UNIT 4 TECHNIQUES OF COUNTING

Structure

4.1 INTRRODUCTION

Suppose a boy/girl has number lock in his/her cycle having 3 wheels each containing 10 digits from 0 to 9. Suppose the boy/girl forgets his/her 3 digits lock number. Then is there any technique which helps him/her to open the lock without breaking the lock? The answer is yes. This answer is provided by the techniques of counting. In case of above situation, techniques of counting tell us how many different locking options (the three digits codes) are possible with 3 wheels each containing 10 digits from 0 to 9. Out of these options, there is only one correct option. If the boy/girl starts to try them definitely at some stage, lock will get opened (because total number of options is finite in number). In our day to day life, there are many situations, where we need to count the number of ways a particular event can take place.

In this unit, we will discuss two such techniques known as permutation and combination based on fundamental principles of counting. We will introduce concept of factorial and binomial theorem also in this unit.

Objectives

After completing this unit, you should be able to:

- get the idea of factorial and its notations;
- get the logic of fundamental principles of addition and multiplication;
- define linear permutation and solve simple problems based on it;
- define circular permutation and solve simple problems based on it;
- define combination and solve simple problems based on it; and
- get an idea of binomial theorem.

4.2 FACTORIAL AND ITS NOTATIONS

The product of first n natural numbers is denoted by n! or $\underline{|n}$ and read as 'n factorial'.

i.e $n! = n\times (n-1) \times\ldots\times 3\times 2\times 1$

If $n = 0$, then we define $0! = 1$

Remark: We see that $n! = n\times (n-1) \times (n-2) \times \ldots \times 3\times 2 1\times$
$= n\times \lfloor n-1$
$= n\times (n-1) \times \lfloor n-2$ and so on.

Let us now consider some examples.

Example 1: Evaluate the following

(i) 8! (ii) (4!) (3!) (iii) $\frac{10!}{8!.4!}$ (iv) 5! + 4! (v) 6! – 4!

Solution:

(i) $8! = 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1 = 40320$

(ii) $(4!)(3!) = (4\times 3\times 2\times 1)(3\times 2\times 1) = 24\times 6 = 144$

(iii) $\frac{10!}{8!\times 4!} = \frac{10.9.8!}{8!(4\times 3\times 2\times 1)} = \frac{10.9}{4.3.2.1} = \frac{15}{4}$

(iv) 5! + 4! = 120 + 24 = 144

(v) 6! – 4! = 720 – 24 = 696

Example 2: Express the following in terms of factorial:

(i) 5.6.7.8.9.10 (ii) 4.8.12.16.20.24

(iii) 1.3.5.7.9.11 (iv) 2n.4n.6n.8n

Solution:

(i) $5.6.7.8.9.10 = \frac{1.2.3.4.5.6.7.8.9.10}{1.2.3.4} = \frac{10!}{4!}$

(ii) $4.8.12.16.20.24 = 4^6(1.2.3.4.5.6) = 4^6(6!)$

(iii) $1.3.5.7.9.11 = \frac{1.2.3.4.5.6.7.8.9.10.11}{2.4.6.8.10} = \frac{11!}{2^5(1.2.3.4.5)} = \frac{11!}{32(5!)}$

(iv) $2n.4n.6n.8n = (2n)^4(1.2.3.4) = (2n)^4(4!)$

Example 3: Solve for n, $n \in N$

(i) $(n+2)! = 42.n!$ (ii) $\frac{1}{8!}+\frac{1}{9!}=\frac{n}{10!}$ (iii) $\frac{2.(n!)}{(n-2)!}=\frac{(n+1)!}{(n-1)!}$, $n \neq 1, 2$

Solution:

(i) $(n+2)(n+1)(n!) = 42(n!) \Rightarrow (n+2)(n+1) = 42$

$\Rightarrow n^2 + 3n + 2 - 42 = 0 \Rightarrow n^2 + 3n - 40 = 0$

$\Rightarrow n^2 + 8n - 5n - 40 = 0 \Rightarrow n(n+8) - 5(n+8) = 0$

$\Rightarrow (n+8)(n-5) = 0$

$\Rightarrow n = -8, 5$

But $n \in N$, therefore n = 5

(ii) $\frac{1}{8!}+\frac{1}{9!}=\frac{n}{10!} \Rightarrow \frac{1}{8!}+\frac{1}{9.8!}=\frac{n}{10.9.8!} \Rightarrow 1+\frac{1}{9}=\frac{n}{90}$

$\Rightarrow \frac{10}{9}=\frac{n}{90} \Rightarrow n = \frac{10\times 90}{9} = 100$

$\therefore n = 100$

(iii) $\frac{2.n.(n-1).(n-2)!}{(n-2)!} = \frac{(n+1).n.(n-1)!}{(n-1)!}$

$\Rightarrow 2n(n-1) = (n+1)n$

$\Rightarrow 2n^2 - 2n = n^2 + n \Rightarrow n^2 - 3n = 0$

$\Rightarrow n(n-3) = 0 \Rightarrow n = 0, 3$

But $n \in N$

$\therefore n = 3$

Now, you can try the following exercises.

E 1) Evaluate the following

(i) $\dfrac{22!}{19!}$

(ii) $\dfrac{15!}{10! \times 5!}$

E 2) Express the following in terms of factorial.

(i) 3.6.9.12.15

(ii) 7.8.9.10.11.12

E 3) Solve for n, $n \in N$

(i) (n – 2)! = 12 (n – 4)!

(ii) n! = 72 (n–2)!

4.3 FUNDAMENTAL PRINCIPLES OF COUNTING

There are two fundamental principles of counting. These two principles solve the problems of counting. So it becomes necessary for us first to define what is the counting problem? According to Grinstead and Snell (2006) it is defined as if you

"Consider an experiment that takes place in several stages and is such that the number of outcomes m at the nth stage is independent of the outcomes of the previous stages. The number m may be different for different stages. We want to count the number of ways that the entire experiment can be carried out."

Let us take an example.

Example 4: Statistics discipline wanted to book the lunch in the IGNOU guest house for the experts during an expert committee meeting. The incharge of the guest house explain the lunch menu like this:

(a) there are two choices for appetizers: soup and juice

(b) there are two choices for main course: veg and non-veg

(c) there are three choices for dessert: sponge rashgulla, gulab jamun and ice cream.

How many options were there for statistics discipline for complete meal?

Solution: If we compare this situation with the counting problem we note that

(i) Entire experiment means complete meal.

(ii) Number of stages of the entire experiment are three in numbers.

(iii) Options (number of outcomes) for first, second, third stages are 2, 2, 3 respectively.

The above situation in the form of a tree diagram can be represented as shown on the next page.

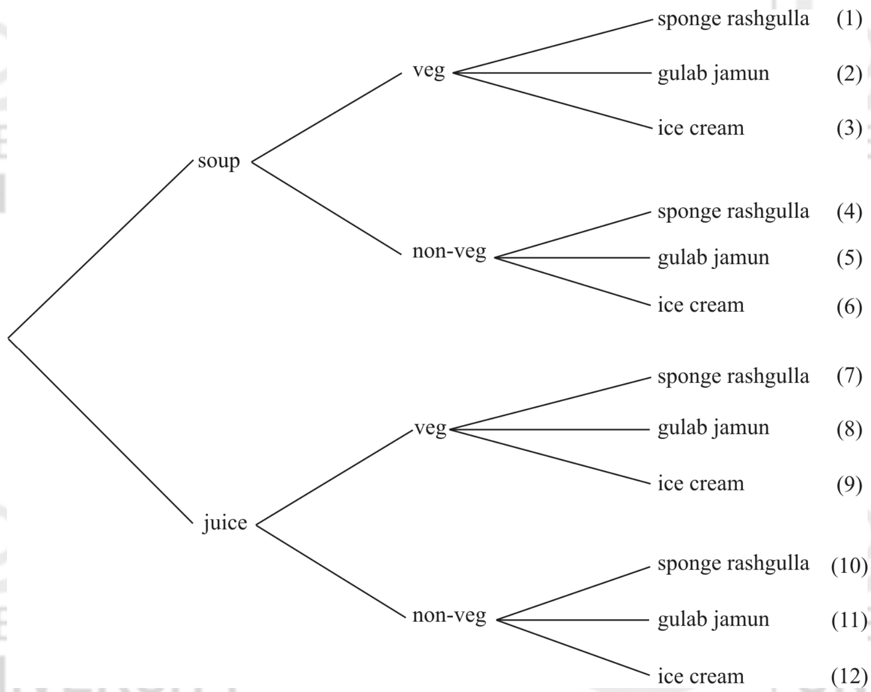


Fig 4.1 Tree Diagram of Guest House Menu

From the above tree diagram, we see that total number of options for complete meal is given by:

(total choices at first stages) $\times$(total choices at second stage) $\times$(total choices at third stage)

$= 2 \times 2 \times 3$

$= 12$

Hence there were 12 options for statistics discipline for complete meal. Each of these 12 options is numbered from 1 to 12 in the right margin of the tree diagram. For example option 7 is 'juice followed by veg followed by sponge rashgulla', option 12 is 'juice followed by non-veg followed by ice-cream', etc.

Now we discuss the two fundamental principles of counting in coming two sub-sections.

4.3.1 Fundamental Principle of Multiplication (FPM)

Suppose we want to complete two jobs, where first job can be done in m distinct ways, second job can be done in n distinct ways then both jobs can take place (one followed by other) in $m \times n$ distinct ways.

In general, suppose we want to complete n jobs, where
first job can be done in m_1 distinct ways,
second job can be done in m_2 distinct ways,
third job can be done in m_3 distinct ways,
and so on
n^{th} job can be done in m_n distinct ways.

Then these n jobs can take place (in succession) in $m_1 \times m_2 \times m_3 \times ... \times m_n$ distinct ways.

For example, suppose a teacher wants to select one boy and one girl student out of a class having 15 boys and 10 girls students, then teacher can make such selection in $15 \times 10 = 150$ distinct ways.

4.3.2 Fundamental Principle of Addition (FPA)

Suppose we want to complete one job out of two jobs, where first job can be done in m distinct ways and second independent job can be done in n distinct ways. Then one of the two jobs can take place in m + n distinct ways.

In general, suppose we want to complete one job out of n jobs, where

first job can be done in m_1 distinct ways,

second job can be done in m_2 distinct ways,

third job can be done in m_3 distinct ways,

and so on

n^{th} job can be done in m_n distinct ways.

Then one of the n jobs (any two or any three… or all of these can not occur simultaneously) can take place in $m_1 + m_2 + m_3 + ... + m_n$ distinct ways.

In case of 'and' we multiply and in case of 'or' we add

For example, suppose a teacher wants to select either a boy or a girl student out of a class having 15 boys and 10 girls, then teacher can select either a boy or a girl student in 15 + 10 = 25 distinct ways.

Now we take some examples based on these two principles of counting.

Example 5: In a college, there are 40 male and 30 female faculties. The principal of that college wants to select one male and one female faculty to accompany with the students of the college going for a picnic. In how many ways can principal do this selection?

Solution: For this selection, principal has to complete two jobs:

(i) Selection of a male faculty

(ii) Selection of a female faculty

First job can be done in 40 ways and second job can be done in 30 ways.
$\therefore$ by fundamental principle of multiplication required number of ways

$= 40 \times 30 = 1200$

Example 6: There are 40 male and 30 female faculties in a college. The principal of the college wants to select one faculty (either male or female) for an examination duty. In how many ways this selection can be done?

Solution: For this selection, principal of the college do either of the following two jobs:

(i) Selection of a male faculty

(ii) Selection of a female faculty

First job can be done in 40 ways and second job can be done in 30 ways.
$\therefore$ by fundamental principal of addition, required number of ways

= 40 + 30 = 70

Example 7: How many different number plates of vehicles are possible using two different letters of English alphabet followed by four different digits 0 to 9?

Solution: In order to complete this job, we have to fill up six positions in succession, where

First position can be filled up in 26 ways,

Second position can be filled up in 25 ways $\left[\begin{array}{l}\because \text{one letter has been} \\ \text{used in first place}\end{array}\right]$

Third position can be filled up in 10 ways [With one of the digits from 0 to 9]

Fourth position can be filled up in 9 ways $\begin{bmatrix}\text{With one of the 9 digits}\\ \text{leaving the one already used}\end{bmatrix}$

Fifth position can be filled up in 8 ways $\begin{bmatrix}\text{With one of the 8 digits}\\ \text{leaving the two already used}\end{bmatrix}$

Sixth position can be filled up in 7 ways $\begin{bmatrix}\text{With one of the 7 digits}\\ \text{leaving the three already used}\end{bmatrix}$

$\therefore$ by fundamental principal of multiplication required numbers of ways

$= 26 \times 25 \times 10 \times 9 \times 8 \times 7$

= 3276000

Note: If in the above example repetition of digits 0 to 9 is allowed (which in practice happens) then required number of ways $= 26 \times 25 \times (10 \times 10 \times 10 \times 10 - 1)$

$$\begin{bmatrix}\text{1 is substracted because we have ignore}\\ \text{the case containg all zeros, i.e. 0000}\end{bmatrix}$$

Now, you can try the following exercises.

E 4) In an examination there are 10 multiple choice questions. First five questions have 4 choices each and last five questions have 5 choices each. How many sequences of answers are possible?

E 5) How many four-letter words can be formed by using letters a, b, g, h, k, if

(i) Repetition is not allowed (ii) Repetition is allowed

4.4 PERMUTATION

Permutation is related to the arrangement of things. Things arranged in a line come under the heading of **linear permutation**, while arrangement of things in a circle comes under the heading of **circular permutation**. Let us discuss these two heading one by one.

4.4.1 Linear Permutation

Possible arrangements in a line of a number of things taken some or all at a time are called the permutation. Before giving the general formula, let us consider an example, where we are to arrange say three books of different colours (Red, Green and Orange):

Permutations of three books when taken one at a time are R, G, W, i.e.

the number of permutations = 3 = $\dfrac{3!}{(3-1)!} = {}^{3}P_{1}$ or P(3, 1)

Permutations of three books when taken two at a time are RG, GR, RW, WR, GW, WG, i.e.

the number of permutations = 6 = $\dfrac{3!}{(3-2)!} = {}^{3}P_{2}$ or P(3, 2)

Permutations of three books when taken all at a time are RGW, RWG, GRW, GWR, WRG, WGR, i.e.

the number of permutations = 6 = $\dfrac{3!}{(3-3)!} = {}^{3}P_{3}$ or P(3, 3)

In general, the total number of permutations of n things taken r $(1 \le r \le n)$ at a time is denoted by ${}^{n}P_{r}$ or P(n, r) and is defined as

$$^nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)\ldots(n-(r-1))$$

i.e. $^nP_r = n(n-1)(n-2)\ldots$up to r factors

For example,

(i) Total number of permutations of a, b, c taken 2 at a time are given by ab, ba, bc, cb, ca, ac.

Also $^3P_2 = 3.2 = 6$

(ii) Total number of permutations of a, b, c taken all at a time are given by abc, acb, bca, bac, cab, cba.

Also $^3P_3 = 3.2.1 = 6$

1. We define $^nP_0 = 1$
2. Always remember the following result

$$^nP_r = \frac{n!}{(n-r)!}, \quad 0 \le r \le n$$

Example 8: Evaluate the following:

(i) $^8P_2$ (ii) $^{20}P_5$ (iii) P(10, 4) (iv) P(8, 8) (v) $^5P_0$

Solution:

(i) $^8P_2 = \frac{8!}{(8-2)!} = \frac{8!}{6!} = \frac{8 \times 7 \times 6!}{6!} = 8 \times 7 = 56$

(ii) $^{20}P_5 = \frac{20!}{(20-5)!} = \frac{20!}{15!} = \frac{20 \times 19 \times 18 \times 17 \times 16 \times 15!}{15!}$

$= 20 \times 19 \times 18 \times 17 \times 16 = 1860480$

(iii) $P(10, 4) = \frac{10!}{(10-4)!} = \frac{10!}{6!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{6!}$

$= 10 \times 9 \times 8 \times 7 = 5040$

(iv) $P(8, 8) = \frac{8!}{(8-8)!} = \frac{8!}{0!} = \frac{40320}{1} = 40320$

(v) $^5P_0 = \frac{5!}{(5-0)!} = \frac{5!}{5!} = 1$

Example 9: Find n if $^nP_5 = 30\,^nP_3$.

Solution: $^nP_5 = 30 \times {}^nP_3 \Rightarrow \frac{n!}{(n-5)!} = \frac{30.n!}{(n-3)!}$

$$\Rightarrow \frac{1}{(n-5)!} = \frac{30}{(n-3)!} \Rightarrow \frac{1}{(n-5)!} = \frac{30}{(n-3).(n-4).(n-5)!}$$

$$\Rightarrow \frac{1}{1} = \frac{30}{(n-3)(n-4)} \Rightarrow (n-3)(n-4) = 30$$

$$\Rightarrow n^2 - 7n + 12 = 30 \Rightarrow n^2 - 7n - 18 = 0$$

$$\Rightarrow n^2 - 9n + 2n - 18 = 0 \Rightarrow n(n-9) + 2(n-9) = 0$$

$$\Rightarrow (n-9)(n+2) = 0 \Rightarrow n = 9, -2$$

But n cannot be – 2.

$\therefore n = 9$

Here are some exercises for you.

E6) Find n if $^5P_n = {}^6P_{n-1}$.

E7) Evaluate the following

(i) $^7P_4$ (ii) ^nP_n (iii) $^nP_{n-1}$ (iv) $^nP_1$ (v) $^nP_2$ (vi) $^{16}P_3$

Example 10: How many different words, with or without meaning, can be formed by using all the letters of the word 'EQUATION' (without repetition)?

Solution: There are 8 letters in the word 'EQUATION' which are all different.
$\therefore$ possible number of words = number of arrangement of 8 letters taken all at a time

$$= {}^8P_8 = \frac{8!}{(8-8)!} = \frac{8!}{0!} = 8! = 40320 \quad \text{as } 0! = 1$$

Example 11: How many signals are possible with 4 flags each of different colour?

Solution: Possible number of signals using one flag at a time = $^4P_1 = 4$

Possible number of signals using two flags at a time = $^4P_2 = 4.3 = 12$

Possible number of signals using three flags at a time = $^4P_3 = 4.3.2 = 24$

Possible number of signals using all flags at a time = $^4P_4 = 4! = 24$

$\therefore$ total number of signals = 4 + 12 + 24 + 24= 64

Here is an exercise for you.

E 8) In how many ways can 5 students stand in a queue?

Permutations of Things not all Distinct

So far we have discussed the permutations of things which were all distinct. But in usual it is not always possible that things to be permuted are all distinct.
In case of repetition of things we use following result.
If out of n things p_1 are of one kind, p_2 are of second kind, p_3 are of third kind and so on p_k are of k^{th} kind then total number of possible permutations are given by

$$\frac{\lfloor n}{\lfloor p_1 \times \lfloor p_2 \times \lfloor p_3 \times \ldots \times \lfloor p_k}, \text{ where } p_1 + p_2 + \ldots + p_k = n \text{ and } p_i \geq 1,\ 1 \leq i \leq k$$

Example 12: How many different words, with or without meaning, can be formed by using all the letters of the word "BANANA"?

Solution: There are 6 letters in the word "BANANA"
Out of which A, N occur 3, 2 times respectively.

$$\therefore \text{total number of permutations} = \frac{6!}{3!\ 2!} = \frac{6.5.4.3!}{3!\ 2!} = \frac{120}{2} = 60$$

Here are some exercises for you.

E 9) How many different signals are possible with 3 red, 4 white and2 green flags by using all at a time in a queue?

E10) How many words can be formed with or without meaning by using the letters of the words AMAR?

Permutation when Repetition is Allowed

Example 13: Prove that total number of permutations of n things taken r at a time any thing can repeat any number of times is given by n^r.

Proof: In order to find out total number of permutations, we have to fill up r positions, where
First position can be filled up in n ways,
Second position can also be filled up in n ways, [$\because$ repetition is allowed]
Third position can be filled up in n way, [Same reason]
And so on
r^{th} position can be filled up in n ways.
$\therefore$ required number of permutations = $\underbrace{n \times n \times n \times \ldots \times n}_{\text{r times}} = n^r$.

Example 14: In how many ways 5 letters can be posted in 3 letter boxes?

Solution: Each of the 5 letters can be posted in 3 ways, i.e. each of 5 letters can be posted by using any of the 3 letter boxes.
$\therefore$ required number of ways = $3 \times 3 \times 3 \times 3 \times 3 = 3^5 = 243$

Here is an exercise for you.

E 11) In how many ways can 3 prizes be distributed among 5 students when

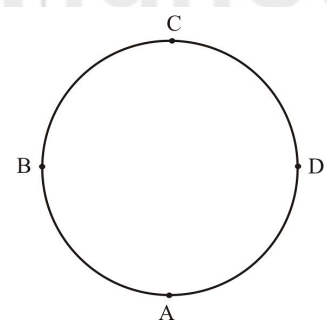
(i) No student gets more than one prize?

(ii) A student may get any number of prizes?

(iii) No student gets all the prizes?

4.4.2 Circular Permutation

Let us consider four letters A, B, C, D. Consider the following arrangements ABCD, BCDA, CDAB, DABC these are 4 different arrangements when arranged in a line. whereas this is a single arrangement when arranged in a circle, in clockwise direction as shown in figure.



$\therefore$ in case of 4 letters, 4 linear arrangements = 1 circular arrangement

1 linear arrangement = $\frac{1}{4}$ circular arrangement

So, 4! Linear arrangements = $\frac{4!}{4} = 3!$ circular arrangements.

In general, if anticlock wise and clock wise order of arrangements makes different permutations then number of circular permutations of n distinct things = $(n-1)!$

And if anti-clock wise and clock wise order of arrangements does not give distinct permutations then total number of permutations of n distinct things

$= \frac{(n-1)!}{2}$

For example, arrangements of flowers in a garland form the same permutation in case of anti clock wise and clockwise order.

Example 15: In how many ways 10 students of a batch can be arrangements in a

(i) Line (ii) Circle

Solution:

(i) Total number of arrangements of 10 students in a line

$= 10! = 3628800$ [Linear permutation]

(ii) Total number of arrangements of 10 students in a circle

$$= (10-1)! = 9! = 362880 \quad \text{[Circular permutation]}$$

Here is an exercise for you.

E 12 (i) How many different garlands are possible with 8 flowers?

(ii) In how many ways 20 members of the management of a college can sit on a round table in a meeting, if president and vice president always sit together.

4.5 COMBINATION

In Sec. 4.4 of this unit we have discussed permutation. We have seen that in case of permutation we want to know the possible number of arrangements of n things taken some or all at a time. But sometimes we are interested in forming only groups or making selections or drawing items without bothering about the arrangements. These are called combinations.

Before giving the general formula, let us consider an example, where we are to form the groups of say three books of different colours (Red, Green, Orange). Combinations of three books when taken one at a time are R, G, W, i.e.

the number of combinations = 3 = $\dfrac{3!}{(3-1)!\times 1!}$ = ${}^3C_1$ or C(3, 1)

Combinations of three books when taken two at a time are RG, RW, GW, i.e.

the number of combinations = 3 = $\dfrac{3!}{(3-2)!\times 2!}$ = ${}^3C_2$ or C(3, 2)

Combination of three books when taken all at a time is RGW, i.e.

the number of combination = 1 = $\dfrac{3!}{(3-3)!\times 3!}$ = ${}^3C_3$ or C(3, 3)

In general, the total number of combinations of n things taken r $(1 \le r \le n)$ at a time is denoted by ${}^nC_r$ or C(n, r) and is defined as

$${}^nC_r = \frac{n!}{(n-r)!\times r!}$$

For example,

(i) Total number of combinations of a, b, c taken 2 at a time are given by ab, bc, ca

Also $${}^3C_2 = \frac{3!}{(3-2)!\times 2!} = \frac{3\times 2\times 1}{1\times 2} = 3$$

(ii) Total number of combinations of a, b, c taken all at a time are given by abc,

Also $${}^3C_3 = \frac{3!}{(3-3)!\times 3!} = \frac{3!}{1!\times 3!} = 1$$

In this section we shall discuss another important technique of counting known as combination. Total number of groups that can be formed of n things taken $r (0 \le r \le n)$ at a time is called combination, denoted by ${}^nC_r$ or C(n, r) or $\begin{pmatrix} n \\ r \end{pmatrix}$ and is defined as

$$^nC_r = \frac{n!}{r! \times (n-r)!}$$

For example, suppose there are six cricket teams and every team has to play one match with each other team. Then total number of matches which are to be played = combinations of six teams when taken two at a time

$$= {}^6C_2 = \frac{6!}{(6-2)! \times 2!} = \frac{6 \times 5}{2} = 15$$

4.6 SELECTION OF PERMUTATION OR COMBINATION

Following points help you in deciding which of the two permutation or combination should be used in a given situation.

- Permutation is used when the order of the selection also matters.
- Combination is used when the order does not matter, but only the selection or group formation or draw of items is taken into consideration.

Example 16: Prove the following

(i) $^nC_r = {}^nC_{n-r}, \quad 0 \le r \le n$ (ii) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iii) $^nC_0 = {}^nC_n = 1$ (iv) $^nC_1 = {}^nC_{n-1} = n$

Solution:

(i) R.H.S. $= {}^nC_{n-r} = \dfrac{n!}{(n-(n-r))! \times (n-r)!} = \dfrac{n!}{r! \times (n-r)!}$

$$= \frac{n!}{(n-r)!.r!} = {}^nC_r = \text{L.H.S.}$$

(ii) L.H.S. $= {}^nC_r + {}^nC_{r-1} = \dfrac{n!}{r!(n-r)!} + \dfrac{n!}{(r-1)!(n-r+1)!}$

$$= \frac{n!}{r.(r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1).(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right] = \frac{n!}{(r-1)!(n-r)!}\left[\frac{n-r+1+r}{r(n-r+1)}\right]$$

$$= \frac{(n+1).n!}{r.(r-1)!(n-r+1).(n-r)!} = \frac{(n+1)!}{r!(n+1-r)!} = {}^{n+1}C_r = \text{R.H.S.}$$

(iii) $\underset{\text{I}}{^nC_0} = \underset{\text{II}}{^nC_n} = \underset{\text{III}}{1}$

$$\text{I} = {}^nC_0 = \frac{n!}{0!.(n-0)!} = \frac{n!}{n!} = 1 = \text{III} \qquad \text{as } 0! = 1$$

$$\text{II} = {}^nC_n = \frac{n!}{n!.(n-n)} = \frac{n!}{n!.0!} = \frac{n!}{n!} = 1 = \text{III}$$

(iv) $\underset{\text{I}}{^nC_1} = \underset{\text{II}}{^nC_{n-1}} = \underset{\text{III}}{n}$

$$\text{I} = {}^nC_1 = \frac{n!}{1!.(n-1)!} = \frac{n.(n-1)!}{(n-1)!} = n = \text{III}$$

$$II = {}^nC_{n-1} = \frac{n!}{(n-1)!.(n-n+1)!} = \frac{n.(n-1)!}{(n-1)!\ 1!} = \frac{n}{1!} = \frac{n}{1} = n = III$$

Example 17: If four cards are chosen from a pack of 52 playing cards then find the number of ways that all the four cards are

(i) of different suit

(ii) of same suit

(iii) face cards

(iv) either red or black

Solution: We know that in a pack of playing cards there are 4 suits namely, spade, diamond, heart, and club each containing 13 cards.

(i) Required number of ways = ${}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1$

$= 13\times13\times13\times13 = 28561$

(ii) Required number of ways = ${}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4$

$= 4 \times {}^{13}C_4 = \frac{4.13.12.11.10}{4!} = 2860$

(iii) We know that there are 12 face cards.

$\therefore$ required number of ways = ${}^{12}C_4 = \frac{12.11.10.9}{4!} = 495$

(iv) We know that there are 26 red and 26 black cards in a pack of playing cards.

$\therefore$ required number of ways = ${}^{26}C_4 + {}^{26}C_4$

$= 2 \times {}^{26}C_4 = 2\frac{26.25.24.23}{4!} = 29900$

Example 18: A bag contains 4 red and 7 white balls. Find the number of ways in which 2 red and 3 white balls can be drawn.

Solution: Out of 4 red balls 2 can be drawn in ${}^4C_2$ ways = $\frac{4.3}{2!} = 6$

Out of 7 white balls 3 can be drawn in ${}^7C_3$ ways = $\frac{7.6.5}{3!} = 35$

$\therefore$ required number of ways = ${}^4C_2 \times {}^7C_3 = 6\times35 = 210$

Here is an exercise for you.

E 13) There are 21 cricket players including 11 batsmen, 7 bowlers and 3 wicket keepers. In how many ways 11 players can be selected having 6 batsmen, 4 bowlers and 1 wicket keeper.

4.7 SOME IMPORTANT RESULTS

In this section, we will discuss some examples based on the following two important results.

Result I Total number of permutations of n distinct things taken r at a time such that

(i) s (0 < s < r) particular things are always included = ${}^{n-s}P_{r-s} \times {}^rP_s$

(ii) s (0 < s < r) particular things are always excluded = ${}^{n-s}P_r$

Result II Total number of combination of n distinct things taken r at a time such that

(i) s (0 < s < r) particular things are always included = ${}^{n-s}C_{r-s}$

(ii) s (0 < s < r) particular things are always excluded = ${}^{n-s}C_{r}$

Example 19: Find the total number of ways of selection of 15 players out of 21 players such that

(i) 3 particular players are always included

(ii) 2 particular players are always excluded.

Solution:

(i) 3 particular players are always included $\Rightarrow$ we have to select 15 – 3 = 12 players out of 21 – 3 = 18 players.

$$\therefore \text{required number of ways} = {}^{18}C_{12} = \frac{18!}{12!(18-12)!}$$

$$= \frac{18\times17\times16\times15\times14\times13\times12!}{12!\times6!}$$

$$= \frac{18\times17\times16\times15\times14\times13}{6\times5\times4\times3\times2\times1} = 18564$$

Alternatively

Here n = 21, r = 15, s = 3

$\therefore$ required number of selections = ${}^{n-s}C_{r-s}$ [Refer part (i) of Result II]

$= {}^{21-3}C_{15-3}$

$= {}^{18}C_{12} = 18564$ [Already calculated]

(ii) 2 particular players are always excluded $\Rightarrow$ we have to select 15 players out of 21 – 2 = 19 players

$$\therefore \text{required number of ways} = {}^{19}C_{15} = \frac{19!}{15!\times(19-15)1}$$

$$= \frac{19\times18\times17\times16\times15!}{15!\times4!} = \frac{19\times18\times17\times16}{4\times3\times2\times1} = 3876$$

Alternatively

Here n = 21, r = 15, s = 2

$\therefore$ required number of selection = ${}^{n-s}C_{r}$ [Refer part (ii) of Result II]

$= {}^{21-2}C_{15}$

$= {}^{19}C_{15} = 3876$ [Already calculated]

Example 20: How many 5 letters words are possible using 8 letters a, b, c, d, e, f, g, h such that

(i) Two letters a, b are always included

(ii) Three letters a, c, d are always excluded

Solution: Here concept of permutation will be used, because we have to form arrangement not groups.

(i) Here n = 8, r = 5, s = 2

$\therefore$ required number of words = ${}^{n-s}P_{r-s}\times{}^{r}P_{s}$ [Refer part (i) of Result I]

$$= {}^{8-2}P_{5-2} \times {}^{5}P_2$$
$$= {}^{6}P_3 \times {}^{5}P_2 = 120 \times 20 = 2400$$

(ii) Here n = 8, r = 5, s = 3

$\therefore$ required number of words = ${}^{n-s}P_r$ [Refer part (ii) of Result I]

$$= {}^{8-3}P_5 = {}^{5}P_5 = 120$$

Now, you can try the following exercises.

E 14) Find the total number of ways of selection of 11 players out of 15 players such that

(i) captain and vice captain are always included

(ii) one particular injured players is always excluded.

E 15) How many 4 digits numbers are possible using 9 digits 1, 2, 3, …, 9 such that

(i) Three digits 1, 6, 8 are always included

(ii) Two digits 3, 8 are always excluded.

4.8 BINOMIAL THEOREM

Following two sub-sections are devoted to the discussion of this theorem.

4.8.1 Binomial Theorem for Positive Integral Index

From your school days, you are familiar what we mean by monomial, binomial, trinomial and multinomial expressions. Let us recall your memory. An expression having one term, two terms, three terms, more than three terms is known as monomial, binomial, trinomial, multinomial respectively.

For example,

(i) 7, x, 9x, $3x^2$, 5y, x^2y all are monomial, as there is only one term in each expression.

(ii) a + b, a – b, 3a + 2b, $a^2 + 3b$, x – y, x – 4y all are binomial, as there are only two terms in each expression.

(iii) a + b + c, x – 2y +z, 3x + 2y – z all are trinomial, as there are only three terms in each expression.

(iv) a + b + c + d, x – 2y + 5z – w + 3u both are multinomial, as there are more than three terms in each expression.

Let us recall another memory of your school days. You have met with the identities:

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

In school days students cram these identities. But here you have become familiar with the concept of combination in Sec. 4.5, using the knowledge of Sec. 4.5 these identities can be written in a systematic manner. In fact our aim of this section is to obtain the expression for $(a+b)^n$, n = 1, 2, 3,… known as binomial theorem (for positive integral index).

Let us see how this very interesting expression can be generated by using the knowledge what we know up to this point.

Obviously $(a+b)^1 = a+b = {}^1C_0\, a^{1-0}\, b^0 + {}^1C_1\, a^{1-1}\, b^1 \quad [\because {}^1C_0 = 1,\ {}^1C_1 = 1]$

$(a+b)^2 = a^2 + 2ab + b^2 \ = {}^2C_0\, a^{2-0}\, b^0 + {}^2C_1\, a^{2-1}\, b^1 + {}^2C_2\, a^{2-2}\, b^2$

$[\because {}^2C_0 = 1,\ {}^2C_1 = 2,\ {}^2C_2 = 1]$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$= {}^3C_0\, a^{3-0}\, b^0 + {}^3C_1\, a^{3-1}\, b^1 + {}^3C_2\, a^{3-2}\, b^2 + {}^3C_3\, a^{3-3}\, b^3$$
$$\left[\because {}^3C_0 = 1 = {}^3C_3,\ \ {}^3C_1 = 3 = {}^3C_2\right]$$

$$(a+b)^4 = (a+b)(a+b)^3$$
$$= (a+b)(a^3 + 3a^2b + 3ab^2 + b^3)$$
$$= a^4 + 3a^3b + 3a^2b^2 + ab^3 + a^3b + 3a^2b^2 + 3ab^3 + b^4$$
$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$= {}^4C_0\, a^{4-0}b^0 + {}^4C_1\, a^{4-1}\, b^1 + {}^4C_2\, a^{4-2}b^2 + {}^4C_3\, a^{4-3}b^3 + {}^4C_4 a^{4-4}\ b^4$$
$$\left[\because {}^4C_0 = {}^4C_4 = 1,\ {}^4C_1 = {}^4C_3 = 4,\ {}^4C_2 = 6\right]$$

…

…

…

$$(a+b)^n = {}^nC_0\, a^{n-0}b^0 + {}^nC_1\, a^{n-1}b^1 + {}^nC_2\, a^{n-2}b^2 + {}^nC_3\, a^{n-3}b^3 + \ldots$$
$$+ {}^nC_{n-1}\, a^{n-(n-1)}b^{n-1} + {}^nC_n\ a^{n-n}b^n$$
$$= {}^nC_0\, a^n + {}^nC_1\, a^{n-1}\, b + {}^nC_2\, a^{n-2}b^2 + {}^nC_3\, a^{n-3}\, b^3 + \ldots$$
$$+ {}^nC_{n-1}\, ab^{n-1} + {}^nC_n\, b^n$$

Some important points related to the above expression which will help you to easily remember it are given below.

1. ${}^nC_0,\ {}^nC_1,\ {}^nC_2, \ldots, {}^nC_n$ are known as binomial coefficients.
2. Exponents of a in successive terms are n, n – 1, n – 2, n – 3, …, 1, 0, i.e. difference of super and sub subscript of C, i.e. exponent of a in the term with binomial coefficient ${}^nC_r$ will be n – r.
3. Exponents of b in successive terms are 0, 1, 2, 3, …, n – 1, n, i.e. equal to the sub subscript of C, i.e. exponent of b in the term with binomial coefficient ${}^nC_r$ will be r.
4. Sum of the exponents of a and b in each term is equal to the actual exponent of the given binomial expression.
5. If n = 0, then $(a+b)^0 = 1 = {}^0C_0\, a^{0-0}b^0$ as ${}^0C_0 = 1$

To become user friendly with this expression, let us do some examples based on it.

Example 21: Expend $\left(x + \dfrac{1}{x}\right)^6$ by binomial theorem.

Solution: Comparing $\left(x + \dfrac{1}{x}\right)^6$ with $(a+b)^n$, we get

$$a = x, b = \frac{1}{x}, n = 6$$

$\therefore$ by binomial theorem

$$\left(x+\frac{1}{x}\right)^6 = {}^6C_0\, x^6\left(\frac{1}{x}\right)^0 + {}^6C_1\, x^5\left(\frac{1}{x}\right) + {}^6C_2\, x^4\left(\frac{1}{x}\right)^2 + {}^6C_3\, x^3\left(\frac{1}{x}\right)^3$$

$$+ {}^6C_4\, x^2\left(\frac{1}{x}\right)^4 + {}^6C_5\, x\left(\frac{1}{x}\right)^5 + {}^6C_6\left(\frac{1}{x}\right)^6$$

$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

$$[\because {}^6C_0 = {}^6C_6 = 1,\ {}^6C_1 = {}^6C_5 = 6,\ {}^6C_2 = {}^6C_4 = 15,\ {}^6C_3 = 20]$$

Example 22: Expand $(1-x)^{10}$ by binomial theorem.

Solution: Comparing $(1-x)^{10}$ with $(a+b)^n$, we get

$a = 1, b = -x,\ n = 10$

$\therefore$ by binomial theorem

$$(1-x)^{10} = (1+(-x))^{10}$$

$$= {}^{10}C_0\,(1)^{10}(-x)^0 + {}^{10}C_1\,(1)^9(-x)^1 + {}^{10}C_2\,(1)^8(-x)^2 + {}^{10}C_3\,(1)^7(-x)^3$$

$$+ {}^{10}C_4\,(1)^6(-x)^4 + {}^{10}C_5\,(1)^5(-x)^5 + {}^{10}C_6\,(1)^4(-x)^6 + {}^{10}C_7\,(1)^3(-x)^7$$

$$+ {}^{10}C_8\,(1)^2(-x)^8 + {}^{10}C_9\,(1)^1(-x)^9 + {}^{10}C_{10}\,(1)^0(-x)^{10}$$

$$= 1 - 10x + 45x^2 - 120x^3 + 210x^4 - 252x^5$$

$$+ 210x^6 - 120x^7 + 45x^8 - 10x^9 + x^{10}$$

$$\left[\begin{array}{l}\because {}^{10}C_0 = {}^{10}C_{10} = 1,\ {}^{10}C_1 = {}^{10}C_9 = 10, {}^{10}C_2 = {}^{10}C_8 = 45, \\ {}^{10}C_3 = {}^{10}C_7 = 120, {}^{10}C_4 = {}^{10}C_6 = 210,\ {}^{10}C_5 = 252\end{array}\right]$$

Now you can try the following exercises.

E 16) Expand $(1+\sqrt{2})^5$ by binomial theorem.

E 17) Expand $(3x-y)^7$ by binomial theorem.

4.8.2 Binomial Theorem for any Index

In subsection 4.8.1 we have discussed binomial theorem for index n, where n = 1, 2, 3, 4, …

But sometimes binomial expansion is needed for rational exponent. The aim of this section is to provide an expression which works for rational exponent.

Let us consider the expansion for positive integral index discussed in previous sub section 4.8.1

$$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}\, b^2 + {}^nC_3\, a^{n-3}\, b^3 + \ldots + {}^nC_{n-1}\, ab^{n-1}$$

$$+ {}^nC_n\, b^n$$

In Sec. 4.5, you have seen that ${}^nC_0, {}^nC_1, {}^nC_2, {}^nC_3$, etc. can be written as

$${}^nC_0 = {}^nC_n = 1,\ {}^nC_1 = {}^nC_{n-1} = n,\ {}^nC_2 = {}^nC_{n-2} = \frac{n(n-1)}{2!},$$

$${}^nC_3 = {}^nC_{n-3} = \frac{n(n-1)(n-2)}{3!}, \text{etc.}$$

$$\therefore (a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots + nab^{n-1}$$

$$+ b^n$$

Here we note that if n is not a positive integer, then this expression will never terminate. Also powers of b are increasing term by term, so in case of infinite expansion to get finite sum it becomes necessary that $|b|<1$. In fact, in case of rational exponent n the binomial expansion of $(1+x)^n$, where $|x|<1$ is given by

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-(r-1))}{r!}x^r + ..$$

and binomial expansion of $(a+b)^n$ is given by

$$(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n$$

$$= a^n\left[1+n\frac{b}{a}+\frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{b}{a}\right)^3+\ldots\right], \text{if } \left|\frac{b}{a}\right|<1 \text{ and}$$

$$(a+b)^n = b^n\left(1+\frac{a}{b}\right)^n$$

$$= b^n\left[1+n\frac{a}{b}+\frac{n(n-1)}{2!}\left(\frac{a}{b}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{a}{b}\right)^3+\ldots\right], \text{ if } \left|\frac{a}{b}\right|<1$$

Let us do some examples based on it.

Example 23: Expand $(5+3x)^{-4}$ using binomial theorem for negative index.

Solution: $(5+3x)^{-4} = (5)^{-4}\left(1+\frac{3}{5}x\right)^{-4}$

$$= \frac{1}{5^4}\left[1+(-4)\left(\frac{3}{5}x\right)+\frac{(-4)(-4-1)}{2!}\left(\frac{3}{5}x\right)^2+\frac{(-4)(-4-1)(-4-2)}{3!}\left(\frac{3}{5}x\right)^3+\ldots\right]$$

This expansion is valid if $\left|\frac{3}{5}x\right|<1$, i.e. $|x|<\frac{5}{3}$

$$\therefore\ (5+3x)^{-4} = \frac{1}{625}\left[1-\frac{12}{5}x+\frac{18}{5}x^2-\frac{108}{25}x^3+\ldots\right], \text{ if } |x|<\frac{5}{3}$$

Example 24: Find expansion of $(7-2x)^{2/3}$.

Solution: $(7-2x)^{2/3} = 7^{2/3}\left(1-\frac{2}{7}x\right)^{2/3}$

$$= 7^{2/3}\left[1+\left(\frac{2}{3}\right)\left(-\frac{2}{7}x\right)+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\left(-\frac{2}{7}x\right)^2+\ldots\right]$$

This expansion is valid only if $\left|\frac{2}{7}x\right|<1$, i.e. if $|x|<\frac{7}{2}$

$$\therefore\ (7-2x)^{2/3} = 7^{2/3}\left[1-\frac{4}{21}x-\frac{4}{441}x^2+\ldots\right], \text{ if } |x|<\frac{7}{2}$$

Now, you can try the following exercise.

E 18) Expand $(1-3x)^{1/2}$.

4.9 SUMMARY

Let us summarise the topics that we have covered in this unit:

1) Concept of factorial.
2) Fundamental principles of multiplication and addition.
3) Definition and examples of permutation.
4) Permutation in different situations.
5) Circular permutation.
6) Definition and examples of combination.
7) Binomial theorem for integral index and for any index.

4.10 SOLUTIONS/ANSWERS

E 1) (i) $\dfrac{22!}{19!} = \dfrac{22\times21\times20\times19!}{19!} = 22\times21\times20 = 9240$

(ii) $\dfrac{15!}{10!\times5!} = \dfrac{15\times14\times13\times12\times11\times10!}{10!\times5\times4\times3\times2\times1} = \dfrac{15\times14\times13\times12\times11}{120} = 3003$

E 2) (i) $3.6.9.12.15 = 3^5(1.2.3.4.5) = 3^5(5!)$

(ii) $7.8.9.10.11.12 = \dfrac{1.2.3.4.5.6.7.8.9.10.11.12}{1.2.3.4.5.6} = \dfrac{12!}{6!}$

E 3) (i) $(n-2)! = 12(n-4)!$

$\Rightarrow (n-2)\,(n-3)\,(n-4)! = 12(n-4)!$

$\Rightarrow (n-2)(n-3) = 12 \Rightarrow n^2 - 5n + 6 - 12 = 0$

$\Rightarrow n^2 - 5n - 6 = 0 \Rightarrow n^2 - 6n + n - 6 = 0$

$\Rightarrow n(n-6) + 1(n-6) = 0 \Rightarrow (n-6)(n+1) = 0$

$\Rightarrow n = 6, -1$

But n cannot be negative

$\therefore n = 6$

(ii) $n! = 72(n-2)!$

$\Rightarrow n(n-1)(n-2)! = 72(n-2)! \Rightarrow n(n-1) = 72$

$\Rightarrow n^2 - n - 72 = 0 \Rightarrow n^2 - 9n + 8n - 72 = 0$

$\Rightarrow n(n-9) + 8(n-9) = 0 \Rightarrow (n-9)(n+8) = 0$

$\Rightarrow n = 9, -8$

But n cannot be negative

$\therefore n = 9$

E 4) In order to solve this problem, we have to perform 10 jobs, where each of first five jobs can be done in 4 ways and each of last five jobs can be done in 5 ways.

$\therefore$ by fundamental principle of multiplication required possible sequences of answers

$= 4\times4\times4\times4\times4\times5\times5\times5\times5\times5 = 4^5\times5^5 = 3200000$

E 5) (i) When repetition is not allowed. First, second, third and fourth positions can be filled up in 5, 4, 3, 2 ways respectively.

$\therefore$ total number of 4-letter words that can be formed be using the letters a, b, g, h, k
$= 5\times4\times3\times2 = 120$

(ii) When repetition is allowed. Each of first, second, third and fourth positions can be filled up in 5 ways.

$\therefore$ total number of 4-letter words that can be formed by using the letters a, b, g, h, k
$= 5\times5\times5\times5 = 5^4 = 625$

E 6) $^5P_n = {}^6P_{n-1} \Rightarrow \frac{5!}{(5-n)!} = \frac{6!}{(6-n+1)!}$

$$\Rightarrow \frac{5!}{(5-n)!} = \frac{6.5!}{(7-n)!} \Rightarrow \frac{1}{(5-n)!} = \frac{6}{(7-n).(6.-n).(5-n)!}$$
$$\Rightarrow (7-n)(6-n) = 6 \Rightarrow 42 - 13n + n^2 = 6 \Rightarrow n^2 - 13n + 36 = 0$$
$$\Rightarrow n^2 - 9n - 4n + 36 = 0 \Rightarrow n(n-9) - 4(n-9) = 0$$
$$\Rightarrow (n-9)(n-4) = 0 \Rightarrow n = 9, 4$$

But if n = 9, then $^5P_n = {}^5P_9$ becomes meaning less

$$\left[\begin{array}{l}\because \text{selection of 9 things out of 5} \\ \text{does not make any sense.}\end{array}\right]$$

$\therefore$ n = 4

E 7) (i) $^7P_4 = \frac{7!}{(7-4)!} \quad \left[\because \ ^nP_r = \frac{n!}{(n-r)!}\right]$

$$= \frac{7.6.5.4.3!}{3!} = 7.6.5.4 = 840$$

(ii) $^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n! \quad$ as 0! = 1

(iii) $^nP_{n-1} = \frac{n!}{(n-(n-1))!} = \frac{n!}{1!} = \frac{n!}{1} = n!$

(iv) $^nP_1 = \frac{n!}{(n-1)!} = \frac{n.(n-1)!}{(n-1)!} = n$

(v) $^nP_2 = \frac{n!}{(n-2)!} = \frac{n(n-1)(n-2)!}{(n-2)!} = n(n-1)$

(vi) $^{16}P_3 = \frac{16!}{(16-3)!} = \frac{16!}{(16-3)!} = \frac{16!}{13!} = \frac{16.15.14.13!}{13!} = 16.15.14 = 3360$

E 8) Possible number of ways = Total number of arrangement of 5 things taken all at a time

$$= {}^5P_5 = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5!}{1} = 120$$

E 9) Total number of flags = 3 + 4 + 2 = 9
Out of which 3 are of one kind, 4 are of second kind and 2 are of

third kind.

$\therefore$ required number of signals $= \dfrac{9!}{4!\,3!\,2!} = \dfrac{9.8.7.6.5.4!}{4! \times 6 \times 2} = \dfrac{9.8.7.6.5}{12} = 1260$

E 10) There are 4 letters in the word "AMAR". Out of which A occur twice.

$\therefore$ total number of permutations $= \dfrac{4!}{2!} = \dfrac{4 \times 3 \times 2!}{2!} = 4 \times 3 = 12$

E 11) (i) First prize can be given to any of the 5 students. Second prize can be given any of the remaining 4 students and similarly third prize can be given in 3 ways. $\left[\because \begin{array}{l}\text{no student gets}\\ \text{more than one prize}\end{array}\right]$

$\therefore$ required number of ways $= 5 \times 4 \times 3 = 60$

(ii) First prize can be given to any of the 5 students, i.e in 5 ways. Second and third each can also be given in 5 ways.

$\left[\because \begin{array}{l}\text{a student may get any}\\ \text{number of prizes}\end{array}\right]$

$\therefore$ required number of ways $= 5 \times 5 \times 5 = 5^3 = 125$

(iii) There are 5 ways that all the prizes come to the same student.
$\therefore$ required number of ways = 125 – 5 = 120

E 12) (i) Possible number of garlands with 8 flowers $= \dfrac{(8-1)!}{2} = \dfrac{7!}{2}$

$= \dfrac{5040}{2} = 2520$

(ii) Let P and V denote president and vice president respectively. Therefore if we consider these two members as a single member then we are left with 19 members. These 19 members can sit in a round table in (19–1)! ways. But president and vice president can change their seats in two ways (i.e. PV or VP).

$\therefore$ required number of ways of sitting the members in a meeting
$= (19-1)! \times 2! = (18!)\ (2!)$

E 13)

11 Batsmen 7 Bowlers 3 Wicket keepers.

Out of 11 batsmen 6 can be selected in ${}^{11}C_6$ ways.

Out of 7 bowlers 4 can be selected in ${}^{7}C_4$ ways.

Out of 3 wicket keepers 1 can be selected in ${}^{3}C_1$ ways.

$\therefore$ required number of ways $= {}^{11}C_6 \times {}^{7}C_4 \times {}^{3}C_1$

$= \dfrac{11.10.9.8.7.6}{6!} \times \dfrac{7.6.5.4}{4!} \times 3$

= 48510

E 14) Here concept of combination will be used, because we have to form possible groups not arrangement.

(i) Here n = 15, r = 11, s = 2

$\therefore$ required number of ways = ${}^{n-s}C_{r-s}$ [Refer part (i) of Result II]

$$= {}^{15-2}C_{11-2}$$

$$= {}^{13}C_9 = \frac{13\times 12\times 11\times 10}{4!} = 715$$

(ii) Here n = 15, r = 11, s = 1

$\therefore$ required number of ways = ${}^{n-s}C_r$ [Refer (ii) of Result II]

$$= {}^{15-1}C_{11} = {}^{14}C_{11} = \frac{14\times 13\times 12}{3!} = 364$$

E 15) (i) Here n = 9 r = 4, s = 3

$\therefore$ possible 4 digits numbers that can be formed using 9 digits 1 to 9 subject to the condition that three digits 1, 6, 8 are always included

$= {}^{n-s}P_{r-s}\times {}^{r}P_s = {}^{9-3}P_{4-3}\times {}^{4}P_3 = {}^{6}P_1\times {}^{4}P_3 = 6\times 4 = 24$

(ii) Here n = 9, r = 4, s = 2

$\therefore$ required possible numbers = ${}^{n-s}P_r = {}^{9-2}P_4 = {}^{7}P_4 = \dfrac{7!}{(7-4)!}$

$$= \frac{7\times 6\times 5\times 4\times 3!}{3!} = 840$$

E 16) Comparing $(1+\sqrt{2})^5$ with $(a+b)^n$, we get

$a = 1,\ b = \sqrt{2},\ n = 5$

$\therefore$ by binomial theorem

$$(1+\sqrt{2})^5 = {}^5C_0\,(1)^{5-0}(\sqrt{2})^0 + {}^5C_1\,(1)^{5-1}\,(\sqrt{2})^1 + {}^5C_2\,(1)^{5-2}\,(\sqrt{2})^2$$
$$+{}^5C_3\,(1)^{5-3}\,(\sqrt{2})^3 + {}^5C_4\,(1)^{5-4}\,(\sqrt{2})^4 + {}^5C_5\,(1)^{5-5}(\sqrt{2})^5$$
$$= 1 + 5(\sqrt{2}) + 10(2) + 10(2\sqrt{2}) + 5(4) + 1(4\sqrt{2})$$
$$\left[\because {}^5C_0 = {}^5C_5 = 1,\ {}^5C_1 = {}^5C_4 = 5,\ {}^5C_2 = {}^5C_3 = 10\right]$$
$$= 1 + 5\sqrt{2} + 20 + 20\sqrt{2} + 20 + 4\sqrt{2}$$
$$= 41 + 29\sqrt{2}$$

E 17) Comparing $(3x - y)^7$ with $(a+b)^n$, we get

$a = 3x,\ \ b = -y\ \ n = 7$

$\therefore$ by binomial theorem

$$(3x-y)^7 = {}^7C_0\,(3x)^{7-0}(-y)^0 + {}^7C_1\,(3x)^{7-1}(-y)^1 + {}^7C_2\,(3x)^{7-2}(-y)^2$$
$$+{}^7C_3\,(3x)^{7-3}(-y)^3 + {}^7C_4\,(3x)^{7-4}(-y)^4 + {}^7C_5\,(3x)^{7-5}(-y)^5$$
$$+\ {}^7C_6\,(3x)^{7-6}(-y)^6 + {}^7C_7\,(3x)^{7-7}(-y)^7$$
$$= 2187x^7 + 7(729x^6)(-y) + 21(243x^5)(y^2) + 35(81x^4)(-y^3)$$
$$+ 35(27x^3)(y^4) + 21(9x^2)(-y^5) + 7(3x)(y^6) - y^7$$
$$\left[\because {}^7C_0 = {}^7C_7 = 1,\ {}^7C_1 = {}^7C_6 = 7,\ {}^7C_2 = {}^7C_5 = 21,\ {}^7C_3 = {}^7C_4 = 35\right]$$
$$= 2187x^7 - 5103x^6y + 5103x^5y^2 - 2835x^4y^3 + 945x^3y^4 - 189x^2y^5$$
$$+ 21xy^6 - y^7$$

E 18) $$(1-3x)^{1/2} = 1 + \left(\frac{1}{2}\right)(-3x) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(-3x)^2 + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-3x)^3 + \ldots$$

This expansion is valid only if $|3x| < 1$, i.e. if $|x| < \frac{1}{3}$

$$\therefore \; (1-3x)^{1/2} = 1 - \frac{3}{2}x - \frac{9}{8}x^2 - \frac{27}{16}x^3 + \ldots \text{ , if } |x| < \frac{1}{3}$$