UNIT 3 PROGRESSIONS

Structure

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3.1 INTRODUCTION

In day to day life many times people use the word sequence. So we are familiar with the dictionary meaning of the word sequence, in fact to put the things in a particular order means we have put the things in a particular sequence. For example, natural numbers which are multiple of 5 can be put in the following sequence.

5, 10, 15, 20, 25, …

In Unit 2 of this block, we have defined functions. In this unit we will define sequence mathematically and it will be interesting to know that sequences are also special types of functions. Then we will see arithmetic progression (A.P.) and geometric progression (G.P.) are special types of sequences. In fact in this unit we will focus on the n^{th} term of A.P and G.P., and sum of first n terms of A.P. and G.P. Finally, we will discuss what we mean by summation and how the initiation (origin) of summation can be changed.

Objectives

After completing this unit, you should be able to:

- define sequence;
- define and recognize arithmetic progression (A.P.);
- give formula for n^{th} term and sum of first n terms of on A.P.;
- define geometric progression (G.P.);
- give formula for n^{th} term and sum of n terms of a G.P.;
- find sum of infinite G.P.; and
- become familiar with the concept of summation.

3.2 SEQUENCE

In the introduction of this unit we have indicated that sequences are special types of functions. So let us first give a mathematical definition of sequence.

Sequence: A sequence is a function whose domain is the set of all natural numbers and range may be any set. A sequence is generally denoted by writing $a_1, a_2, a_3, \ldots, a_n, \ldots$

Or simply by $\{a_n\}$ or $< a_n >$, where a_n denotes the n^{th} term of the sequence, i.e. a_1 is first term of the sequence, a_2 is second term of the sequence, a_3 is third term of the sequence, and so on.

For example, define a function $f : N \to R$ by

$$f(n) = \frac{1}{n}, \quad n \in N$$

This function represents a sequence which can be written as $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

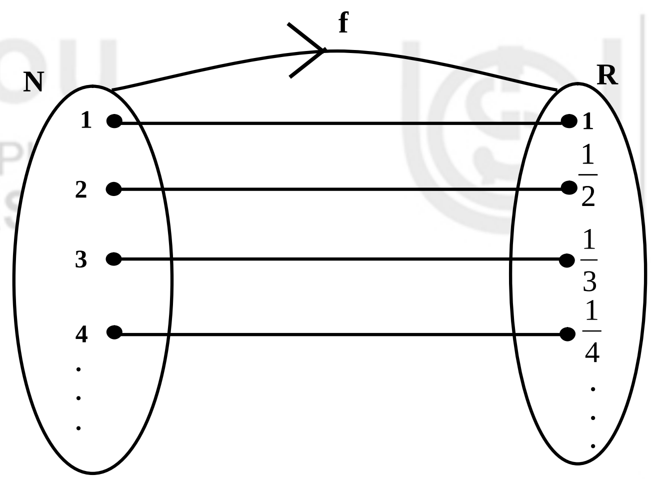

Fig. 4.1

Remark 1:

(i) Sequences are special types of functions because here domain always remains set of natural numbers whereas in case of real functions domain may be any subset of real numbers.

(ii) If range of a sequence is subset of R, then we say that sequence is real.

(iii) Here we will discuss only real sequences.

(iv) Here geometrical representation of the sequence is given just to realize you that sequences are special types of functions. In future we will not give geometrical representation and it is neither required.

Next question which may strike your mind is that what are the commonly used methods to represent a sequence? This question is addressed in the following discussion:

Ways of Representing a Sequence

A sequence may be represented by any of the following ways:

(a) One of the ways of representing a sequence is writing down the first few terms of the sequence till a definite rule for writing down other terms becomes clear.

For example,

(i) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$; is a sequence having n^{th} term $\frac{1}{n^2}$.

(ii) 5, 10, 15, 20,…; is a sequence having n^{th} term 5n.

(b) A sequence can also be represented by giving a formula for its n^{th} term.

For example,

(i) If $a_n = n^2$, then this represents the sequence 1, 4, 9, 16, …

(ii) If $a_n = \frac{n}{n+1}$, then this represents the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ...$

(c) Recursive Relation: A sequence can also be represented by writing its first few terms and a formula to write down the other terms of the sequence. Such way of representing a sequence is known as recursive relation.

For example, if $a_1 = a_2 = 1$, and $a_{n+1} = a_n + a_{n-1}, \quad n \ge 2$

then terms of this sequence are 1, 1, 2, 3, 5, 8,…
i.e. each term (except first and second) is equal to the sum of its preceding two terms. This sequence is known as Fibonacci sequence.

(d) Sometimes, the nature of the sequence is such that it cannot be represented by giving a single formula for its n^{th} term. So, to avoid this difficulty we have another way of representing a sequence by writing more than one relation for its n^{th} term.

For example, if $a_n = \begin{cases} \frac{n-1}{2}, & \text{if } n = 1, 3, 5, ... \\ \frac{n}{2}, & \text{if } n = 2, 4, 6, ... \end{cases}$

then terms of this sequence are 0, 1, 1, 2, 2, 3, 3,…

We have discussed different methods of representing a sequence, so it is the right place to provide an example to obtain some terms of the sequence given by using either of the ways.

Example 1: For the given sequences write down the given terms:

(i) $a_n = n^2 - 2n + 2$, find a_1, a_2, a_r.

(ii) $a_n = \frac{1+(-1)^n}{2}$, find a_1, a_2, a_3, a_4.

(iii) $a_1 = 2,\ a_2 = 3,\ a_n = 2a_{n-1} + 4a_{n-2}, \quad n \ge 3$, find a_3, a_4.

(iv) $a_n = \begin{cases} n^2, & n = 1, 3, 5, ... \\ \frac{1}{n+1}, & n = 2, 4, 6, ... \end{cases}$, find $a_1, a_2, a_3, a_4, a_5, a_6$.

(v) $a_n = \frac{(n-1)(n-2)}{2(n+1)}$, find a_1, a_2, a_3, a_4.

Solution:

(i) $a_n = n^2 - 2n + 2$

For n = 1, 2, r, we have

$a_1 = (1)^2 - 2 \times 1 + 2 = 1,\ a_2 = 2^2 - 2 \times 2 + 2 = 2,\ a_r = r^2 - 2r + 2$

(ii) $a_n = \dfrac{1+(-1)^n}{2}$

For n = 1, 2, 3, 4, we have

$a_1 = \dfrac{1+(-1)^1}{2} = \dfrac{1-1}{2} = \dfrac{0}{2} = 0$, $a_2 = \dfrac{1+(-1)^2}{2} = \dfrac{1+1}{2} = \dfrac{2}{2} = 1$

$a_3 = \dfrac{1+(-1)^3}{2} = \dfrac{1-1}{2} = \dfrac{0}{2} = 0$, $a_4 = \dfrac{1+(-1)^4}{2} = \dfrac{1+1}{2} = \dfrac{2}{2} = 1$

(iii) $a_1 = 2,\ \ a_2 = 3$

$a_n = 2a_{n-1} + 4a_{n-2},\ \ n \geq 3$

For n = 3, 4, we have

$a_3 = 2a_2 + 4a_1 = 2\times 3 + 4\times 2 = 6 + 8 = 14$

$a_4 = 2a_3 + 4a_2 = 2\times 14 + 4\times 3 = 28 + 12 = 40$

(iv) For $n = 1, 2, 3, 4, 5, 6,$ we have

$a_1 = (1)^2 = 1,\quad a_2 = \dfrac{1}{2+1} = \dfrac{1}{3},\ \ a_3 = (3)^2 = 9$

$a_4 = \dfrac{1}{4+1} = \dfrac{1}{5},\ a_5 = (5)^2 = 25,\ a_6 = \dfrac{1}{6+1} = \dfrac{1}{7}$

(v) For n = 1, 2, 3, 4, we have

$a_1 = \dfrac{(1-1)(1-2)}{2(1+1)} = \dfrac{0}{4} = 0,\quad a_2 = \dfrac{(2-1)(2-2)}{2(2+1)} = \dfrac{0}{6} = 0$

$a_3 = \dfrac{(3-1)(3-2)}{2(3+1)} = \dfrac{2}{8} = \dfrac{1}{4},\ a_4 = \dfrac{(4-1)(4-2)}{2(4+1)} = \dfrac{6}{10} = \dfrac{3}{5}$

Here is an exercise for you.

E 1) (i) If $a_n = \sqrt{2n}$ then find a_1, a_2, a_3, a_4.

(ii) If $a_n = \dfrac{2}{\sqrt{n}}$ then find a_1, a_2, a_3, a_4, a_8.

(iii) If $a_n = \dfrac{2^n \times 3^n}{2^n + 3^n}$ then find a_1, a_2, a_3.

3.3 ARITHMETIC PROGRESSION (A.P.)

Some sequences follow certain pattern. Arithmetic progression (A.P.) is also a sequence which follows a particular pattern as defined below.

Arithmetic progression (A.P.): A sequence $\{a_n\}$ or $< a_n >$ is said to be arithmetic progression (A.P.) if

$a_{n+1} - a_n = d,\ \ \forall\, n, n = 1, 2, 3, \ldots$ where d is a fixed constant known as common difference of the A.P.

i.e. difference of any term to its preceding term always remains constant.

For example, 7, 11, 15, 19, … is an A.P. with first term = 7 and common difference = 11 – 7 = 4.

Remark 2:

(i) If a sequence is given by listing its first few terms and we want to know whether it is an A.P. or not, for this first of all we calculate
$a_2 - a_1,\ a_3 - a_2,\ a_4 - a_3,$ etc.
If $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = ... = d$, then we say that it is an A.P. with d as common difference, otherwise it is not an A.P.

(ii) If a sequence is given by writing its n^{th} term a_n then we calculate $a_{n+1} - a_n$. If this difference is independent of n, it represents A.P. and if the differences $a_{n+1} - a_n$ involve n then it is not an A.P.

Following example is based on the two points discussed in the Remark 2 given above.

Example 2: In which of the following cases given sequence is an A.P.:

(i) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4},...$ (ii) 1, 4, 9, 16,… (iii) 1, 4, 7, 10,…

(iv) $8, 7\frac{1}{3}, 6\frac{2}{3}, 6, ...$ (v) $a_n = 4n+5$ (vi) $a_n = n^2 + n$

Solution:

(i) $a_2 - a_1 = \frac{1}{2} - 1 = \frac{-1}{2};\ a_3 - a_2 = \frac{1}{3} - \frac{1}{2} = \frac{2-3}{6} = \frac{-1}{6}$
$\therefore a_2 - a_1 \neq a_3 - a_2 \Rightarrow$ it is not an A.P.

(ii) $a_2 - a_1 = 4 - 1 = 3;\ a_3 - a_2 = 9 - 4 = 5$
$\therefore a_2 - a_1 \neq a_3 - a_2 \Rightarrow$ it is not an A.P.

(iii) $a_2 - a_1 = 4 - 1 = 3;\ a_3 - a_2 = 7 - 4 = 3;\ a_4 - a_3 = 10 - 7 = 3$
and so on
$\therefore a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = ... = 3(= \text{constant})$
$\Rightarrow$ it is an A.P. with first term 1 and common difference 3.

(iv) $a_2 - a_1 = 7\frac{1}{3} - 8 = \frac{22}{3} - 8 = \frac{22-24}{3} = -\frac{2}{3}$

$a_3 - a_2 = 6\frac{2}{3} - 7\frac{1}{3} = \frac{20}{3} - \frac{22}{3} = -\frac{2}{3}$

$a_4 - a_3 = 6 - 6\frac{2}{3} = 6 - \frac{20}{3} = \frac{18-20}{3} = -\frac{2}{3}$

and so as

$\therefore a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = ... = -\frac{2}{3}$ (= constant)

$\Rightarrow$ it is an A.P. with first term 8 and common difference = $-\frac{2}{3}$.

(v) $a_n = 4n + 5$
Replace n by n+1, we get
$a_{n+1} = 4(n+1) + 5 = 4n + 9$
$a_{n+1} - a_n = 4n + 9 - (4n+5)$
$= 4n + 9 - 4n - 5$
= 4 which is independent of n
$\therefore$ it is an A.P. with first term 9 and common difference = 4.

(vi) $a_n = n^2 + n$

Replacing n by n+1, we get

$a_{n+1} = (n+1)^2 + (n+1) = n^2 + 2n + 1 + n + 1 = n^2 + 3n + 2$

$a_{n+1} - a_n = n^2 + 3n + 2 - (n^2 + n) = n^2 + 3n + 2 - n^2 - n$

$= 2n + 2$ which is not free from n

$\therefore$ it is not an A.P.

Here is an exercise for you.

E 2) Show that sequence $\log a,\ \log\dfrac{a^2}{b},\ \log\dfrac{a^3}{b^2},$ form an A.P.

So far in this section we have defined A.P. and also learned how to check whether a given sequence is an A.P. or not? But now question arise can we find any term of a given A.P.? and second question can we find the sum of any number of terms of an A.P.? Answers of both the questions are yes and we will discuss these questions separately in two subsections 3.3.1 and 3.3.2.

3.3.1 Standard A.P. and its General Term

A sequence defined by a, a + d, a + 2d, a + 3d,… … (1)
is known as standard A.P. with first term = a and common difference = d.

Standard A.P.: A.P. defined by (1), i.e.
a, a + d, a + 2d, a + 3d,…
is known as standard A.P.

General Term: From standard A.P. given by (1) we see that

First term = $a_1 = T_1 = a = a + (1-1)d$

Second term = $a_2 = T_2 = a + d = a + (2-1)d$

Third term = $a_3 = T_3 = a + 2d = a + (3-1)d$

Forth term = $a_4 = T_4 = a + 3d = a + (4-1)d$

…

…

…

$\therefore$ $\boxed{n^{th}\text{ term} = a_n = T_n = a + (n-1)d}$

Remark 3: Keep this formula always in mind and is known as formula for n^{th} term or general term of an A.P. with first term 'a' and common difference 'd'.

Example 3: Find indicated term(s) in each case:

(i) If a = 4, d = 3, find T_n, T_{17}.

(ii) Find T_n of the A.P. $5, 5+\sqrt{3}, 5+2\sqrt{3}, 5+3\sqrt{3}, \ldots$

(iii) $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots$. Find T_{20} of this A.P.

(iv) Which term of the of the A.P. 43, 38, 33, 28,…is – 457?

(v) Which term of the A.P. $17, 16\frac{1}{2}, 16, 15\frac{1}{2}, \ldots$ is the first negative term?

(vi) If 7^{th} and 31^{st} terms of an A.P. are 29, 125 respectively, then find the A.P.

Solution:

(i) $T_n = a + (n-1)d = 4 + (n-1)3 = 3n + 1$

For n = 17, $T_{17} = 3 \times 17 + 1 = 51 + 1 = 52$

(ii) Here, a = 5, d = $5 + \sqrt{3} - 5 = \sqrt{3}$

$\therefore T_n = a + (n-1)d = 5 + (n-1)\sqrt{3}$

(iii) Here, a = 20, d = $19\frac{1}{4} - 20 = \frac{77}{4} - 20 = -\frac{3}{4}$

$\therefore T_{20} = a + 19d = 20 + 19\left(-\frac{3}{4}\right) = \frac{80-57}{4} = \frac{23}{4} = 5\frac{3}{4}$

(iv) Here a = 43, d = $38 - 43 = -5$

Let $T_n = -457$

$\Rightarrow a + (n-1)d = -457$

$\Rightarrow 43 + (n-1)(-5) = -457$

$\Rightarrow 43 - 5n + 5 = -457$

$\Rightarrow -5n = -457 - 48 = -505$

$\Rightarrow n = 101$

(v) Here a = 17, d = $16\frac{1}{2} - 17 = \frac{33}{2} - 17 = -\frac{1}{2}$

Let $T_n < 0$

$\Rightarrow a + (n-1)d < 0 \Rightarrow 17 + (n-1)\left(-\frac{1}{2}\right) < 0$

$\Rightarrow 17 - \frac{n}{2} + \frac{1}{2} < 0 \Rightarrow -\frac{n}{2} < -17 - \frac{1}{2}$

$\Rightarrow -\frac{n}{2} < -\frac{35}{2} \Rightarrow \frac{n}{2} > \frac{35}{2}$ $\quad [\because -x > -a \Rightarrow x < a \text{ Or } -x < -a \Rightarrow x > a]$

$\Rightarrow n > 35 \Rightarrow n = 36$

$\therefore$ 36 is the first negative term of this A.P.

(vi) Let a, d be the first term and common difference, respectively, of the given A.P.

According to the problem,

$$\left.\begin{matrix} T_7 = 29 \\ T_{31} = 125 \end{matrix}\right\rangle \Rightarrow \begin{cases} a + 6d = 29 & \dots (1) \\ a + 30d = 125 & \dots (2) \end{cases}$$

(2) – (1) gives

24d = 96 $\Rightarrow$ d = 4

Putting d = 4 in (1), we get

a + 24 = 29 $\Rightarrow$ a = 5

$\therefore$ given A.P. is 5, 9, 13, 17, …

Here is an exercise for you.

E 3) (i) If 5k + 1, 6k + 5 and 10k + 3 are three consecutive terms of an A.P. then find k.

(ii) Is 121 a term of the sequence 3, 9, 15, 21, …?

(iii) How many terms are there in the A.P. $-1, \frac{-1}{4}, \frac{1}{2}, \frac{5}{4}, \dots, 14$?

3.3.2 Sum of n Terms of an A.P.

Standard A.P. is $a, a+d, a+2d, a+3d, \ldots$

We know that $T_n = a+(n-1)d$

$\therefore T_{n-1} = a+(n-2)d$

Let S_n denotes the sum of first n terms of the above A.P., then

$S_n = T_1 + T_2 + \ldots + T_{n-1} + T_n$

Or $S_n = a + (a+d) + \ldots + [a+(n-2)d] + [a+(n-1)d]$ … (1)

Writing the terms of R.H.S. in reverse order we get

$S_n = [a+(n-1)d] + [a+(n-2)d] + \ldots + (a+d) + a$ … (2)

(1) + (2) gives

$$2S_n = \underbrace{[2a+(n-1)d]+[2a+(n-1)d]+\ldots+[2a+(n-1)d]+[2a+(n-1)d]}_{\text{n times}}$$

$\Rightarrow 2S_n = n[2a+(n-1)d]$

$$\Rightarrow \boxed{S_n = \frac{n}{2}[2a+(n-1)d]}$$

Remark 4:

(i) This formula can also be written as

$$S_n = \frac{n}{2}[a+a+(n-1)d] = \frac{n}{2}(a+l), \text{ where } l = a+(n-1)d = \text{last term}$$

(ii) Keep this formula always in mind and is known as formula for sum of first n terms of an A.P. with first term ‘a’ and common difference ‘d’.

Example 4: Find the following sums:

(i) 1 + 4 + 7 + 10 + … to 40 terms

(ii) 0.8 + 0.81 + 0.82 + … to 101 terms

(iii) 3 + 7 + 11 + … + 79

Solution:

(i) Here a = 1, d = 4 –1 = 3, n = 40

We know that

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$\therefore S_{40} = \frac{40}{2}[2\times1+(40-1)3] = 20(2+117) = 2380$$

(ii) Here a = 0.8, d = 0.81 – 0.8 = 0.01, n = 101

We know that

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$\therefore S_{101} = \frac{101}{2}[2\times0.8+(101-1)\times0.01] = \frac{101}{2}[2\times0.8+100\times0.01]$$

$$= \frac{101}{2}[1.6+1] = \frac{101}{2}\times2.6 = 101\times1.3 = 131.3$$

(iii) Here a = 3, d = 7 – 3 = 4, $T_n = 79$

We know that

$T_n = a+(n-1)d \Rightarrow a+(n-1)d = 79 \Rightarrow 3+(n-1)4 = 79$

$\Rightarrow 4n-1 = 79 \Rightarrow 4n = 80 \Rightarrow n = 20$

Now, we also know that

$$S_n = \frac{n}{2}(a + l), \text{ where } l \text{ is last term}$$

$$\therefore S_{20} = \frac{20}{2}(3 + 79) = 10 \times 82 = 820$$

Alternatively

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$\therefore \ S_{20} = \frac{20}{2}[2 \times 3 + (20-1) \times 4] = 10(6 + 76) = 820$$

3.4 GEOMETRIC PROGRESSION (G.P.)

A sequence $\{a_n\}$ is said to be geometric progression (G.P.) if

$$\frac{a_{n+1}}{a_n} = r \qquad \forall n \in N$$

i.e. ratio of any term to its preceding term is same (remains constant).
where r is non zero fixed constant and is known as common ratio

For example, 3, 6, 12, 24, 48, … is G.P. $\left[\because \frac{6}{3} = \frac{12}{6} = \frac{24}{12} = \frac{48}{24} = ... = 2\right]$

Remark 5:

(i) In case of G.P. neither a_n (for all n) nor r can be zero.
i.e. $a_n \neq 0, \quad \forall \ n \in N$ and $r \neq 0$

(ii) If a sequence is given by listing its first few terms and we want to know whether it is a G.P. or not, for this first of all we calculate
$\frac{a_2}{a_1}, \ \frac{a_3}{a_2}, \ \frac{a_4}{a_3}$, etc.
If $\frac{a_2}{a_1}, \frac{a_3}{a_2}, \frac{a_4}{a_3} = ... = r$, then we say that it is a G.P. with common ratio r, otherwise it is not a G.P.

For example, we have seen just before Remark 5 that the sequence 3, 6, 12, 24, 48, … is a G.P. by using this procedure.

So far in this section we have defined G.P. and also learned how to check whether a given sequence is a G.P. or not? But now question arise can we find any term of a given G.P.? and second question can we find the sum of any number of terms of G.P.? Answers of both the questions are yes and we will discuss these questions separately in two subsections 3.4.1 and 3.4.2.

3.4.1 Standard G.P. and its General Term

A sequence defined by $a, ar, ar^2, ar^3, ...$ … (1)
is known as standard G.P. with first term = a and common ratio = r.

Standard G.P.: G.P. defined by (1), i.e.

$$a, ar, ar^2, ar^3, ...$$

is known as standard G.P.

General Term: From standard G.P. given by (1) we see that

First term = $a_1 = T_1 = a = ar^{1-1}$

Second term = $a_2 = T_2 = ar = ar^{2-1}$

Third term = $a_3 = T_3 = ar^2 = ar^{3-1}$

Forth term = $a_4 = T_4 = ar^3 = ar^{4-1}$

…

…

…

$\therefore$ $\boxed{n^{th} \text{ term} = a_n = T_n = ar^{n-1}}$

Remark 6: Keep this formula always in mind and is known as formula for n^{th} term or general term of the G.P. with first term 'a' and common ratio 'r'.

Example 5: Which of the following sequences are G.P.. If a sequence is a G.P., write its first term, common ratio and n^{th} term.

(i) 2, 6, 18, 54,…

(ii) $\frac{1}{8}, -\frac{1}{4}, \frac{1}{2}, -1, ..$

(iii) $\sqrt{2},\ 3\sqrt{2},\ 6\sqrt{2},\ 12\sqrt{2},\ ...$

Solution:

(i) $\frac{T_2}{T_1} = \frac{6}{2} = 3, \quad \frac{T_3}{T_2} = \frac{18}{6} = 3, \quad \frac{T_4}{T_3} = \frac{54}{18} = 3, \ldots$

$\therefore \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3} = ... = 3$

$\Rightarrow$ it is a G.P with first term = a = 2 and common ratio r = 3.

n^{th} term $= T_n = ar^{n-1} = 2(3)^{n-1}$

(ii) $\frac{T_2}{T_1} = \frac{-1/4}{1/8} = -2, \quad \frac{T_3}{T_2} = \frac{1/2}{-1/4} = -2, \quad \frac{T_4}{T_3} = \frac{-1}{1/2} = -2, \ldots$

$\therefore \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3} = ... = -2$

$\Rightarrow$ it is a G.P. with first term = a = $\frac{1}{8}$ and common ratio = r = – 2.

$\therefore n^{th} \text{term} = T_n = ar^{n-1} = \frac{1}{8}(-2)^{n-1}$

(iii) $\frac{T_2}{T_1} = \frac{3\sqrt{2}}{\sqrt{2}} = 3, \quad \frac{T_3}{T_2} = \frac{6\sqrt{2}}{3\sqrt{2}} = 2$

$\therefore \frac{T_2}{T_1} \neq \frac{T_3}{T_2}$

$\Rightarrow$ it is not a G.P.

Here is an exercise for you.

E 4) (i) Find the 10^{th} term of the G.P. 128, 32, 8, 2, …

(ii) 4^{th} and 7^{th} terms of a G.P. are 24 and 192 respectively. Find the G.P.

3.4.2 Sum of n Terms of a G.P.

Standard G.P. is $a, ar, ar^2, ar^3, \ldots$

We know that

$T_n = ar^{n-1}$

$\therefore T_{n-1} = ar^{n-2}$

Let S_n denotes the sum of first n terms of the above G.P., then

$S_n = T_1 + T_2 + \ldots + T_{n-1} + T_n$

$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}$ … (1)

Multiply on both sides by r, we get

$rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n$ … (2)

(1) – (2) gives

$(1-r)S_n = a - ar^n$ [All other terms cancel out in pairs]

$$\Rightarrow S_n = \frac{a(1-r^n)}{1-r}$$

By taking negative sign common from numerator as well as denominator we can also write the above formula as

$$S_n = \frac{(-1)a(r^n-1)}{(-1)(r-1)} = \frac{a(r^n-1)}{r-1}$$

So, you can use either form of the formula for S_n result will be same, but we will use the formula depending on the value of common ratio r as given in the box below.

$$\boxed{\begin{array}{ll} S_n = \dfrac{a(1-r^n)}{1-r}, & |r| < 1 \\ S_n = \dfrac{a(r^n-1)}{r-1}, & |r| > 1 \end{array}}$$

Remark 7: Keep this formula always in mind and is known as formula for sum of first n terms of a G.P. with first term ‘a’ and common ratio ‘r’.

Let us evaluate the sum of a given G.P. with the help of above formula in the following example.

Example 6: Find the sum of the following, G.P.:

(i) 2, 4, 8, 16, …to 10 terms (ii) $-1 + \frac{2}{3} - \frac{4}{9} + \ldots$to 8 terms

Solution:

(i) Here a = 2, d = $\frac{4}{2}$ = 2, n = 10

We know that

$$S_n = \frac{a(r^n-1)}{r-1}, \quad \text{as } r = 2 > 1$$

$$\therefore S_{10} = \frac{2(2^{10}-1)}{2-1} = 2(1024-1) = 2046$$

(ii) Here a = – 1, r = $\frac{2/3}{-1} = -\frac{2}{3}$, $n = 8$

We know that

$$S_n = \frac{a(1-r^n)}{1-r}, \quad \text{as } |r| = \frac{2}{3} < 1$$

$$\therefore S_8 = \frac{(-1)\left[1-\left(\frac{-2}{3}\right)^8\right]}{1-\left(\frac{-2}{3}\right)} = \frac{-\left[1-\frac{256}{6561}\right]}{1+\frac{2}{3}}$$

$$= \frac{-(6561-256)}{6561\left(\frac{3+2}{3}\right)} = \frac{-6305\times 3}{6561\times 5} = -\frac{1261}{2187}$$

Here is an exercise for you.

E 5) Find the sum $\frac{2}{9}+\frac{2}{3}+2+6+...+486$.

3.5 SUM OF INFINITE G.P.

A geometric progression (G.P.) is said to be infinite G.P. if number of terms in it are infinite. That is, G.P. given by

$a, ar, ar^2, ar^3, ...$ to ∞ … (1)

is an infinite G.P.

We note that sum of an infinite G.P. will be finite if common ratio is less than 1 in magnitude. Let S denotes the sum of the infinite G.P. given by (1)

i.e. $S = a + ar + ar^2 + ar^3 + ...$ to ∞ … (2)

Multiplying on both sides of (2) by r (common ratio), we get

$rS = ar + ar^2 + ar^3 + ...$ to ∞ … (3)

$(2)-(3)$ gives

$(1-r)S = a, \quad -1 < r < 1$, i.e $|r| < 1$ [All other terms cancel out in pairs]

$S = \frac{a}{1-r}, \quad -1 < r < 1$, i.e. $|r| < 1$

If you are interested to know the details related to the above formula, refer the remark given below.

Remark 8:

(i) If n approaches to infinity, i.e. $n \to \infty$, then behaviour of x^n is given below

$$x^n = \begin{cases} \infty, & \text{if } |x| > 1 \\ 0, & \text{if } |x| < 1 \\ 1, & \text{if } x = 1 \\ 1, & \text{if } x = -1 \text{ and } n \text{ is even} \\ -1, & \text{if } x = -1 \text{ and } n \text{ is odd} \end{cases}$$

But $(1)^\infty$ is not defined

For example, let x = 4, then for n = 1, 2, 3, 4, 5, … we have

$4^1 = 4, 4^2 = 16, 4^3 = 64, 4^4 = 256, 4^5 = 1024, ...$

That is, we observe that as n increases then x^n increases very fast and hence we write $x^n \to \infty$, as $n \to \infty$.

Similarly, let x = 0.2, then for n = 1, 2, 3, 4, 5, … we have

$(0.2)^1 = 0.2, (0.2)^2 = 0.04, (0.2)^3 = 0.008, (0.2)^4 = 0.0016, (0.2)^5 = 0.00032, \ldots$

That is, we observe that as n increases then x^n decreases very fast and reaches nearer and nearer to zero and hence we write $x^n \to 0$, as $n \to \infty$.

And if x = 1, then we define $x^n = 1$ for finite values of n = 1, 2, 3, . . ., $< \infty$. whereas x^n is not defined in the case $n = \infty$, and we handle this type of situation by using some results from limit, etc.

(ii) $S_n = \frac{a(1-r^n)}{1-r}$, if $-1 < r < 1$, then

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} \qquad \left[\because \lim_{n\to\infty} r^n = 0 \text{ as discussed in part (i)}\right]$$

(iii) Concept of limit will be discussed in Unit 5 of this course, i.e. MST-001.

Example 7: Find the following sums:

(i) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \text{to } \infty$ (ii) $1 - \frac{1}{5} + \frac{1}{25} - \frac{1}{125} + \ldots \text{to } \infty$

Solution:

(i) Here a = 1, r = $\frac{1/2}{1} = \frac{1}{2} < 1$

$$\therefore S = \frac{a}{1-r} = \frac{1}{1-1/2} = \frac{1}{1/2} = 2 \qquad \left[\because \text{ sum of infinite G.P.} = \frac{a}{1-r}\right]$$

(ii) Here a = 1, r = $\frac{-1/5}{1} = -\frac{1}{5}$

$\therefore |r| < 1$, so sum of infinite G.P. is given by

$$S = \frac{a}{1-r} = \frac{1}{1-(-1/5)} = \frac{1}{1+1/5} = \frac{1}{6/5} = \frac{5}{6}$$

Here is an exercise for you.

E 6) Prove that (i) $4^{1/4}.4^{1/8}.4^{1/16}.4^{1/32}\ldots\text{to } \infty = 2$ (ii) $5^{\frac{1}{3}}.5^{\frac{1}{9}}.5^{\frac{1}{27}}\ldots\text{to } \infty = \sqrt{5}$

3.6 CONCEPT OF SUMMATION

3.6.1 Series:

If $a_1, a_2, a_3, \ldots$ to ∞ is a sequence then expression $a_1 + a_2 + a_3 + \ldots \text{to } \infty$ is known as series.

This series in the form of summation is written as $\sum_{n=1}^{\infty} a_n$

i.e. $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots \text{to } \infty$

In case of finite expression $x_1 + x_2 + x_3 + \ldots + x_n$

We write as $\sum_{i=1}^{n} x_i = x_1 + x_2 + \ldots + x_n$

Remark 9:

(i) The symbol Σ is the Greek letter pronounced as sigma.

(ii) The letters n and i used above are known as dummy variables. These letters have nothing special other letters like m, r, s, k, j, etc. can also be used.

3.6.2 Change of Origin of Summation

A given series can be written in different ways in terms of summation
i.e. series $x_0 + x_1 + x_2 + \ldots \text{to } \infty$
can be written in any of the following ways

$$\sum_{n=0}^{\infty} x_n \quad \text{or} \quad \sum_{n=2}^{\infty} x_{n-2} \quad \text{or} \quad \sum_{x=10}^{\infty} x_{n-10} \quad \text{or} \quad \sum_{n=r}^{\infty} x_{n-r}$$

i.e. origin of the summation $\sum_{n=0}^{\infty} x_n$ is at n = 0 and if we want to shift the origin at n = k then we have to subtract k from the suffixes of terms within summation.

In Case of Finite Terms: Origin of the summation $\sum_{i=0}^{n} x_i$ is at i = 0 and if we want to shift the origin at i= k then we have to subtract k from the suffixes of the terms within summation and we also have to add k in range of the summation

i.e. $\sum_{i=0}^{n} x_i = \sum_{i=k}^{n+k} x_{i-k}$

3.7 SUM OF SOME SPECIAL SEQUENCES

Following are given sum of some special sequences as they will be helpful at various occasions during study of the programme. Keep these always in mind

(1) $\sum n = \sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ = sum of first n natural numbers.

(2) $\sum n^2 = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

= sum of squares of first n natural numbers.

(3) $\sum n^3 = \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + .. + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

= sum of cubes of first n natural numbers.

3.8 SUMMARY

Let us summarise the topics that we have covered in this unit:

1) Definition of sequence.
2) Ways of representation a sequence.
3) Arithmetic progression (A.P.).
4) Geometric progression (G.P.).
5) Sum of infinite G.P.
6) Concept of summation and change of initiation of summation.
7) Sum of some special sequences.

3.9 SOLUTIONS/ANSWERS

E 1) (i) $a_n = \sqrt{2n}$

For n = 1, 2, 3, 4, we have

$a_1 = \sqrt{2\times1} = \sqrt{2}$, $a_2 = \sqrt{2\times2} = \sqrt{4} = 2$

$a_3 = \sqrt{2\times3} = \sqrt{6}$, $a_4 = \sqrt{2\times4} = \sqrt{8} = \sqrt{2\times2\times2} = 2\sqrt{2}$

(ii) $a_n = \dfrac{2}{\sqrt{n}}$

For n = 1, 2, 3, 4, 8, we have

$a_1 = \dfrac{2}{\sqrt{1}} = \dfrac{2}{1} = 2$, $a_2 = \dfrac{2}{\sqrt{2}} = \dfrac{2}{\sqrt{2}}\times\dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{2\sqrt{2}}{2} = \sqrt{2}$

$a_3 = \dfrac{2}{\sqrt{3}} = \dfrac{2}{\sqrt{3}}\times\dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$, $a_4 = \dfrac{2}{\sqrt{4}} = \dfrac{2}{2} = 1$

$a_8 = \dfrac{2}{\sqrt{8}} = \dfrac{2}{2\sqrt{2}} = \dfrac{1}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}\times\dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

(iii) $a_n = \dfrac{2^n \times 3^n}{2^n + 3^n}$

For n = 1, 2, 3, we have

$a_1 = \dfrac{2^1\times3^1}{2^1+3^1} = \dfrac{6}{5}$, $a_2 = \dfrac{2^2\times3^2}{2^2+3^2} = \dfrac{36}{13}$, $a_3 = \dfrac{2^3\times3^3}{2^3+3^3} = \dfrac{8\times27}{8+27} = \dfrac{216}{35}$

E 2) $a_2 - a_1 = \log\dfrac{a^2}{b} - \log a = \log\dfrac{a^2}{b}\times\dfrac{1}{a} = \log\dfrac{a}{b}$ $\left[\because \log m - \log n = \log\dfrac{m}{n}\right]$

$a_3 - a_2 = \log\dfrac{a^3}{b^2} - \log\dfrac{a^2}{b} = \log\dfrac{a^3}{b^2}\times\dfrac{b}{a^2} = \log\dfrac{a}{b}$ [Same reason]

and so on

$\therefore a_2 - a_1 = a_3 - a_2 = ... = \log\dfrac{a}{b}$

$\therefore$ it is an A.P.

E 3) (i) Since 5k + 1, 6k + 5 and 10k + 3 are three consecutive terms of an A.P.

$\therefore$ (6k + 5) – (5k + 1) = (10k + 3) – (6k + 5) $[\because T_2 - T_1 = T_3 - T_2]$

$\Rightarrow$ k + 4 = 4k – 2 $\Rightarrow$ 6 = 3k $\Rightarrow$ k = 2

(ii) Here a = 3, d = 9 – 3 = 6

Let $T_n = 121 \Rightarrow a + (n-1)d = 121 \Rightarrow 3 + (n-1)6 = 121$

$\Rightarrow 3 + 6n - 6 = 121 \Rightarrow 6n - 3 = 121 \Rightarrow 6n = 124 \Rightarrow n = \dfrac{124}{6} = \dfrac{62}{3}$

$\Rightarrow n = 20\dfrac{2}{3}$

This is not possible because value of n is always a natural number.

$\therefore$ 121 cannot be a term of this A.P.

(iii) Here a = –1, d = $-\dfrac{1}{4} - (-1) = -\dfrac{1}{4} + 1 = \dfrac{3}{4}$

Let $T_n = 14 \Rightarrow a + (n-1)d = 14 \Rightarrow -1 + (n-1)\dfrac{3}{4} = 14$

$$\Rightarrow -1 + \frac{3}{4}n - \frac{3}{4} = 14 \Rightarrow \frac{3}{4}n = 14 + 1 + \frac{3}{4} \Rightarrow \frac{3}{4}n = \frac{63}{4} \Rightarrow n = 21$$

$\therefore$ number of terms = 21. That is 14 is the 21^{st} term of the given A.P.

E 4) (i) Here a = 128, $r = \frac{32}{128} = \frac{1}{4}$

We know that

$T_n = ar^{n-1}$

$$\therefore T_{10} = 128\left(\frac{1}{4}\right)^{10-1} = 2^7\left(\frac{1}{4}\right)^9 = \frac{2^7}{2^{18}} = \frac{1}{2^{11}} = \frac{1}{2048}$$

(ii) Let a, r be the first term and common ratio of the given G.P. respectively.

According to the problem

$$\left.\begin{matrix} T_4 = 24 \\ T_7 = 192 \end{matrix}\right\rangle \Rightarrow \begin{cases} ar^3 = 24 & \quad ...(1) \\ ar^6 = 192 & \quad ...(2) \end{cases}$$

(2) $\div$ (1) gives

$$r^3 = \frac{192}{24} = 8 \Rightarrow r^3 = 2^3 \Rightarrow r = 2$$

Putting r = 2 in (1), we get

8a = 24 $\Rightarrow$ a = 3

$\therefore$ G.P. is 3, 6, 12, 24, 48, …

E 5) Here $a = \frac{2}{9}$, $r = \frac{2/3}{2/9} = \frac{2}{3} \times \frac{9}{2} = 3$

$$T_n = 486 \Rightarrow ar^{n-1} = 486 \Rightarrow \frac{2}{9}(3)^{n-1} = 486 \Rightarrow 3^{n-1} = 486 \times \frac{9}{2}$$

$$\Rightarrow 3^{n-1} = 243 \times 9 \Rightarrow 3^{n-1} = 3^7 \Rightarrow n - 1 = 7 \Rightarrow n = 8$$

We know that

$$S_n = \frac{a(r^n - 1)}{r - 1}, \quad \text{as } r = 3 > 1$$

$$\therefore S_8 = \frac{\frac{2}{9}[(3)^8 - 1]}{3 - 1} = \frac{\frac{2}{9}[6561 - 1]}{2} = \frac{6560}{9}$$

E 6) (i) L.H.S. = $4^{1/4}.4^{1/8}.4^{1/16}.4^{1/32}$ ...to ∞ = $4^{(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \text{to } \infty)}$

$$= 4^{\left(\frac{1/4}{1-1/2}\right)} \quad \left[\because \text{sum of infinite G.P.} = \frac{a}{1-r} \text{ here } a = \frac{1}{4} \text{ and } r = \frac{1}{2}\right]$$

$$= 4^{\frac{1}{2}} = (2^2)^{\frac{1}{2}} = 2 = \text{R.H.S.}$$

(ii) L.H.S. = $5^{(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots \text{to } \infty)}$

$$= 5^{\left(\frac{1/3}{1-1/3}\right)} \quad \left[\because \text{sum of infinite G.P.} = \frac{a}{1-r} \text{ here } a = \frac{1}{3} \text{ and } r = \frac{1}{3}\right]$$

$$= 5^{\left(\frac{1/3}{2/3}\right)} = 5^{1/2} = \sqrt{5} = \text{R.H.S.}$$