UNIT 2 FUNCTIONS

Structure

2.1 INTRODUCTION

Many times, we observe the association of the elements of one set with the elements of another set. For example, roll numbers of students in an examination of a university are associated with their corresponding marks. Such type of associations are discussed under the heading "Function". In this unit, we will focus on definition of function, its classification and various types.

In this unit, we will use some concepts related to sets discussed in the preceding unit.

Objectives

After completing this unit, you should be able to:

- define constant quantity, variable, interval, give some examples of each;
- define function, some particular functions;
- evaluate the value of some particular functions at given points;
- get an idea of one-one, onto and one to one correspondence and their geometrical interpretation; and
- define countable and uncountable sets.

2.2 QUANTITY

Before defining function, let us first explain what we mean by quantity, constant, variable and intervals. See Fig. 2.1

Quantity

Here, by quantity we mean those things on which four basic mathematical operations addition, subtraction, multiplication and division can be applied. For example, temperature, height, weight, and time all these are quantities but they are continuous in nature, where as books in a library, number of trees, number of balls is discrete in nature.

Note 1: If the nature of the quantity is such that it can take any possible value between two certain limits then such a quantity is known as continuous in nature.

Let us consider the following example:

Suppose at the time of birth, height of a baby was 1.5 ft and after 15 years, the height of the same baby is 5 ft, then we know that height of this baby took all possible values between 1.5ft. to 5ft. That is, this is not the case that at the time of birth the height was 1.5 ft and in the next moment it reached to1.6 ft and in successive moment to 1.7 ft. In fact, there are infinitely many values between 1.5 ft and 1.6 ft and all these values were taken by the height of that baby.

Note 2: If the nature of the quantity is such that it can take at most countable values between two certain limits then such a quantity is known as discrete in nature. Countable set is defined in Sec. 2.6 of this unit.

Let us consider an example:

Number of children per family in a locality is an example of discrete quantity.

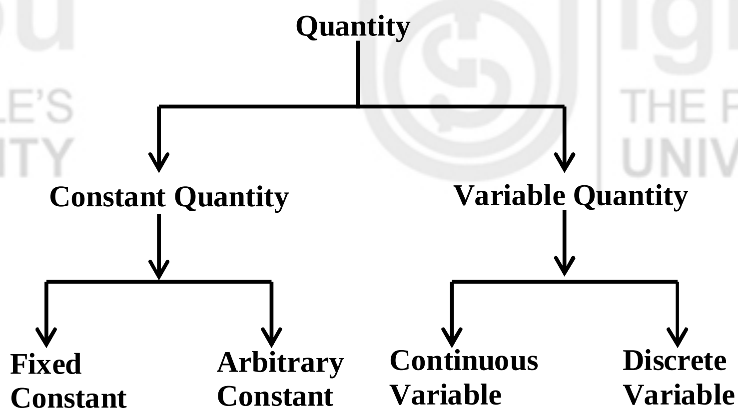


Fig. 2.1

Constant Quantity

A quantity which remains same (unchanged) throughout a particular investigation is known as a constant quantity.

Fixed Constant

Those types of constants which always remain same (unchanged) independent of the purpose of user, place and time are known as fixed constants.

For example,

(i) $2, 5, \frac{3}{2}, \sqrt{13}, -\sqrt{17},$, etc.

(ii) quotient of circumference of a circle and its diameter which is always equal to π.

Arbitrary Constants

Those types of constants which remain same in one problem but may vary from problem to problem are known as arbitrary constants and are generally denoted by a, b, c, l, m, n, $\alpha, \beta, \gamma,$ etc.

For example, heights of the houses constructed by the different owners of the plots in a locality as per their own choice, is an example of arbitrary constant because heights of the houses vary from house to house as it depends on the choice of the owners.

Variable

A quantity which may change its value even in a particular problem is known as variable.

For example, blood pressures of a person as it vary time to time.

2.3 INTERVAL

Let R be the set of all real numbers. Then a set $I \subseteq R$ is said to be an interval if whenever $a, b \in I$ and $a < x < b$ then $x \in I$

For example, the set of all real numbers satisfying $2 \le x \le 3$ is an interval where x can take any real value between 2 and 3 including 2 and 3.

Open Interval

An open interval $I \subseteq R$ with end points a and b (a < b) is denoted by (a, b) and is defined by

$(a, b) = \{x \in R : a < x < b\}$

i.e. an open interval contains each value between the end points but does not include the end points.

For example, open interval (2, 5) contains each real number lying between 2 and 5 but does not contain 2 and 5.

Closed Interval

A closed interval $I \subseteq R$ with end points a and b (a < b) is denoted by [a, b] and is defined by

$[a, b] = \{x \in R : a \le x \le b\}$

i.e. a closed interval contains each value between and including extreme values.

For example, closed interval [2, 5] contains each real number lying between 2 and 5. It also contains its end points. i.e.

$x \in [2,5] \ \ \forall \ x, \ 2 < x < 5$ and

also $2 \in [2,5], \ 5 \in [2,5]$

Left Open and Right Closed Interval

A left open and right closed interval $I \subseteq R$ with end points a and b (a < b) is denoted by (a, b] and is defined as

$(a, b] = \{x \in R : a < x \le b\}$

In this case $x \in (a,b], \forall x, \ a < x \le b$ and $a \notin (a,b]$, but $b \in (a,b]$

Left Closed and Right Open Interval

A left closed and right open interval $I \subseteq R$ with end points a and b (a < b) is denoted by [a, b) and is defined as

$[a, b) = \{x \in R : a \le x < b\}$

In this case $x \in [a,b), \forall x, \ a \le x < b$ and $a \in [a,b)$, but $b \notin [a,b)$

Length of an Interval

Length of each of the intervals (a, b), [a, b], (a, b], [a, b) is defined as $b - a, a < b$

i.e. length of the interval = difference of the end points

For example, if I = (2, 7) then $l(I) = 7 - 2 = 5$, where $l(I)$ denotes the length of the interval I.

Finite Interval

An interval is said to be finite if its length is finite.

For example, if I = (– 3, 5) then $l(I) = 5 - (-3) = 5 + 3 = 8$ which is finite.
$\therefore$ interval I is finite.

Infinite Interval

An interval is said to be infinite interval if its length is not finite.

For example,

(i) The set $\{x \in R : x > a, a \in R\}$ is an infinite interval and is denoted by (a, ∞)

(ii) The set $\{x \in R : x < a, a \in R\}$ is an infinite interval and is denoted by $(-\infty, a)$

Similarly, infinite intervals $[a, \infty)$, $(-\infty, a]$ are defined as
$[a, \infty) = \{x \in R : x \geq a\}$, where a is a fixed real number
$(-\infty, a] = \{x \in R : x \leq a\}$, where a is a fixed real number

Remark 1:

(i) Each interval contains infinitely many elements.

(ii) Each interval is an infinite set but an infinite set may or may not be an interval. For example N, W, Z, Q are infinite sets but are not intervals.

(iii) A set may or may not be an interval.

(iv) R, set of real numbers, is an infinite interval given by $R = (-\infty, \infty)$

(v) Remember that $-\infty$ and ∞ are not included in the set of real numbers. Also, if extreme value is ∞ or $-\infty$, then open bracket is used on the side having extreme value.

(vi) When we say that x is a finite number/real number it means that $-\infty < x < \infty$

Now we are in a position to define function.

2.4 FUNCTION

Definition of Function

There are two conditions for a rule to be a function.
(i) Each element of X must be associated to some element of Y.
(ii) There is unique element of Y corresponding to each element of X.

Let X and Y be two sets. Then a rule which associates each element of X to a unique element of Y is called a function.
X is called domain of the function.
Y is called co-domain of the function and set of only those values of Y for which function is defined is called range of the function.
That is, subset $\{y \in Y : y = f(x) \text{ for some } x \in X\}$ of Y is called range of the function.

Notation:

(i) A function is generally denoted by f, g, h, etc., in the case of above definition we write f: $X \to Y$ and read as f is a function from X to Y.

(ii) A function f: $X \to Y$ is generally described by writing
$y = f(x), \quad x \in X$, where $f(x)$ is an expression in terms of x.

Pictorial Presentation of a Function

A function can be presented diagrammatically. As shown in the example of mothers and daughters discussed below.

To get a clear cut idea of the definition of a function without cramming and for a long time memory. Let us consider a real life example.

Let X = set of daughters and Y = set of mothers

Then consider the following situations given in (a), (b), (c) and (d) with the help of the diagrams.

(a)

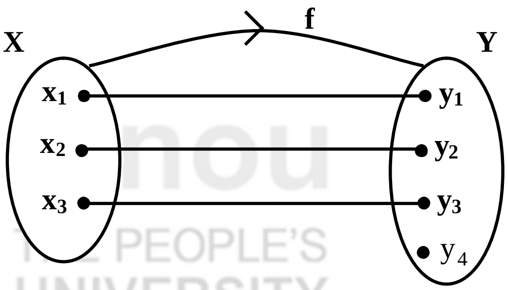


Fig. 2.2

The rule f shown in Fig. 2.2 is a function because each daughter has unique mother. [i.e. both conditions mentioned in the box are satisfied] and domain of the function $f =$ set of all daughters = X = $\{x_1, x_2, x_3\}$

co-domain of this function = set of all mothers = Y = $\{y_1, y_2, y_3, y_4\}$

range of this function = set of those mothers who has at least one daughter $= \{y_1, y_2, y_3\}$

Note 3: One point which may come in your mind is that if y_4 is a mother then there should be at least one daughter of y_4. But as we know that to become a mother it is not necessary that there should be a daughter. A mother may have only one son or only two sons or more than two sons without a daughter.

(b)

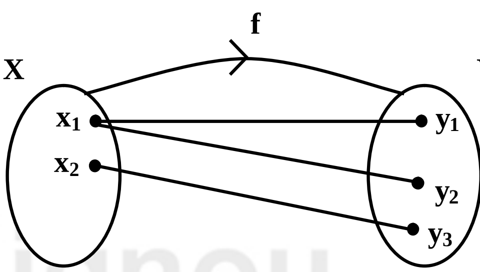


Fig. 2.3

The rule f shown in Fig. 2.3 is not a function because x_1 has two mothers y_1, y_2 which is not possible. [i.e. condition (ii) given in the box is not satisfied]

(c)

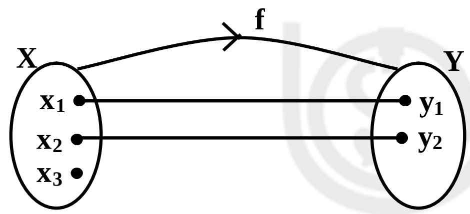


Fig. 2.4

The rule f shown in Fig. 2.4 is not a function because daughter x_3 has no mother. If x_3 came in this world then there should be some mother of x_3. [i.e. condition (i) given in the box is not satisfied]

(d)

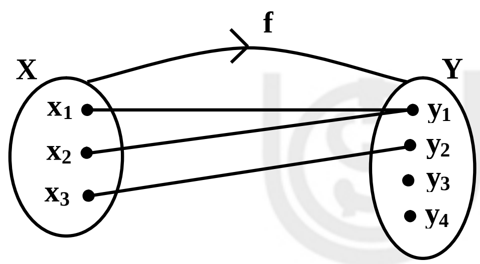


Fig. 2.5

The rule f shown in Fig. 2.5 is a function because each daughter has unique mother. We see that x_1, x_2 both have same mother, no problem it is possible. Further mothers y_3, y_4 have no daughters again no problem it is also possible. In this case:

domain of the function $f = \{x_1, x_2, x_3\} = X =$ set of all daughters,

co-domain of the function $f = \{y_1, y_2, y_3, y_4\} = Y =$ set of all mothers and

range of the function $f = \{y_1, y_2\} =$ set of only those mothers who have at least one daughter.

Some more Concepts Related to Function are given as under

(i) If $f : X \to Y$ is a function given by $y = f(x)$ then x is known a pre image of y and y is known as image of x.

(ii) If $y = f(x)$ is a function then values of y depend on values of x. So, y is known as dependent variable and x is known as independent variable.

Let us now take up some examples which will enable you to distinguish as to whether a rule is a function or not.

For example,

(a)

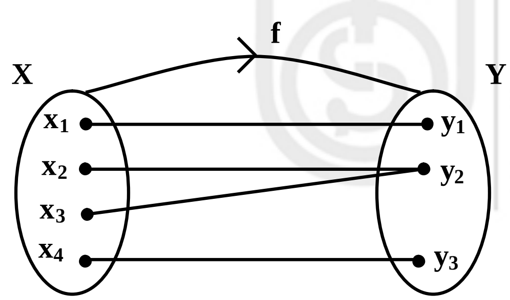


Fig. 2.6

The rule f shown in Fig. 2.6 is a function because it satisfies both conditions i.e.

(i) Each element of X is associated to some element of Y

(ii) There is unique element of Y corresponding to each element of X.

i.e. y_1 is unique element of Y corresponding to $x_1 \in X$

y_2 is unique element of Y corresponding to both $x_2, x_3 \in X$

y_3 is unique element of Y corresponding to $x_4 \in X$

Here domain of f $= \{x_1, x_2, x_3, x_4\}$

Co-domain of f $= \{y_1, y_2, y_3\}$ and range of f $= \{y_1, y_2, y_3\}$

(b)

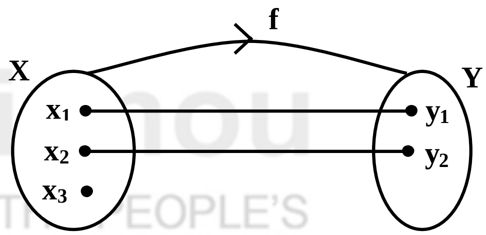


Fig. 2.7

The rule f shown in Fig. 2.7 is not a function because $x_3 \in X$, but it is not associated to the element of Y.

$\left[\because\right.$ out of two restriction for a rule to be a function, first is that each element of X must be associated to some element of Y.$\left.\right]$

(c)

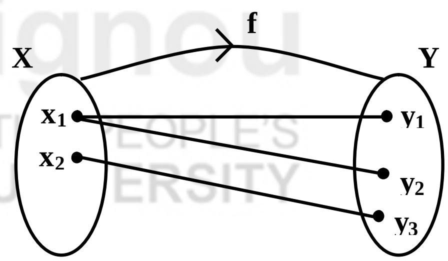


Fig. 2.8

The rule f shown in Fig. 2.8 is also not a function because $x_1 \in X$ is not associated to unique element of Y.

$\left[\because\right.$ for a rule to be a function it is must that each element of X is associated to a unique elemeny of Y.$\left.\right]$

(d)

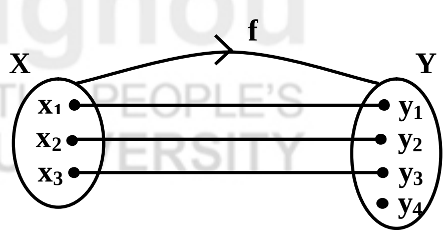


Fig. 2.9

The rule f shown in Fig. 2.9 is a function because it satisfies both the conditions for a rule to be a function, i.e.

(i) Each element of X is associated to some element of Y.

(ii) There is unique element of Y corresponding to each element of X.

Here domain of the function $f = \{x_1, x_2, x_3\} = X$

Co-domain of the function $f = \{y_1, y_2, y_3, y_4\} = Y$

Range of the function $f = \{y_1, y_2, y_3\} \subset Y$

Some Examples of Functions

Example 1: Let $f : N \to N$ defined by
f(n) = 3n, $n \in N$
Express the function diagrammatically. Also write domain, range and co-domain of the function.

Solution: $f : N \to N$ defined by

$f(x) = 3n, \quad n \in N$
$\therefore f(1) = 3,\ f(2) = 6,\ f(3) = 9$ and so on. See Fig. 2.10

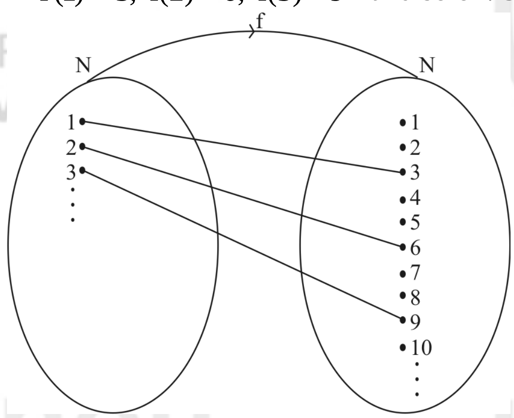


Fig. 2.10

Domain of the function $f = \{1,\ 2,\ 3,\ ...\} = N$
Range of the function f = Set of only those values for which function is define
= {3, 6, 9, …}

Co-domain = Set of all values of Y = {1, 2, 3, …}= N

Example 2: If $f : R \to R$ be a function defined by
$f(x) = 3^x, \quad x \in R$, then obtain (i) Domain of f (ii) Range of f

Solution: $f : R \to R$ is defined by

$f(x) = 3^x, \quad x \in R$

(i) Since $f(x)$ is defined for all $x \in R$

$\therefore$ domain of $f = R =$ set of all real numbers

(ii) We Know that

$3^x > 0 \qquad \forall\ x \in R$

i.e. $f(x) > 0 \qquad \forall\ x \in R$

$\therefore$ range of f = $(0, \infty)$

Example 3: Find the domain of the function $f : R \to R$, defined by

$f(x) = \sqrt{(x-3)(5-x)}\ , \quad x \in R$
Also evaluate f(3), f(4), f(5).

Solution: Given function is

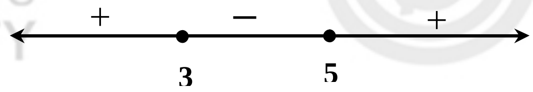
$f(x) = \sqrt{(x-3)(5-x)}, \quad x \in R$

For $f(x)$ to be real, the quantity under the square root should be non negative and hence

$(x-3)(5-x) \geq 0$

$\Rightarrow -(x-3)(x-5) \geq 0$

$\Rightarrow (x-3)(x-5) \leq 0$



Now, when we take x less then 3, the L.H.S. comes out to be
(–ve) (–ve) = + ve hence does not satisfy the inequality.
Also, if we take x greater than 5, the L.H.S. comes out to be
(+ ve) (+ ve) = + ve and hence this value does not satisfy the inequality.
But, if we take 3 < x < 5, the L.H.S. becomes (+ ve) (–ve) = –ve and hence satisfies the inequality.

$\Rightarrow x \in [3,5]$

$\therefore$ domain of $f = [3,5]$

Also $f(3) = \sqrt{(3-3)(5-3)} = \sqrt{0 \times 2} = \sqrt{0} = 0$

$f(4) = \sqrt{(4-3)(5-4)} = \sqrt{1 \times 1} = 1$

$f(5) = \sqrt{(5-3)(5-5)} = \sqrt{2 \times 0} = \sqrt{0} = 0$

Now, you can try the following exercises.

E 1) Find the domain and range of the function f: $R \to R$ given by

$f(x) = 4x + 5, x \in R$. Also, evaluate f(0), $f\left(\frac{1}{2}\right)$.

E 2) Find the domain and range of the function f: $R \to R$ given by

$f(x) = \frac{1}{x-2}, x \in R$. Also, evaluate f(1), $f(3)$, $f(-5)$.

E 3) Find the domain and range of the function f: $R \to R$ given by

$f(x) = \frac{1}{x+3}, x \in R$. Also, evaluate f(0), $f(2)$, $f(-3)$ if possible.

2.5 CLASSIFICATION OF FUNCTIONS WITH THEIR GRAPHS

In previous Sec. we have seen that function is a rule (satisfying two conditions mentioned in the box at page number 30) which associates the elements of one set to the elements of another set. Based on the nature of classification a function may be given some particular names. In this section you will meet some of these commonly used names, their definitions followed by some examples.

Constant Function

Let $X, Y \subseteq R$, then a function $f : X \to Y$ is said to be a constant function if it is defined as

$f(x) = a, \quad \forall x \in R$, where a is a real constant.

i.e. a function is constant if range is a singleton set.
i.e. all elements of the domain are associated to a single element of the co-domain of the function.

For example,

(i) $f : N \to N$ defined by

$f(x) = 3, \qquad n \in N$

is a constant function because all elements of the domain are associated to the single element 3 as shown in the Fig. 2.11

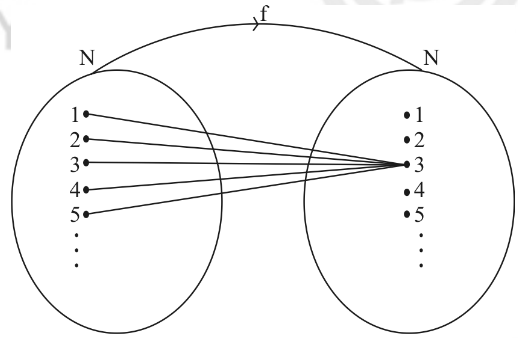


Fig. 2.11

(ii) $f : R \to R$ defined by

$f(x) = 2, \ x \in R$

is also a constant function and its graph is given below in Fig. 2.12.

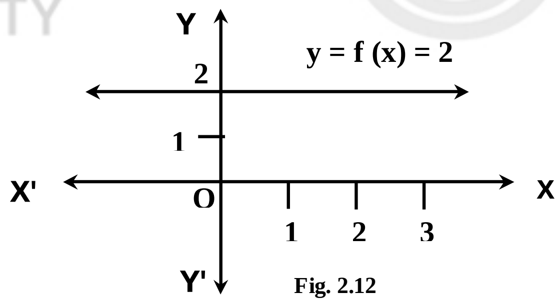


Fig. 2.12

Identity Function

Let $X \subseteq R$, a function $f : X \to X$ is said to be an identity function if it is defined as

$f(x) = x, \qquad x \in X$

i.e. a function is said to be identity function if each element is associated to itself.

For example,

(i) $f : N \to N$ defined by

$f(n) = n, \qquad n \in N$

is an identity function as shown in the Fig. 2.13.

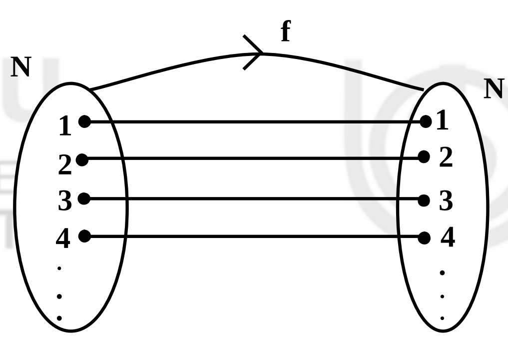


Fig. 2.13

(ii) Function $f : R \to R$ defined by

$f(x) = x, \quad x \in R$

is also an identity function and its graph is given in Fig. 2.14.

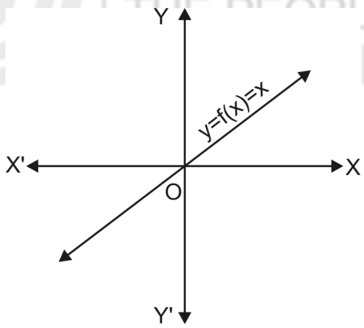


Fig. 2.14

Polynomial Function

A function $f: R \to R$ is said to be a polynomial function of degree n if it is defined as

$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n, x \in R$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n \in R$, $a_0 \neq 0$ are constants

e.g. f(x) = $2x^3 + x^2 - x + 5$, is a polynomial function of degree 3.

Linear Function (Polynomial Function of Degree 1)

A function $f: R \to R$ is said to be linear function if it is defined as

$f(x) = ax + b, \quad x \in R$, where $a, b \in R, a \neq 0$ are real constants.

Graph of the function $f: R \to R$ defined by

$f(x) = 2x + 3, \quad x \in R$

is given in Fig. 2.15.

$y = f(x) = 2x + 3$

$y = 2x + 3$

x	0	1	2
y	3	5	7

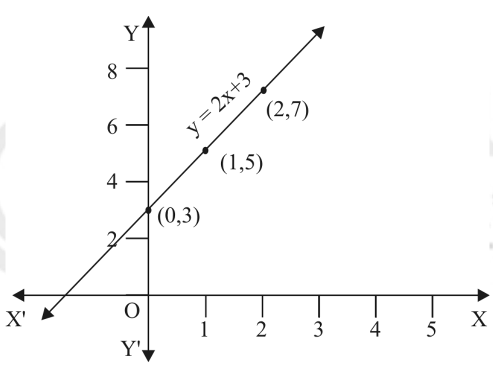


Fig. 2.15

Logarithm Function

A function $f: R^+ \to R$ is said to be logarithm function if it is defined as

$y = f(x) = \log_a x, \quad x \in R^+ =$ set of all positive real numbers.

where $a > 0$ and $a \neq 1$

If $a^m = n$ then in terms of logarithm we write it as $\log_a n = m$

i.e. $2^3 = 8 \Rightarrow \log_2 8 = 3$

$2^4 = 16 \Rightarrow \log_2 16 = 4$

$4^3 = 64 \Rightarrow \log_4 64 = 3$

$16^{\frac{3}{4}} = 8 \Rightarrow \log_{16} 8 = \frac{3}{4}$

Graph of $y = \log_a x, \quad a > 0, \quad a \neq 1, \quad x > 0$ is given in Fig. 2.16 a, b.

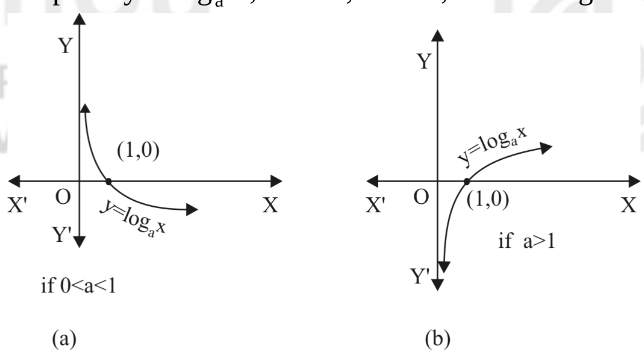


Fig. 2.16

Domain of logarithm function is R^+ and range of logarithm function is R.

Laws of Logarithm

1. $\log_a mn = \log_a m + \log_a n$
2. $\log_a \frac{m}{n} = \log_a m - \log_a n$
3. $\log_a m^n = n \log_a m$
4. $a^{\log_a m} = m$
5. $\log_a a = 1$
6. $\log_a b = \frac{1}{\log_b a}$
7. $\log_a b = \frac{\log_n b}{\log_n a}$ this is known as base change formula, infact we can take any base in place of n.

In mathematics number e is denoted by the sum of an infinite series $1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots$ In general, expansion for e^x is given by $1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots$ where $\lfloor 2$ read as 2 factorial, etc. Factorial and its notations have been discussed in Sec. 4.2 in Unit 4 of this block.

Remark 2:

(i) If base of the logarithm is 10 then it is known as common logarithm.

(ii) If base of the logarithm is e then it is known as natural logarithm and some time is written as In x instead of log x.

(iii) When we write log x it means base is e. That is, in most of the cases base is mentioned only when it is other than e.

Exponential Function

A function $f : R \to R$ defined by

$f(x) = a^x, \quad x \in R, \quad a > 0, \; a \neq 1$

is called exponential function.

i.e. in case of exponential function there is a constant in the base and variable in the exponent.

i.e. nature of exponential function = $(\text{Constant})^{\text{Variable}}$

For example, $f(x) = 2^x$ is an exponential function.

Graph of the exponential function is shown in Fig. 2.17 (a), (b).

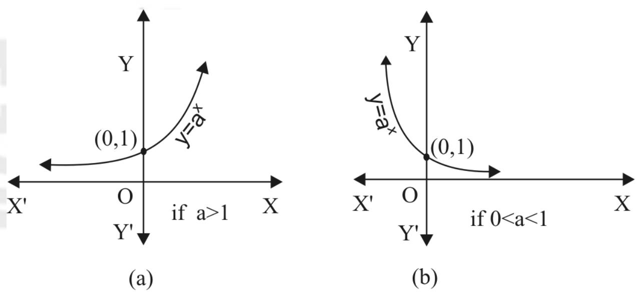


Fig. 2.17

Absolute Value Function or Modulus Function

A function $f : R \to R$ defined by

$y = f(x) = |x|, \quad \text{where } x \in R \text{ and } |x| = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}$

is called absolute value function and graph of this function is given in Fig. 2.18

Domain of this function = R and range of this function = $[0, \infty)$.

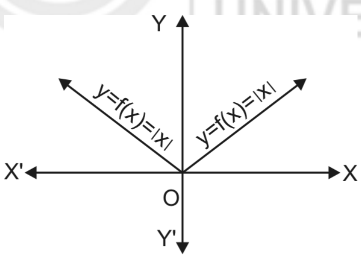


Fig. 2.18

Let us see how we calculate the value of modulus function at some particular point with the help of following example.

Example 4: If $f(x) = |x| - 3$ then evaluate $f(2), f(-2), f(3), f(-3), f(-7)$.

Solution:

$f(2) = |2| - 3 = 2 - 3 = -1 \quad \left[\begin{array}{l} \because 2 > 0, \text{ so by definition of} \\ \text{modulus function } |2| = 2 \end{array}\right]$

$f(-2) = |-2| - 3 = -(-2) - 3 = 2 - 3 = -1 \left[\begin{array}{l} \because -2 < 0, \text{so by definition of} \\ \text{modulus function } |-2| = -(-2) \end{array}\right]$

$f(3) = |3| - 3 = 3 - 3 = 0$

$f(-3) = |-3| - 3 = -(-3) - 3 = 3 - 3 = 0$

$f(-7) = |-7| - 3 = -(-7) - 3 = 7 - 3 = 4$

Here is an exercise for you.

E 4 If $f(x) = 5 - |x - 3|$ then evaluate $f(2), f(-2), f(6), f(-5), f(12)$.

Even Function

A function $f(x)$ is said to be even function if it satisfies

$f(-x) = f(x)$, for all points x of the domain of the function f.

i.e. the value of function remains unchanged on changing x to – x.

For example,

(i) $f(x) = x^2 + x^4$

$f(-x) = (-x)^2 + (-x)^4 = x^2 + x^4 = f(x)$

$\therefore$ it is an even function.

(ii) $f(x) = |x|$

$f(-x) = |-x| = |(-1)(x)| = |(-1)||x| = (1)|x|$ as $|-1| = -(-1) = 1$

$= |x|$

$= f(x)$

$\therefore$ it is an even function

(iii) $f(x) = x^2 + x^3$

$f(-x) = (-x)^2 + (-x)^3 = x^2 - x^3 \neq f(x)$

$\therefore$ it is not an even function.

Odd Function

A function $f(x)$ is said to be odd function if it satisfies

$f(-x) = -f(x)$, for all points x of the domain of the function f

i.e. the value of the function becomes – ve on changing x to –x.

For example,

(i) $f(x) = x^3 + x$

$f(-x) = (-x)^3 + (-x) = -x^3 - x = -(x^3 + x) = -f(x)$

$\therefore$ it is an odd function.

(ii) $f(x) = \dfrac{1}{x^3}$

$f(-x) = \dfrac{1}{(-x)^3} = \dfrac{1}{-x^3} = -\dfrac{1}{x^3} = -f(x)$

$\therefore$ it is an odd function.

(iii) $f(x) = x^2 + x^3$

$f(-x) = (-x)^2 + (-x)^3 = x^2 - x^3 = -(x^3 - x^2) \neq -f(x)$

$\therefore$ it is not an odd function

We see that if $f(x) = x^2 + x^3$ then neither $f(-x) = f(x)$ nor $f(-x) = -f(x)$

$\therefore f(x) = x^2 + x^3$ is neither even nor odd function.

2.6 TYPES OF FUNCTIONS

One-One Function

A function $f : X \to Y$ is said to be 1-1 or **injective** function if distinct elements of X are associated to distinct elements of Y under f.

i.e. if $x_1, x_2 \in X$ be s.t.

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Or if $x_1, x_2 \in X$ and $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

If we compare this definition with example of daughters and mothers then one-one function means each daughter must have different mother, i.e. there cannot be two daughters having same mother for a function to be one-one.

For example,

(i)

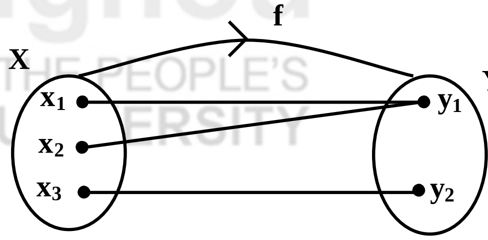


Fig. 2.19

The function f shown in Fig. 2.19 is not one-one functions because two different elements x_1, x_2 of X have the same image y_1.

(ii)

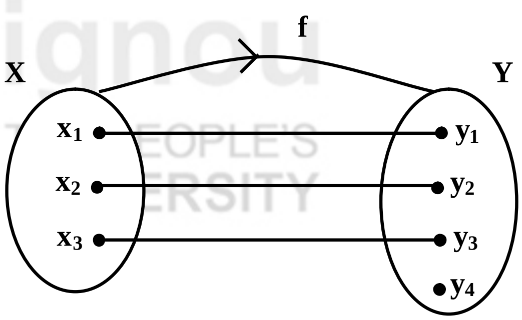


Fig. 2.20

The function f shown in Fig. 2.20 is one-one function because all the three elements of X have distinct images in Y.

Remark 3: If we want to show that a function $f(x)$ is one-one then we take $x_1, x_2 \in X$ s.t.

$f(x_1) = f(x_2)$ and we have to show that $x_1 = x_2$.

For example,

(i) Show that the function $f : R \to R$ defined by $f(x) = 7x + 5$ is 1-1 function.

Solution: Let x_1, x_2 be s.t.

$f(x_1) = f(x_2) \Rightarrow 7x_1 + 5 = 7x_2 + 5 \Rightarrow 7x_1 = 7x_2 \Rightarrow x_1 = x_2$

$\Rightarrow$ f is 1-1 function

(ii) Check whether the function $f : R \to R$ defined by $f(x) = x^2$ is 1-1 or not.

Solution: $f(x) = x^2$

Let $x_1 = 2$, $x_2 = -2$ then $x_1 \neq x_2$

But $f(x_1) = f(2) = (2)^2 = 4$ and $f(x_2) = f(-2) = (-2)^2 = 4$

$\therefore f(2) = f(-2)$, i.e. $f(x_1) = f(x_2)$ but $x_1 \neq x_2$

$\Rightarrow f$ is not 1-1 function.

Onto Function

A function $f : X \to Y$ is said to be onto or **surjective** if each element of Y has at least one pre image in X.

i.e. for each $y \in Y$, there exists at least one $x \in X$ such that

$f(x) = y$

If we compare this definition with example of daughters and mothers then onto function means each mother must have at least one daughter.

For example,

(i)

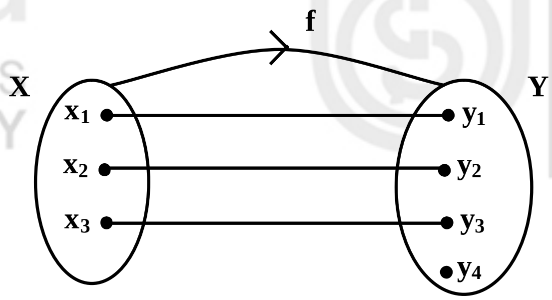


Fig. 2.21

The function f shown in Fig. 2.21 is not onto function because $y_4 \in Y$ but there is no $x \in X$ such that

$y_4 = f(x)$

(ii)

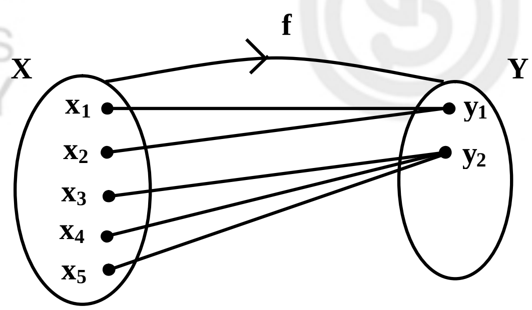


Fig. 2.22

The function f shown in Fig. 2.22 is onto function because each element of Y has at least one pre image, i.e. y_1 has two pre images and y_2 has three pre images.

Remark 4: If we want to show that a function $f(x)$ is onto then first we take an element y in Y and we have to show that there exists an element x is X such that $f(x) = y$

For example, show that the function $f : R \to R$ defined by $f(x) = 7x + 5$ is onto function.

Solution: Here X = R, Y = R

Let $y \in Y = R$, then $\frac{y-5}{7} \in X = R$ s.t.

$$f\left(\frac{y-5}{7}\right) = 7\left(\frac{y-5}{7}\right) + 5 = (y-5) + 5 = y$$

$\therefore f$ is an onto function.

One-One and Onto Function

A function $f : X \to Y$ is said to be one-one and onto or **bijective** or **one-one correspondence** if

(i) f is one-one

(ii) f is onto

If we compare this definition with example of daughters and mothers then one-one and onto function means each mother have exactly one daughter and there is no mother who does not have any daughter. The function f shown in Fig.2.23 represents a situation of one-one and onto function.

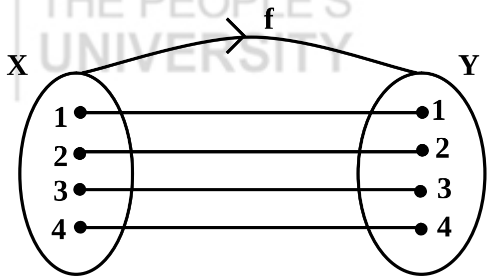


Fig. 2.23

Also the function $f : R \to R$ defined by
$f(x) = 7x + 5, \qquad x \in R$
is one-one and onto (already shown)
$\therefore f$ is one-one and onto function.

Geometrical Meaning of Injective, Surjective and Bijective Functions

One-One Function

If a function is 1-1 then geometrically it will satisfy the following condition. Each horizontal line either does not intersect the graph of the function or if it intersects it will intersect exactly at one point.

For example,

(i) Graph shown in Fig. 2.24 (b) is the graph of a one-one function because each horizontal line either does not intersect the graph or if it intersects, it will intersect exactly at one point, i.e. each horizontal line above x-axis will intersect the graph exactly at one point and each horizontal line below x- axis not intersect the graph at all.

(ii) Graph shown in the Fig. 2.24 (c) is also the graph of a one-one function because each horizontal line intersects the graph exactly at one point.

(iii) But the graph shown in the Fig. 2.24 (a) is not the graph of a one-one function because if we draw any horizontal line below the x-axis, then it will intersect the graph at two points. Similarly graph shown in Fig. 2.24 (d) is also not a one-one function.

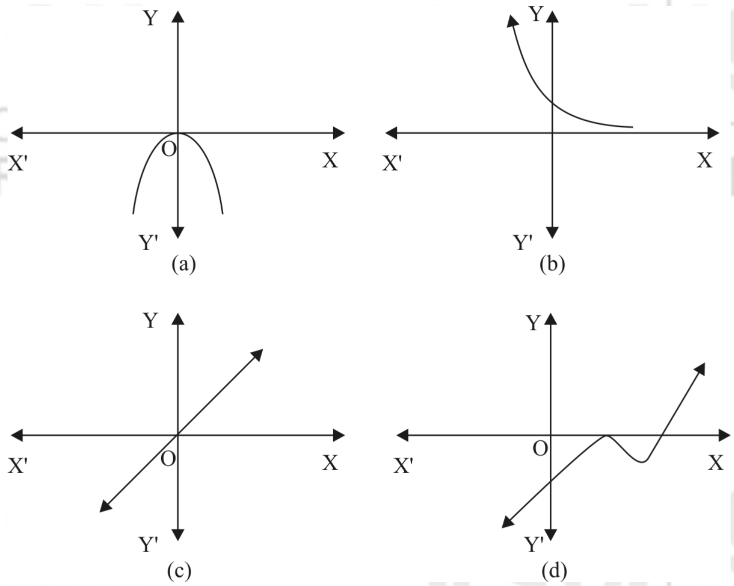


Fig. 2.24

Onto or Surjective Function

If a function is onto then geometrically it will satisfy the following condition. Each horizontal line must intersect the graph of the function at least at one point.

For example,

(i) Graph shown in the Fig. 2.24 (d) is the graph of an onto function because each horizontal line intersect the graph of the function at least at one point. Graph shown in Fig. 2.24 (c) is also onto function because each horizontal line intersects the graph exactly at one point.

(ii) But the graph shown in the Fig. 2.24 (b) is not the graph of a surjective function because if we draw any horizontal line below the x-axis, then it will not intersect the graph.

(iii) Similarly the graph shown in the Fig. 2.24 (a) is not the graph of an onto function because if we draw any horizontal line above the x-axis, then it will not intersect the graph.

Bijective Function

If a function is bijective or one-one and onto or one-one correspondence, then each horizontal line must intersect the graph of the function exactly at one point.

For example,

(i) Graph shown in the Fig. 2.24 (c) is the graph of a bijective function because each horizontal line intersects the graph of the function exactly at one point.

(ii) But the graphs shown in the Fig. 2.24 (a) and (b) are not the graphs of a bijective function because they are not onto.

(iii) Similarly the graph shown in the Fig. 2.24 (d) is not the graph of a bijective function because it is not one-one.

Countable Sets

Equivalent Sets: Two sets A and B are said to be equivalent if either there exists a one- one correspondence from A to B or from B to A and is denoted by $A \sim B$.

For example, let A = {1, 2, 3, 4} and B = {1, 4, 9, 16}, then $A \sim B$ because there exists a one-one correspondence $f : A \to B$ defined by
$f(x) = x^2, \quad x \in A$ between A and B as shown in Fig. 2.25

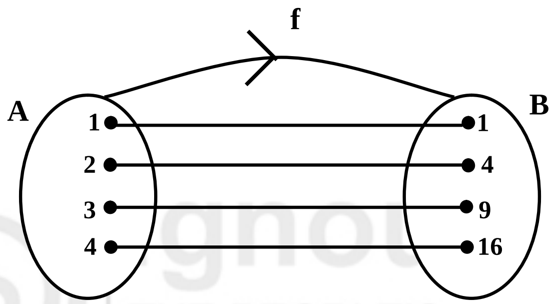


Fig. 2.25

Enumerable Set: A set E is said to be enumerable, if it is equivalent to the set of natural numbers, i.e. if $N \sim E$
i.e. if there exists a one- one correspondence between N and E.
An enumerable set is also known as denumerable set or countably infinite set.

For example,

(i) Let E = $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$

Define a map $f : N \to E$ by

$f(n) = \frac{1}{n}, \quad n \in N$

Then f is both 1–1 and onto as shown in Fig. 2.26.
$\Rightarrow N \sim E \Rightarrow$ E is enumerable.

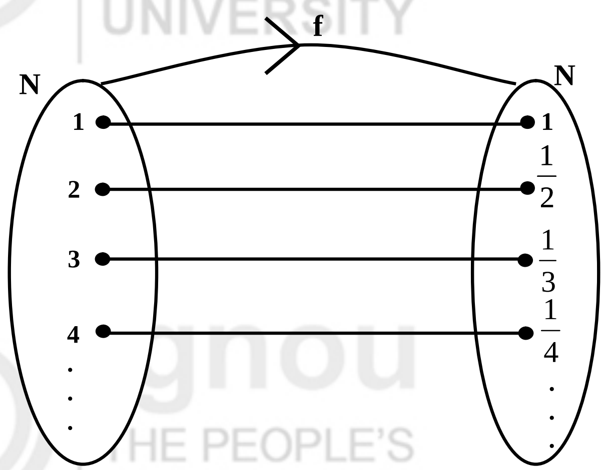


Fig. 2.26

(ii) Let A = {3, 6, 9, 12, …}

Define a map $f : N \to A$ by
$f(n) = 3n, \quad n \in N$
Then f is both 1–1 and onto as shown in Fig. 2.27.
$\Rightarrow N \sim A \Rightarrow$ A is enumerable.

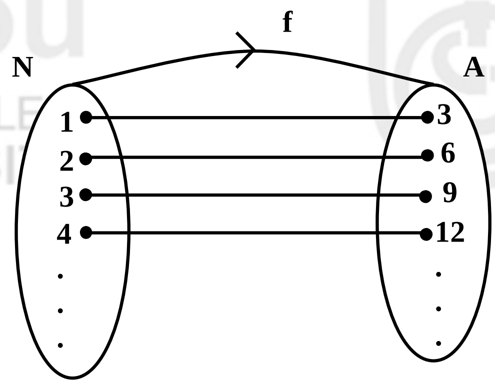


Fig. 2.27

Here is an exercise for you.

E 5 Show that

(i) A = {5, 25, 125, 625,…} (ii) B = {1, 5, 25, 125, 625,…}

(iii) C = {1, 4, 7, 10, 13,…}

all are enumerable sets.

Countable Set: A set is said to be countable if either it is finite or enumerable

For example,

(i) A = {} = ϕ , which is a finite set so it is countable.

(ii) B = {a, b, c}, which is a finite set and hence countable.

(iii) C = $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$, which is an enumerable set (already shown), so it is a countable set.

Remark 5: In the fifth unit of Course-3, i.e. MST-003, you will meet the word countable in the definition of the discrete random variable. So it becomes very important to understand what we mean by countable set.

We close this unit by summarising the topics that we have discussed in this unit:

2.7 SUMMARY

In this unit we have covered following topics:

1) Quantity, constant quantity, variable.
2) Interval, open interval, closed interval, semi-open and closed interval, finite and infinite intervals.
3) Function and its classification with examples and their graphs.
4) Types of functions, i.e.1-1, onto and one-one correspondence with their geometrical interpretation.
5) Equivalent sets, enumerable sets and countable sets.

2.8 SOLUTIONS/ANSWERS

E 1) Since f(x) takes real values for all $x \in R$.
$\therefore$ domain of f = R
Also, as x vary, over R, then 4x + 5 also vary over R, so range of f = R
Now, $f(0) = 4(0) + 5 = 5$ and $f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right) + 5 = 2 + 5 = 7$

E 2) Here f(x), takes real values for all $x \in R$, except at x = 2
i.e. f(x) is defined for all real values of x, except at x = 2
$\therefore$ domain of f = R – {2} and
f(x) cannot be zero at any real number,
$\therefore$ range of f = R – {0}
Also, $f(1) = \frac{1}{1-2} = \frac{1}{-1} = -1$, $f(3) = \frac{1}{3-2} = \frac{1}{1} = 1$ and
$f(-5) = \frac{1}{-5-2} = \frac{1}{-7} = -\frac{1}{7}$

E 3) Here f(x) takes real values for all $x \in R$, except at x = –3
$\therefore$ domain of f = R – {– 3}
and f(x) cannot takes zero values at any real number,
$\therefore$ range of f = R– {0}
Also, $f(0) = \frac{1}{0+3} = \frac{1}{3}$, $f(2) = \frac{1}{2+3} = \frac{1}{5}$
f(– 3) is not defined because $x = -3$ is not a point of the domain of f.

E 4) $f(x) = 5 - |x-3|$

$f(2) = 5 - |2-3| = 5 - |-1| = 5 - (-(-1)) = 5 - 1 = 4$

$f(-2) = 5 - |-2-3| = 5 - |-5| = 5 - (-(-5)) = 5 - (5) = 5 - 5 = 0$

$f(6) = 5 - |6-3| = 5 - |3| = 5 - 3 = 2$

$f(-5) = 5 - |-5-3| = 5 - |-8| = 5 - (-(-8)) = 5 - (8) = 5 - 8 = -3$

$f(12) = 5 - |12-3| = 5 - |9| = 5 - 9 = -4$

E 5) (i) Define a map $f : N \to A$ by
$f(n) = 5^n, \quad n \in N$
f is both 1–1 and onto as shown in Fig. 2.28.
$\Rightarrow N \sim A$
$\Rightarrow$ A is enumerable.

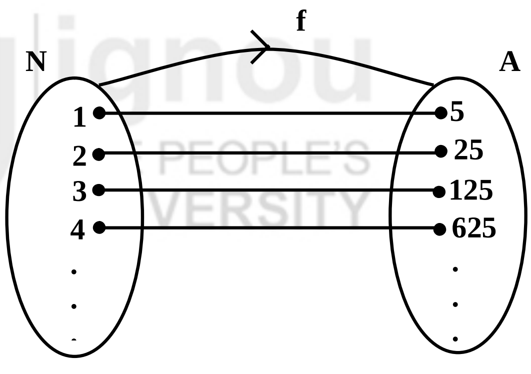


Fig. 2.28

(ii) Define a map $f : N \to B$ by
$f(x) = 5^{n-1}, \quad n \in N$
f is both 1–1 and onto as shown in Fig. 2.29.
$\Rightarrow N \sim B \Rightarrow B$ is enumerable.

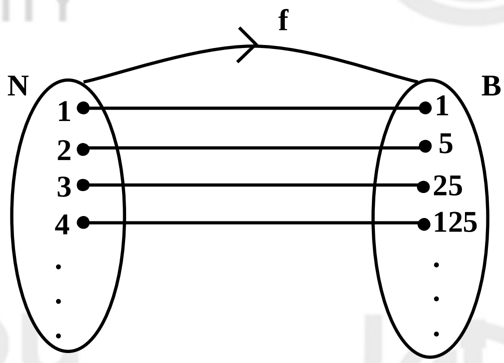


Fig. 2.29

(iii) Define a map $f : N \to C$ by
$f(x) = 3n - 2, \quad n \in N$
f is both 1–1 and onto as shown in Fig. 2.30.
$\Rightarrow N \sim C \Rightarrow C$ is enumerable.

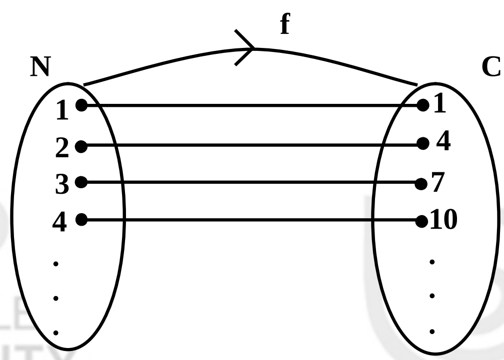


Fig. 2.30