cccc|}\hline 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \cline{1-2} 0 & 0 & \cdots & 0 \\ \cline{1-1} 0 & 0 & \cdots & 0\end{array}\right\} \qquad 1+3+5+\cdots+(2n-1) = n^2$$

The truth is that though such visual evidence can be **useful as an aid** for proving the relevant statement rigorously, mathematicians do not accept it as proof. This is because we have to remember that in mathematics, what we demand of a proof is that **it should be valid in all situations where the conditions of the statement we are proving are valid**. It would often be quite impossible to visually consider all the possible situations in which the conditions of a statement are true. In fact, what is even worse is that we may draw a diagram in which a particular statement is true and not even realise that there are other possible situations where all the conditions of the statement are satisfied and yet the statement is actually false.

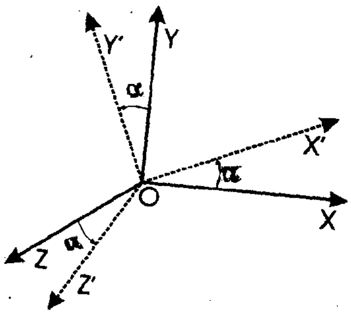


Fig.5

For example, recall what happens when children represent a rotation of three mutually perpendicular axes in three-dimensional space on paper. Very often they show all three axes as having rotated through the same angle α (see Fig.5), something that is not possible (as will be stressed in Unit 7).

If the lines are all in a plane, this is possible, but not otherwise. And this false generalisation has come about entirely because of the examples through which our mathematical intuition was built up.

We can find many other such examples related to functions and other topics. The other point that comes out from these examples is that if we find that we are arriving at a result that appears to be going against our common (and mathematical) sense, then we probably need to pause and re-examine our work carefully. But, sometimes our intuition or common sense may be **wrong** because this may be limited by what one sees in a few particular instances. That is, our generalisation from particular instances may be wrong.

Here's a related question for you.

E12) Give an example of mathematical concepts or processes being understood by your students due to excessive weightage given by them to visual aids.

Let us end this section with a brief overview of what mathematics is. It is a world of abstract objects and relationships between them, based on undefined terms and axioms about them. It is extended further by the processes of generalisation, abstraction and some laid down rules of mathematical logic. What is very important is that everything has to be consistent with what is known earlier.

Therefore, if even one of the axioms is changed, the whole theory that is built around it will change. A very good example of this is given in the following section.

2.5 CHANGE A POSTULATE, AND THE WORLD CHANGES!

"From nothing I have created another new world".

(Janos Bolyai in a letter to his father.)

Most people are unaware that around a century and a half ago a revolution took place in the field of geometry that was as scientifically profound as the Copernican revolution in astronomy and, in its impact, as philosophically important as the Darwinian theory of evolution. "The effect of the discovery of hyperbolic geometry on our ideas of truth and reality has been so profound," writes the great Canadian geometer H.S.M. Coxeter, "that we can hardly imagine how shocking the possibility of a geometry different from Euclid's must have seemed in 1820." Today, however, when it is known that the space-time continuum is closely related to the non-Euclidean geometries, some knowledge of these geometries is an essential prerequisite for a proper understanding of relativistic cosmology.

Mathematics is a deductive system in which one starts from some definitions, some undefined terms and some self-evident truths (which may be based on experience) called **axioms**. Using these as a basis, we move to further results by a process of deductive logic. This is perhaps most evident when we study Euclidean geometry, starting in high school.

Fig.6: Euclid, who produced the definitive treatment of Greek geometry and number theory in his 13-volume 'Elements' around 300BC.

2.5.1 Euclid's Postulates

As in all mathematical theories, Euclid built a theory involving some abstract objects and relationships between them. Some of the objects are undefined, for example, 'point', 'line', etc. Others are defined in terms of these objects. Of course, the effort is to keep the number of undefined terms to a minimum.

Each theorem in Euclid's geometry is proved from some preceding results. Of course, he started with a set of five assumptions about the undefined terms, which are the axioms or **postulates** of the theory. Any set of statements can be laid down as postulates so long as they do not lead to any logical contradictions or inconsistencies. Obviously, the fewer the postulates, the better. Mathematicians try to derive more and more theorems from fewer and fewer postulates.

Euclid's five postulates are:

1. Given two distinct points, there is a unique straight line that passes through them.
2. A line segment can be prolonged indefinitely.
3. For every point O and every point A distinct from O a circle can be constructed with centre O and radius OA.
4. All right angles are congruent to each other.
5. **The Parallel Postulate** (sometimes called **Playfair's axiom**) **:** If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

Remark: The fifth postulate can also be worded as: For every line l and for every point P not lying on l, there exists a unique line m through P that is parallel to l.

In the first of his thirteen books, Euclid made no use of parallel lines, defined as lines in a plane that do not meet. He proved several theorems and propositions without using his fifth postulate. Later mathematicians have added to this number, and together they are now known as **absolute geometry**, a term first used by Janos Bolyai.

It was almost as if Euclid sensed that his fifth postulate was on a different footing from the other four. He may have not been totally sure about whether or not it could be derived from the other four. In fact, Euclid himself proved that if AB is any straight line and P is a point in the plane of AB but not on it, then **at least** one line parallel to AB can be drawn through P. However, he could not prove that there is only one such line. Had he proved this, his fifth postulate would not have been required.

Why don't you try an exercise now?

E13) Show that the two ways of presenting the 5th postulate given above (Pt.5 and the remark) are equivalent.

Several mathematicians, since Euclid, have tried to prove the uniqueness of the parallel line passing through P using only the other four postulates and the theorems derived from them. But no one was successful. However, these efforts led to a great achievement — the creation of several non-Euclidean geometries. This is considered a landmark in the history of thought because till then everyone had believed that Euclid's was the only geometry, and that the world itself was Euclidean. Now the geometry of the universe we live in has been demonstrated to be non-Euclidean.

In the process of creating different geometries, two important ideas have been established. They are

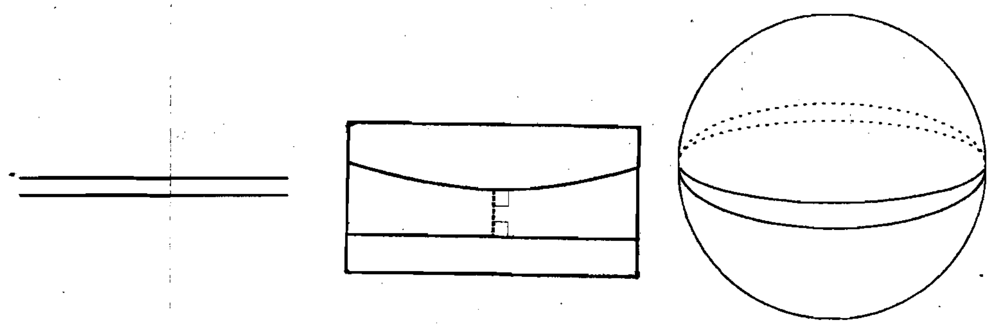
i) there is no single 'correct' geometry; and
ii) mathematical theories are not necessarily real.

Let us take a glimpse of some of these geometries.

2.5.2. Non-Euclidean Geometries

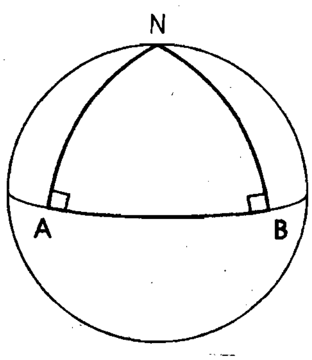
There are several non-Euclidean geometries now. Each of them is built on Euclid's postulates, except for the second one or the fifth one. All these geometries fall into one of two categories: hyperbolic or elliptic. Hyperblic geometry was discovered by Gauss, J.Bolyai and Lobachevsky. Elliptic geometry was discovered by Riemann.

Consider two straight lines drawn perpendicular to another straight line AB at A and B on the same side of AB. In Euclidean geometry, the mutual distance between the two straight lines will remain constant. In hyperbolic geometries, the two straight lines will grow further apart (as in Fig. 7(b)). In elliptic geometries they will come closer together (as in Fig. 7(c)).

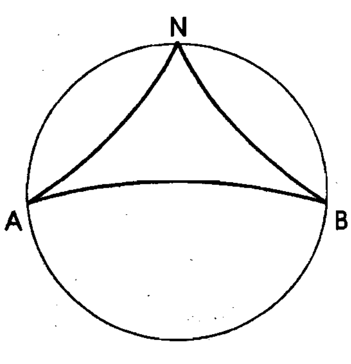


(a) (b) (c)

Fig.7: 'Parallel lines' in a)Euclidean geometry, b) hyperbolic geometry, c) elliptic geometry.



(a)



(b)

Fig.8: The sum of the interior angles of the triangle NAB
(a) is greater than 180° in elliptic geometry,
(b) is less than 180° in

In hyperbolic geometries, if the perpendicular at A is replaced by a straight line making a slightly smaller angle with AB, then this new straight line will at first converge towards the straight line perpendicular to AB at B, come to some minimum distance and then diverge. Therefore, Euclid's fifth postulate is no longer valid in this case.

In elliptic geometries, the parallel postulate (in the form stated by Euclid) is satisfied trivially, but his second postulate is violated because now every straight line closes on itself like a circle. Note that **in any elliptic geometry, any two straight lines, both of which are perpendicular to a third straight line, intersect.** This means that these three straight lines form a triangle. Therefore, **in elliptic geometry the sum of the interior angles of a triangle must be greater than two right angles** (see Fig.8).

On the other hand **in hyperbolic geometry the sum of the angles of the triangle will be less than two right angles.**

Why don't you try an exercise now?

E14) What activities would you give your learners to help them see the differences pointed out between hyperbolic and Euclidean geometry?

As a concrete example of a non-Euclidean geometry, let us now consider the geometry on the surface of a sphere, called **spherical geometry**. This is one example of an elliptic geometry.

2.5.3. Spherical Geometry

Let us imagine that the surface of the earth is a perfect sphere and we are restricted all the time to move only along this surface. If we take as our definition of a straight line that path which has the shortest distance between two points, then a straight line joining any two points on the surface of the sphere is an arc of the great circle passing through these points (see Fig.9). Thus, on the surface of a perfectly spherical earth, the equator and all the lines of longitude are great circles, i.e., straight lines. Since these great circles are of finite circumference, the straight lines cannot be of infinite length. So, **Euclid's second postulate is not valid for spherical geometry.**

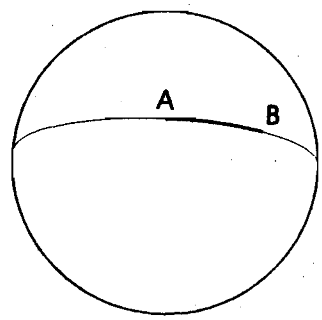


Fig.9

A number of corollaries follow from these statements:

1. Since all the lines of longitude are perpendicular to the line of the equator, they must all be parallel to each other. Yet they all intersect at the North and South poles. Therefore, an infinite number of parallel lines, all distinct from each other, can pass through the poles. Similarly, they can pass through any point on the surface of a sphere.

2. Other than the equator, none of the lines of latitude are great circles. Therefore, none of these lines are straight lines. Consequently, they cannot be considered to be straight lines on the surface of the sphere.

3. Since two lines of longitude intersect at the North and the South poles, between them they enclose a region. This shows that in spherical geometry, unlike in Euclidean geometry, two straight lines can enclose a region between them.

4. Consider two lines of longitude NAS and NBS which are perpendicular to each other, i.e., $\angle ANB = 90°$. Here A and B are the points of intersection of these lines with the equator. Then $\angle NAB = \angle NBA = 90°$. Further each of the sides of the triangle NAB, namely, NA = NB = AB = $1/4^{th}$ the circumference of a great circle. Thus, triangle NAB is an equilateral triangle on the surface of the sphere. So, we immediately have two results that are different from Euclidean geometry.

   i) First, each angle of an equilateral triangle in spherical geometry is 90°, and not 60° as in plane geometry. An immediate consequence of this is that **Pythagoras' theorem is not meaningful in spherical geometry.**

   ii) Second, we have the result that **the sum of the interior angles of a triangle on the surface of a sphere is greater than two right angles,** instead of equal to two right angles, as in plane geometry. Incidentally, this second result establishes that spherical geometry is an example of elliptic geometry since the sum of the angles of a spherical triangle has been shown to be greater than two right angles.

Some more features are given in the following exercises.

E15) Show that the sum of the angles of a spherical triangle can vary between π and 3π, the actual value depending on the triangle.

E16) Show that, in spherical geometry the ratio of the circumference of a circle to its diameter can vary between π and 2.

E17) Imagine you are standing on the Earth, and you walk one mile due South, then one mile due east, then one mile due north and find yourself back at your starting point with a bear staring you in the face. What colour is the bear?

With this we end our short discussion on one of the most exciting mathematical advances of the 19^{th} century. We shall look into other aspects of mathematical thinking in some more detail in the next unit and Block 5. For now, let us summarise what we have done in this unit so far.

2.6 SUMMARY

In this unit we have covered the following points.

1. We have seen that the world of mathematics is made up of abstract objects, and relations between these objects. There are certain rules and conventions that we agree that these objects will follow.

2. The essence of mathematical reasoning is generalising on the basis of patterns observed in particular instances. These generalisations should be valid. Validity means that they should hold true **for every case** that fits the conditions under which the generalisation is made.

3. We have seen what a proof is in mathematics and what disproving a statement involves.

4. We have studied an example of the implications of changing the basis of a mathematical theory, and hence developing new theories which are consistent within themselves. The example presented here is that of Euclidean and non-Euclidean geometries.

2.7 COMMENTS ON EXERCISES

E1) For example, mathematical concepts could be 'closed and open figures'. Examples of non-mathematical ideas could be 'heat' or sharpness.

Think about how each of these concepts develop in our minds. As infants, we don't know what 'heat' is. What kind of experiences gradually make us understand this concept? How we learn to compare and find out which is hotter?

E2) One form of abstraction is abstracting from the concrete. For example, from pens, ball pens, pencils, etc, the idea of things that write is abstracted. Think of more examples of this kind of abstraction.

The second kind of abstraction is abstraction from ideas that are not actually available in reality. For example, abstracting the concept of negative numbers and imaginary numbers.

E3) 100 (base 10) = $4 \times 25 + 0 \times 5 + 0 \times 1 = 400$ (base 5)
= 20 (base 50)
= 1 (base 100)

For writing in base 101, you need to first have 101 symbols, say $a_0, a_1, \ldots, a_{100}$. Then
100 (base 10) = a_{100} (base 101).

E4) We know that $2n^2 + n^1 + 3n^0 = 3 \times 10^2 + 0 \times 10^1 + 3 \times 10^0$. Solving this for n, we get n = 12.

E5) For example, if one knows how to make one kind of 'daal', the same procedure of using a pressure cooker to cook, can be used for cooking other 'daals' as well. The procedure for cooking a vegetable can also be similarly generally developed. You should think of more examples of procedures that are generalised, including some from mathematics.

E6) We know that, in general, people smile when they are pleased or when they want to be friends. Therefore, in particular, when we meet a new person and he/she smiles at us, we assume that the person wants to be friends. This is an example of particularisation.

E7) There could be many examples in which the generalisation could be wrong. For example, since every linear polynomial has a real root, many children generalise this fact to quadratic and cubic polynomials wrongly. Give other examples of possible generalisations that are erroneous.

E8) The steps are

1) Writing P(n) $\forall\ n \geq 1$.
2) Checking that P(1) is true. (This is called the basis of induction, though its proof is by deductive logic.)
3) For some $m \geq 1$, assuming P(m) is true, and then showing P(m+1) is true. (This is called the inductive step, and again, it is proved by deductive logic.)
4) Hence, P(n) is true $\forall\ n \geq 1$. (Inductive logic)

E9) You know that each interior angle of an equilateral triangle is 60°, which is $[180 - \frac{360}{3}]^\circ$. Now, use what you have proved earlier about the sum of the interior angles of a polygon. Also, use the definition of a regular polygon — all its sides are of the same length, and hence, all its interior angles have the same degree measure. Then see if you can prove the given statement.

E10) You need to give the activities you use for developing the abilities of using inductive and deductive logic. You also need to present the methods you use for helping them develop their mathematical intuition, particularly what kind of conjectures to make, and how to verify if they 'make sense'.
The method that **will not** work is to give a series of proofs to be rote learnt, and coughed up at exam time.

E11) For example, consider the statement: Every planar section of a sphere is a circle. Prove or disprove it.

E12) Several misconceptions regarding skew and parallel lines in 3D are because of this factor. Think of other examples.

E13) You need to show that Playfair's axiom implies the Remark, and vice versa.

E14) Can you think of some results in Euclidean geometry that are based on the parallel postulate? Ask them the results would alter if this postulate does not hold any/more.

You could go further and ask them what would happen if the first postulate didn't hold, and so on.

There are several websites on hyperbolic geometry that you can ask your students to access to visually see the difference triangles, perpendicular lines, etc., in this geometry and Euclidean geometry.

E15) Look again at Fig.8(a). In Δ NAB, what happens to $\angle$ANB if we move the points A and B closer together? As these points approach each other, $\angle$ANB will become smaller and smaller, nearer and nearer to zero. And then the sum of the interior angles of the triangle will become nearer and nearer to two right-angles. If, on the other hand, we move the points A and B further and apart, $\angle$ANB will become larger and larger, with a maximum value of 360°, as the length of the side AB approaches the circumference of the great circle. In such a situation, the sum of the interior angles of ΔNAB will approach six right-angles.

E16) Consider a sphere of radius R. Note that the diameter of any circle on the sphere, being a straight line, has to be an arc of a great circle. Let A be a point on the surface of the sphere, which we shall take as the pole, and let O be the centre of the sphere (see Fig.10). Consider a line of latitude with A as the pole and α as the co-latitude (angle between OA and a line joining O to any point B on the line of latitude). Then the diameter of the circle forming the latitude is $2\alpha R$, since this is an arc of a great circle subtending an angle 2α at the centre O. On the other hand, the circumference of the circle of latitude is $2\pi R\sin\alpha$ (as this is also the circumference of a planar circle of diameter $2R\mathrm{Sin}\,\alpha$).

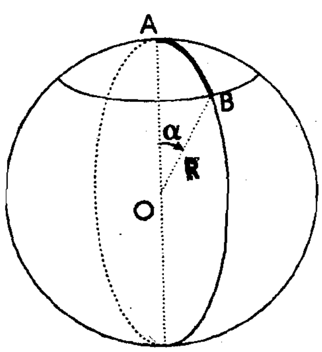


Fig.10

Thus the ratio of the circumference of a spherical circle to its diameter is $\pi_\alpha = 2\pi R\sin\alpha / 2\alpha R = \pi\,\mathrm{Sin}\,\alpha/\alpha$.

The largest circle that one can draw on the surface of the sphere is a great circle (like the equator) for which $\alpha = \pi/2$, so that $\pi_\alpha = 2$.
The smallest circle is one of radius 0, for which $\alpha = 0$, so that $\pi_\alpha = \pi$.
Thus in spherical geometry, π_α varies between 2 and π.

E17) White, because you are near the North Pole, since no bear exists around the South Pole. But if the question had been 'where are we?', there are an infinite number of locations close to the South Pole which fit the bill!