UNIT 7 INDEFINITE INTEGRATION

Structure

7.1 INTRODUCTION

In the previous unit, we have studied the differentiation of some functions. Here, in this unit we are going to discuss the reverse process of differentiation known as integration.

In this unit, we will study the integration of some commonly used functions in section 7.3, integration by substitutions in section 7.4, integration by using partial fractions in section 7.5 and integration by parts in section 7.6.

Objectives

After completing this unit, you should be able to:

- evaluate the integration of some commonly used functions;
- evaluate the integration by substitution method;
- evaluate the integration using partial fractions; and
- evaluate the integration by parts.

7.2 MEANING AND TERMINOLOGY USED

Notations

You have become familiar with the concept of summation discussed in Unit 3 of this course, i.e. MST-001. In fact, summation is convenient way to represent the sum of discrete values only. If the variable is continuous, then the summation cannot be used in the way it is used for discrete values. Summation is obtained by the process of integration in case of continuous variable. Origin of integration lies in the process of summation. In mathematics, the words “Summation” and “Integration” are used for the words “to unite”.

In previous unit we have studied differentiation. The integration is just the reverse process of differentiation. Actually, it is an antiderivative of a function. That is, if $f'(x)$ is derivative of f(x) and hence $f'(x)$ is derivative of f(x) + c ($\because$ derivative of constant is zero). And therefore, f(x) + c is the integration of $f'(x)$.

Integral of a function f(x) w.r.t. x is denoted by $\int f(x)dx$

,where f(x) is known as integrand, dx reflects the message that integrand is to be integrated w.r.t. to the variable x and the entire process of finding the integral of integrand is known as integration. The symbol $\int$ has its origin from the letter S, which was used for summation.

Let us consider a simple example first and then give a list of the formulae.

We know that the function x is the differentiation of $\frac{x^2}{2}+c$ w.r.t. x.

$\therefore \frac{x^2}{2}+c$ is integration of x.

i.e. $\int x\,dx = \frac{x^2}{2}+c$, where c is known as constant of integration.

Similarly, integration of other functions can be obtained. Integrations of some commonly used functions are listed in the following table.

List of Formulae of Integration

S. No.	Function f(x)	$\int f(x)dx$
1	k (constant function)	kx + c, where c is constant of integration
2	x^n	$\frac{x^{n+1}}{n+1}+c,\ n \neq -1$
3	$\frac{1}{x}$	$\log\lvert x\rvert + c$
4	$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)}+c,\ n \neq -1$
5	$\frac{1}{ax+b}$	$\frac{1}{a}\log\lvert ax+b\rvert + c$
6	Exponential functions (i) a^{mx} (ii) a^{mx+n} (iii) e^{ax} (iv) e^{ax+b}	(i) $\frac{a^{mx}}{m\log_e\lvert a\rvert}+c$ (ii) $\frac{a^{mx+n}}{m\log\lvert a\rvert}+c$ (iii) $\frac{e^{ax}}{a}+c$ (iv) $\frac{e^{ax+b}}{a}+c$

Remark 1:

If f, g are integral functions such that f + g, f – g, are defined and a, b are real constants, then

(i) $\int a(f(x)dx = a\int f(x)dx$

(ii) $\int \left(af(x) \pm bg(x)\right)dx = a\int f(x)dx \pm b\int g(x)dx$

7.3 INTEGRATION OF SOME PARTICULAR FUNCTIONS

In this section, we learn how the formulae mentioned in the table on previous page are used.

Example 1: Evaluate the following integrals:

(i) $\int \sqrt{5}dx$ (ii) $\int 0dx$ (iii) $\int \pi\, dx$ (iv) $\int x^3 dx$

(v) $\int x^{7/2}dx$ (vi) $\int \frac{1}{x^5}dx$ (vii) $\int \frac{1}{x^{7/2}}dx$ (viii) $\int \frac{1}{\sqrt[3]{x}}dx$

(ix) $\int (x^3+x+5)dx$ (x) $\int (x^2+1)(x-1)dx$ (xi) $\int \frac{x^6-x^4+1}{x^2}dx$

(xii) $\int \frac{x^4+x^3+3}{\sqrt{x}}dx$ (xiii) $\int \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$ (xiv) $\int \left(x+\frac{1}{x}\right)^2 dx$

(xv) $\int \left(8x^3+2x-3+\frac{1}{x^2}+\frac{2}{x^3}\right)dx$

Solution:

(i) $\int \sqrt{5}dx = \sqrt{5}x + c$ $\left[\because \sqrt{5}\text{ is a constant and if } k \text{ is constant then } \int k dx = kx + c\right]$

where c is constant of integration.

Note: Constant of integration c is added everywhere, so in future we will not write 'where c is constant of integration'.

(ii) $\int 0dx = 0x + c = c$ as 0 is constant

(iii) $\int \pi\, dx = \pi x + c$ as π is constant

(iv) $\int x^3 dx = \frac{x^{3+1}}{3+1} + c = \frac{x^4}{4} + c$ $\left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$

(v) $\int x^{7/2}dx = \frac{x^{\frac{7}{2}+1}}{\frac{7}{2}+1} + c = \frac{2}{9}x^{\frac{9}{2}} + c$ $\left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$

(vi) $\int \frac{1}{x^5}dx = \int x^{-5}dx = \frac{x^{-5+1}}{-5+1} + c = -\frac{1}{4}x^{-4} + c$

$\left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$

(vii) $\int \frac{1}{x^{7/2}}dx = \int x^{-7/2}dx = \frac{x^{-5/2}}{-5/2} + c = -\frac{2}{5}x^{-7/2} + c$

$\left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$

(viii) $\int \frac{1}{\sqrt[3]{x}}dx = \int \frac{1}{x^{1/3}}dx = \int x^{-1/3}dx = \frac{x^{2/3}}{2/3} + c = \frac{3}{2}x^{2/3} + c$

$$\left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$$

(ix) $\int (x^3 + x + 5)dx = \int x^3 dx + \int x\, dx + \int 5\, dx = \frac{x^4}{4} + \frac{x^2}{2} + 5x + c$

(x) $\int (x^2 + 1)(x - 1)dx = \int (x^3 - x^2 + x - 1)dx$

$$= \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x + c \quad \left[\begin{array}{l}\because \int x^n dx = \frac{x^{n+1}}{n+1} + c \\ \text{and } \int k dx = kx + c\end{array}\right]$$

(xi) $\int \frac{x^6 - x^4 + 1}{x^2} dx = \int \left(\frac{x^6}{x^2} - \frac{x^4}{x^2} + \frac{1}{x^2}\right)dx = \int (x^4 - x^2 + x^{-2})dx$

$$= \frac{x^5}{5} - \frac{x^3}{3} + \frac{x^{-1}}{-1} + c = \frac{x^5}{5} - \frac{x^3}{3} - \frac{1}{x} + c$$

(xii) $\int \frac{x^4 + x^3 + 3}{\sqrt{x}} dx = \int \left(\frac{x^4}{\sqrt{x}} + \frac{x^3}{\sqrt{x}} + \frac{3}{\sqrt{x}}\right)dx = \int \left(x^{7/2} + x^{5/2} + 3x^{-1/2}\right) dx$

$$= \frac{x^{9/2}}{9/2} + \frac{x^{7/2}}{7/2} + 3\frac{x^{1/2}}{1/2} + c = \frac{2}{9}x^{9/2} + \frac{2}{7}x^{7/2} + 6\sqrt{x} + c$$

(xiii) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx = \int \left(x^{1/2} + x^{-1/2}\right)dx = \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + c$

$$= \frac{2}{3}x^{\frac{3}{2}} + 2\sqrt{x} + c$$

(xiv) $\int \left(x + \frac{1}{x}\right)^2 dx = \int \left(x^2 + \frac{1}{x^2} + 2.x.\frac{1}{x}\right)dx$

$$= \int (x^2 + x^{-2} + 2)dx = \frac{x^3}{3} + \frac{x^{-1}}{-1} + 2x + c = \frac{x^3}{3} - \frac{1}{x} + 2x + c$$

(xv) $\int \left(8x^3 + 2x - 3 + \frac{1}{x^2} + \frac{2}{x^3}\right)dx = \int (8x^3 + 2x - 3 + x^{-2} + 2x^{-3})dx$

$$= \frac{8x^4}{4} + \frac{2x^2}{2} - 3x + \frac{x^{-1}}{-1} + \frac{2x^{-2}}{-2} + c$$

$$= 2x^4 + x^2 - 3x - \frac{1}{x} - \frac{1}{x^2} + c$$

Now, you can try the following exercise.

E 1) Evaluate the following integrals:

(i) $\int \left(x^2 + \frac{1}{x^2}\right)^2 dx$ (ii) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^3 dx$ (iii) $\int (\alpha - 3)dx$

(iv) $\int \sqrt{x}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx$ (v) $\int (x^a + 1)(x^b + 1)dx$ (vi) $\int \frac{x^m}{x^n} dx$

(vii) $\int \left(x + \frac{1}{x}\right)\left(x^3 + \frac{1}{x^3}\right)dx$

Example 2: Evaluate the following integrals:

(i) $\int (2x+3)^5 dx$ (ii) $\int (5-9x)^6 dx$ (iii) $\int (9x+5)^{3/2} dx$

(iv) $\int \sqrt[5]{8-3x}\, dx$ (v) $\int \frac{(3+2x)^{7/2}}{\sqrt{3+2x}} dx$ (vi) $\int \frac{1}{(7x+2)^3} dx$

(vii) $\int \frac{1}{\sqrt{3x+5}} dx$ (viii) $\int e^{5\log\sqrt{3x-5}} dx$ (ix) $\int a^{\log_a \sqrt{4x+5}} dx$

(x) $\int a^{3x} dx$ (xi) $\int e^{3x} dx$ (xii) $\int e^{5x+7} dx$

(xiii) $\int a^{3-2x} dx$ (xiv) $\int (5e^{7x} + x^2) dx$ (xv) $\int (a^x + e^x + a^a + e^a + a^e) dx$

(xvi) $\int 5^x 2^x dx$ (xvii) $\int \frac{2^x}{3^x} dx$ (xviii) $\int \frac{(a^x - b^x)^2}{a^x b^x} dx$

(xix) $\int (e^{a\log x} + a^{x\log_a a} + a^{m\log_a a}) dx$

(xx) $\int \left(\frac{x}{3} + (5x-3)^3 + x\sqrt{x} + \frac{1}{\sqrt{5+2x}} + a^a \right) dx$

Solution:

(i) $\int (2x+3)^5 dx = \frac{(2x+3)^{5+1}}{2(5+1)} + c$ $\left[\because \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \quad \text{Here } a=2,\ n=5 \right]$

$$= \frac{(2x+3)^6}{12} + c$$

(ii) $\int (5-9x)^6 dx = \frac{(5-9x)^7}{7(-9)} + c$ $\left[\because \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \quad \text{Here } a=-9,\ n=6 \right]$

$$= -\frac{1}{63}(5-9x)^7 + c$$

(iii) $\int (9x+5)^{3/2} dx = \frac{(9x+5)^{5/2}}{\frac{5}{2} \times 9} + c$ $\left[\because \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \quad \text{Here } a=9,\ n=3/2 \right]$

$$= \frac{2}{45}(9x+5)^{5/2} + c$$

(iv) $\int \sqrt[5]{8-3x}\, dx = \int (8-3x)^{\frac{1}{5}} dx = \frac{(8-3x)^{\frac{6}{5}}}{\frac{6}{5} \times (-3)} + c = \frac{-5}{18}(8-3x)^{\frac{6}{5}} + c$

$\left[\because \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \quad \text{Here } a=-3,\ n=1/5 \right]$

(v) $\int \frac{(3+2x)^{\frac{7}{2}}}{\sqrt{3+2x}} dx = \int (3+2x)^{\frac{7}{2}-\frac{1}{2}} dx$ $\left[\because \frac{a^m}{a^n} = a^{m-n} \right]$

$$= \int (3+2x)^3 dx$$

$$= \frac{(3+2x)^4}{4\times 2} + c \qquad \left[\begin{array}{l} \because \int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c \\ \text{Here } a = 2,\ n = 3 \end{array}\right]$$

$$= \frac{1}{8}(3+2x)^4 + c$$

(vi) $\displaystyle\int \frac{1}{(7x+2)^3}\,dx = \int (7x+2)^{-3}dx$

$$= \frac{(7x+2)^{-2}}{-2\times 7} + c \left[\begin{array}{l} \because \int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c \\ \text{Here } a = 7,\ n = -3 \end{array}\right]$$

$$= -\frac{1}{14}(7x+2)^{-2} + c$$

(vii) $\displaystyle\int \frac{1}{\sqrt{3x+5}}\,dx = \int (3x+5)^{-1/2}dx$

$$= \frac{(3x+5)^{1/2}}{\frac{1}{2}\times 3} + c \qquad \left[\begin{array}{l} \because \int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c \\ \text{Here } a = 3,\ n = -1/2 \end{array}\right]$$

$$= \frac{2}{3}\sqrt{3x+5} + c$$

(viii) $\displaystyle\int e^{5\log\sqrt{3x-5}}dx = \int e^{\log(3x-5)^{\frac{5}{2}}}dx = \int (3x-5)^{\frac{5}{2}}dx \quad \left[\because a^{\log_a f(x)} = f(x)\right]$

$$= \frac{(3x-5)^{7/2}}{(7/2)(3)} + c = \frac{2}{21}(3x-5)^{7/2} + c$$

(ix) $\displaystyle\int a^{\log_a \sqrt{4x+5}}dx = \int \sqrt{4x+5}dx \qquad \left[\because a^{\log_a f(x)} = f(x)\right]$

$$= \frac{(4x+5)^{3/2}}{(3/2)\times 4} + c = \frac{1}{6}(4x+5)^{3/2} + c$$

(x) $\displaystyle\int a^{3x}dx = \frac{a^{3x}}{3\log|a|} + c \qquad \left[\begin{array}{l} \because \int a^{mx}dx = \dfrac{a^{mx}}{m\log|a|} + c \\ \text{Here } a = a, m = 3 \end{array}\right]$

(xi) $\displaystyle\int e^{3x}dx = \frac{e^{3x}}{3} + c \qquad \left[\begin{array}{l} \because \int e^{ax}dx = \dfrac{e^{ax}}{a} + c \\ \text{Here } a = 3 \end{array}\right]$

(xii) $\displaystyle\int e^{5x+7}dx = \frac{e^{5x+7}}{5} + c \qquad \left[\begin{array}{l} \because \int e^{ax+b}dx = \dfrac{e^{ax+b}}{a} + c \\ \text{Here } a = 5, b = 7 \end{array}\right]$

(xiii) $\displaystyle\int a^{3-2x}dx = \frac{a^{3-2x}}{-2\log|a|} + c \qquad \left[\begin{array}{l} \because \int a^{mx+n}\,dx = \dfrac{a^{mx+n}}{m\log|a|} + c \\ \text{Here } m = -2, n = 3 \end{array}\right]$

(xiv) $\displaystyle\int (5e^{7x} + x^2)dx = \frac{5}{7}e^{7x} + \frac{x^3}{3} + c$

(xv) $\int (a^x + e^x + a^a + e^a + a^e)dx = \dfrac{a^x}{\log|a|} + e^x + a^a x + e^a x + a^e x + c$

$$\left[\begin{array}{l}\text{Here } a^a, e^a, a^e \text{ all are constants} \\ \text{and if } k \text{ is constant then} \int k\,dx = kx + c\end{array}\right]$$

(xvi) $\int 5^x 2^x dx = \int (5.2)^x dx = \int 10^x dx = \dfrac{10^x}{\log 10} + c \quad \left[\begin{array}{l}\because \int a^x dx = \dfrac{a^x}{\log|a|} + c \\ \text{Here } a = 10\end{array}\right]$

(xvii) $\int \dfrac{2^x}{3^x} dx = \int (2/3)^x dx = \dfrac{(2/3)^x}{\log 2/3} + c \quad \left[\begin{array}{l}\because \int a^x dx = \dfrac{a^x}{\log|a|} + c \\ \text{Here } a = 2/3\end{array}\right]$

$$= \frac{(2/3)^x}{\log 2 - \log 3} + c$$

(xviii) $\int \dfrac{(a^x - b^x)^2}{a^x b^x} dx = \int \dfrac{a^{2x} + b^{2x} - 2a^x b^x}{a^x b^x} dx \quad [\because (a-b)^2 = a^2 + b^2 - 2ab]$

$$= \int \left(\frac{a^{2x}}{a^x b^x} + \frac{b^{2x}}{a^x b^x} - \frac{2a^x b^x}{a^x b^x} \right) dx = \int \left(\frac{a^x}{b^x} + \frac{b^x}{a^x} - 2 \right) dx$$

$$= \int \left((a/b)^x + (b/a)^x - 2 \right) dx$$

$$= \frac{(a/b)^x}{\log|a/b|} + \frac{(b/a)^x}{\log|b/a|} - 2x + c \quad \left[\because \int m^x dx = \frac{m^x}{\log|m|} + c\right]$$

(xix) $\int (e^{a\log x} + a^{x\log_a a} + a^{m\log_a a})dx = \int e^{\log x^a} + a^{\log_a a^x} + a^{\log_a a^m})dx$

$$= \int (x^a + a^x + a^m)dx \quad [\because a^{\log_a f(x)} = f(x)]$$

$$= \frac{x^{a+1}}{a+1} + \frac{a^x}{\log|a|} + a^m x + c$$

$$[\because a^m \text{ is a constant quantity}]$$

(xx) $\int \left[\dfrac{x}{3} + (5x-3)^3 + x\sqrt{x} + \dfrac{1}{\sqrt{5+2x}} + a^a \right] dx$

$$= \int \left[\frac{x}{3} + (5x-3)^3 + x^{3/2} + (5+2x)^{-1/2} + a^a \right] dx$$

$$= \frac{x^2}{3 \times 2} + \frac{(5x-3)^4}{4 \times 5} + \frac{x^{5/2}}{5/2} + \frac{(5+2x)^{1/2}}{2 \times 1/2} + a^a x + c$$

$$= \frac{x^2}{6} + \frac{(5x-3)^4}{20} + \frac{2}{5} x^{5/2} + (5+2x)^{1/2} + a^a x + c$$

Now, you can try the following exercise.

E 2) Evaluate the following integral:

(i) $\int\left(a^x + e^x a^x + \frac{x}{a}\right)dx$ (ii) $\int (3^{2\log_3 x} + 3^{x\log_3 a} + a^{a\log_a x} + a^{a\log_a a})dx$

Example 3: Evaluate the following integrals:

(i) $\int \frac{1}{x}dx$ (ii) $\int \frac{3}{x}dx$ (iii) $\int \frac{5}{x+1}dx$ (iv) $\int \frac{7}{5x+2}dx$ (v) $\int \frac{3}{9-2x}dx$

Solution:

(i) $\int \frac{1}{x}dx = \log|x| + c$ [Using formula 3 of the table]

(ii) $\int \frac{3}{x}dx = 3\int \frac{1}{x}dx = 3\log|x| + c$ [Using formula 3 of the table]

(iii) $\int \frac{5}{x+1}dx = 5\log|x+1| + c$ [Using formula 5 of the table]

(iv) $\int \frac{7}{5x+2}dx = \frac{7\log|5x+2|}{5} + c$ [Using formula 5 of the table]

$= \frac{7}{5}\log|5x+2| + c$

(v) $\int \frac{3}{9-2x}dx = \frac{3\log|9-2x|}{(-2)} + c$ [Using formula 5 of the table]

$= -\frac{3}{2}\log|9-2x| + c$

Remark 2: In solving these examples you have noted that integration is in fact anti derivative of a function.

For example, consider (ix) part of Example 1

Let f(x) = $x^3 + x + 5$ then $\int f(x)dx = \frac{x^4}{4} + \frac{x^2}{2} + 5x + c$ (already calculated)

Now, let $F(x) = \frac{x^4}{4} + \frac{x^2}{2} + 5x + c$

Diff. w.r.t.x

$$\frac{d}{dx}(F(x)) = \frac{4x^3}{4} + \frac{2x}{2} + 5 + 0 = x^3 + x + 5$$

Thus, we note that if $\int f(x)dx = F(x)$ then $\frac{d}{dx}(F(x)) = f(x)$

i.e. integral F(x) of f(x) is indefinite because of the presence of arbitrary constant c.

In the next unit you will meet definite integral, where c will be cancel out. (Refer section 8.2 of Unit 8 of this course, i.e. MST-001).

7.4 INTEGRATION BY SUBSTITUTION

In section 7.3, we have taken into consideration the integrations for which a formula can directly be used. But sometimes integrand cannot be directly integrated using standard formula. In order to convert it into a form for which standard formula can be applied, we substitute some function in place of some other function and this technique of obtaining the integration is known as integration by substitution method.

Substitution Method

If integral is of the type $\int (f(x))^n f'(x)\,dx$ or $\int \frac{f'(x)}{(f(x))^n}dx$, then

Step I We substitute f(x) = t … (1)

Step II Differentiate on both sides of (1)

Step III Change the given integral in terms of t

Step IV After simplification, if necessary, we get one of the standard forms discussed in Sec. 7.3. Using appropriate formula we can obtain the integral of given integrand in terms of t.

Step V Replace t in terms of x, we get the desired result after simplification, if required.

Following example will explain the substitution method and the steps involved in it:

Example 4: Evaluate the following integrals:

(i) $\int \frac{x^9}{x^{10}+1}dx$ (ii) $\int \frac{x^{n-1}}{x^n+a}$ (iii) $\int \frac{e^x}{e^x+5}dx$

(iv) $\int \frac{1}{x\log x}dx$ (v) $\int \frac{2ax+b}{ax^2+bx+c}dx$ (vi) $\int \frac{8x^3+4x}{(x^4+x^2+1)^6}dx$

(vii) $\int \frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}dx$ (viii) $\int \frac{1}{x+\sqrt{x}}dx$ (ix) $\int (2ax+b)\sqrt{ax^2+bx+c}\,dx$

(x) $\int \frac{2x}{(1+x^2)\log(1+x^2)}dx$

Solution:

(i) Let $I = \int \frac{x^9}{x^{10}+1}dx$ … (1)

Putting $x^{10}+1=t$

Differentiating

$$10x^9dx = dt \Rightarrow x^9dx = \frac{dt}{10}$$

$\therefore$ (1) becomes

$$I = \int \frac{1}{t}\frac{dt}{10} = \frac{1}{10}\int\frac{dt}{t} = \frac{1}{10}\log|t| + c \quad \left[\because \int \frac{1}{y}dy = \log|y| + c\right]$$

$$= \frac{1}{10}\log|x^{10}+1| + c \quad \text{[Replacing t in terms of x]}$$

$$= \frac{1}{10}\log(x^{10}+1) + c \quad \left[\begin{array}{l}\because x^{10}+1 \text{ cannot} \\ \text{be} - \text{ve for real } x\end{array}\right]$$

Alternatively: We can also put

$x^{10} = t$

Differentiating

$$10x^9 dx = dt \Rightarrow x^9 dx = \frac{dt}{10}$$

$\therefore$ (1) becomes

$$I = \int \frac{1}{t+1} \times \frac{dt}{10} = \frac{1}{10}\int \frac{1}{t+1} dt$$

$$= \frac{1}{10}\log|t+1| + c \qquad \left[\because \int \frac{1}{x+a} dx = \log|x+a| + c\right]$$

$$= \frac{1}{10}\log|x^{10}+1| + c \qquad \text{[Replacing t in terms of x]}$$

$$= \frac{1}{10}\log(x^{10}+1) + c \qquad \left[\begin{array}{l}\because x^{10}+1 \text{ is always} + \text{ve} \\ \text{for real } x\end{array}\right]$$

(ii) Let $I = \int \frac{x^{n-1}}{x^n + a} dx$ … (1)

Putting $x^n + a = t$

Differentiating

$$nx^{n-1}dx = dt \Rightarrow x^{n-1}dx = \frac{dt}{n}$$

$\therefore$ (1) becomes

$$I = \frac{1}{n}\int \frac{dt}{t} = \frac{1}{n}\log|t| + c \qquad \left[\because \int \frac{1}{x} dx = \log|x| + c\right]$$

$$= \frac{1}{n}\log|x^n + a| + c \qquad \text{[Replacing t in terms of x]}$$

(iii) Let $I = \int \frac{e^x}{e^x + 5} dx$ … (1)

Putting $e^x + 5 = t$

Differentiating

$e^x dx = dt$

$\therefore$ (1) becomes

$$I = \int \frac{dt}{t} = \log|t| + c = \log|e^x + 5| + c \qquad \left[\because \int \frac{1}{x} dx = \log|x| + c\right]$$

(iv) Let $I = \int \frac{1}{x \log x} dx$ …(1)

Putting $\log x = t$

Differentiating

$\frac{1}{x} dx = dt$

$\therefore$ (1) becomes

$$I = \int \frac{dt}{t} = \log|t| + c \qquad \left[\because \int \frac{1}{x} dx = \log|x| + c\right]$$

$$= \log|\log x| + c \qquad \text{[Replacing t in terms of x]}$$

(v) Let $I = \int \frac{2ax + b}{ax^2 + bx + c} dx$ … (1)

Putting $ax^2 + bx + c = t$

Differentiating

$(2ax+b)dx = dt$

$\therefore$ (1) becomes

$$I = \int \frac{dt}{t} = \log|t| + k = \log\left|ax^2+bx+c\right| + k$$

where k is constant of integration

(vi) Let $I = \int \frac{8x^3+4x}{(x^4+x^2+1)^6} dx = 2\int \frac{4x^3+2x}{(x^4+x^2+1)^6} dx$ … (1)

Putting $x^4+x^2+1=t$

Differentiating

$(4x^3+2x)dx = dt$

$\therefore$ (1) becomes

$$I = 2\int \frac{dt}{t^6} = 2\int t^{-6} dt = 2\times\frac{t^{-5}}{-5} + c$$

$$= \frac{-2}{5}(x^4+x^2+1)^{-5} + c \quad \text{[Replacing t in terms of x]}$$

(vii) Let $I = \int \frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}} dx$ … (1)

Putting $e^{2x}-e^{-2x}=t$

Differentiating

$(2e^{2x}+2e^{-2x})dx = dt \Rightarrow (e^{2x}+e^{-2x})dx = \frac{dt}{2}$

$\therefore$ (1) becomes

$$I = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log|t| + c \quad \left[\because \int \frac{1}{x}dx = \log|x| + c\right]$$

$$= \frac{1}{2}\log\left|e^{2x}-e^{-2x}\right| + c \quad \text{[Replacing t in terms of x]}$$

(viii) Let $I = \int \frac{1}{x+\sqrt{x}} dx = \int \frac{1}{\sqrt{x}(\sqrt{x}+1)} dx$ … (1)

Putting $\sqrt{x}+1=t$

Differentiating

$\frac{1}{2\sqrt{x}}dx = dt \Rightarrow \frac{dx}{\sqrt{x}} = 2dt$

$\therefore$ (1) becomes

$$I = 2\int \frac{dt}{t} = 2\log|t| + c = 2\log\left|\sqrt{x}+1\right| + c$$

(ix) Let $I = \int (2ax+b)\sqrt{ax^2+bx+c}\, dx$ … (1)

Putting $ax^2+bx+c=t$

Differentiating

$(2ax+b)dx = dt$

$\therefore$ (1) becomes

$$I = \int \sqrt{t}\, dt = \frac{t^{3/2}}{3/2} + k = \frac{2}{3}(ax^2+bx+c)^{3/2} + k,$$

where k is constant of integration

(x) Let $I = \int \frac{2x}{(1+x^2)\log(1+x^2)}dx$ … (1)

Putting $\log(1+x^2) = t$

Differentiating

$$\frac{1}{1+x^2}2x\ dx = dt$$

$\therefore$ (1) becomes

$$I = \int \frac{dt}{t} = \log|t| + c = \log|\log(1+x^2)| + c \text{ [Replacing t in terms of x]}$$

Now, you can try the following exercise.

E 3) Evaluate the following integrals:

(i) $\int \frac{2x+1}{(x^2+x+7)^5}dx$ (ii) $\int x\sqrt{x+a}\ dx$

(iii) $\int \frac{x}{\sqrt{x+a}}dx$ (iv) $\int \frac{1}{(1+x)\log(1+x)}dx$

7.5 INTEGRATION USING PARTIAL FRACTIONS

The integrand may be in the form that it can be integrated only after resolving it into partial fractions. Here, in this section, we are going to deal with integration of such functions:

First of all we discuss the process of resolving such functions into partial fractions:

Important steps for resolving into partial fractions are:

1. Check degree of numerator, if it is less than that of denominator, go to step 2 and if it is greater than or equal to that of denominator, then first divide the numerator by the denominator and then go to step 2.
2. We may have one of the following main types of functions which we will dealt as discussed below:

Type 1 Denominator involve all linear factors with exponent as unity.

e.g. $\frac{x+5}{(x-1)(x-2)(x-3)}$.

Step I Let $\frac{x+5}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$ … (1)

Step II Equate each of the factors of denominator to zero.
i.e. $x - 1 = 0 \Rightarrow x = 1$, $x - 2 = 0 \Rightarrow x = 2$, $x - 3 = 0 \Rightarrow x = 3$

Step III Put x = 1, 2, 3 every where (in the given expression) but not in the factor from which it has come out,

$$A = \frac{1+5}{(1-2)(1-3)} = \frac{6}{2} = 3, \quad \text{[By putting x = 1 in L.H.S. of (1)]}$$

Step IV $B = \frac{2+5}{(2-1)(2-3)} = \frac{7}{-1} = -7$, [By putting x = 2 in L.H.S. of (1)]

Step V and $C = \frac{3+5}{(3-1)(3-2)} = \frac{8}{2\times 1} = 4$ [By putting x = 3 in L.H.S. of (1)]

Thus, we may write $\frac{x+5}{(x-1)(x-2)(x-3)} = \frac{3}{x-1} + \frac{-7}{x-2} + \frac{4}{x-3}$

R.H.S. is nothing but the partial fractions of the given expression. Here we note that integration of R.H.S. is directly available, as we will see in the Example 5 of this unit.

Type 2 Denominator involves all linear factors but some have 2, 3, etc. as exponents

e.g. $\frac{x^2+x+5}{(x+5)(x+2)^3}$

Step I Let $\frac{x^2+x+5}{(x+5)(x+2)^3} = \frac{A}{x+5} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}$

Multiply on both sides by denominator of L.H.S. in this case by $(x+5)(x+2)^3$, we get

$$x^2 + x + 5 = A(x+2)^3 + B(x+5)(x+2)^2 + C(x+5)(x+2) + D(x+5) \quad \ldots(1)$$

Step II Equate each of the factors to zero.

i.e. $x + 5 = 0 \Rightarrow x = -5$, $x + 2 = 0 \Rightarrow x = -2$

Step III Put $x = -5$ in (1) we get value of A, as given below

$$(-5)^2 + (-5) + 5 = A(-5+2)^3 + B(0) + C(0) + D(0)$$

$$\Rightarrow 25 = -27A \Rightarrow \boxed{A = -\frac{25}{27}}$$

Step IV Put $x = -2$ in (1) we get value of D, as given below

$$(-2)^2 + (-2) + 5 = A(0) + B(0) + C(0) + D(-2+5)$$

$$\Rightarrow 7 = 3D \Rightarrow \boxed{D = \frac{7}{3}}$$

Step V In order to find the values of B, C we have to equate the coefficients of different powers of x on both sides of (1).

In present case equating coefficients of x^3 and constant terms, we get

$0 = A + B$ … (2)

$5 = 8A + 20B + 10C + 5D$ … (3)

By putting value of A from Step III and value of D from step IV in equations (2) and (3), we get.

$$0 = -\frac{25}{27} + B = 0 \Rightarrow \boxed{B = \frac{25}{27}}$$

$$5 = 8\left(-\frac{25}{27}\right) + 20B + 10C + 5\left(\frac{7}{3}\right)$$

$$\Rightarrow 10C = 5 + \frac{200}{27} - 20\left(\frac{25}{27}\right) - \frac{35}{3} \Rightarrow \boxed{C = -48}$$

Thus, we may write

$$\frac{x^2+x+5}{(x+5)(x+2)^3}=\frac{-25/27}{x+5}+\frac{25/27}{x+2}+\frac{-48}{(x+2)^2}+\frac{7/3}{(x+2)^3}$$

R.H.S. is nothing but the partial fractions of the given expression. Here we note that integration of R.H.S. is directly available, as we will see in Example 5 of this unit.

Type 3 Denominator involves quadratic expressions. We will not discuss the problems based on this type, because it will involve the integral formulae which are beyond our contents.

Type 4 Denominator involves higher powers of quadratic expressions. This type is also not discussed here because it will involve the integral formulae which are beyond our contents.

Following example will illustrate how these types are used in evaluating integrals.

Example 5: Evaluate the following integrals:

(i) $\int \frac{4x+1}{(x-1)(x-2)}dx$ (ii) $\int \frac{3x+4}{x^2-x-12}dx$ (ii) $\int \frac{8x}{(x+1)(x-3)^2}dx$

(iv) $\int \frac{x^2+x+2}{(x+2)(x+1)^3}dx$ (v) $\int \frac{x^2+1}{x^2-1}dx$

Solution:

(i) Let $I=\int \frac{4x+1}{(x-1)(x-2)}dx$

$$=\int\left[\frac{-5}{x-1}+\frac{9}{x-2}\right]dx \qquad \left[\begin{array}{l}\text{Using type 1 procedure as already}\\ \text{discussed. Put } x=1 \text{ every where}\\ \text{except in } x-1 \text{ and } x=2 \text{ every}\\ \text{where except in } x-2\text{, we have}\\ A=\frac{4.1+1}{1-2}=-5,\ B=\frac{4.2+1}{2-1}=9\end{array}\right]$$

$$=-5\log|x-1|+9\log|x-2|+c$$

(ii) Let $I=\int \frac{3x+4}{x^2-x-12}dx=\int \frac{3x+4}{(x-4)(x+3)}dx$

$$=\int\left(\frac{16/7}{x-4}+\frac{5/7}{x+3}\right)dx \qquad \left[\begin{array}{l}\text{Using partial fractions}\\ \text{as discussed in type 1}\end{array}\right]$$

$$=\frac{16}{7}\log|x-4|+\frac{5}{7}\log|x+3|+c$$

(iii) Let $I=\int \frac{8x}{(x+1)(x-3)^2}dx$ … (1)

Let us first resolve into partial fractions

Let $\frac{8x}{(x+1)(x-3)^2}=\frac{A}{x+1}+\frac{B}{x-3}+\frac{C}{(x-3)^2}$

Multiplying on both sides by $(x+1)(x-3)^2$, we get

$$8x = A(x-3)^2 + B(x+1)(x-3) + C(x+1) \qquad \ldots (2)$$

Putting $x = -1$ in (2), we get $[\because x+1 = 0 \text{ gives } x = -1]$

$$-8 = A(-1-3)^2 + B(0) + C(0) \Rightarrow -8 = 16A \Rightarrow \boxed{A = -\frac{1}{2}}$$

Putting $x = 3$ in (2), we get $[\because x-3 = 0 \text{ gives } x = 3]$

$$24 = A(0) + B(0) + C(3+1) \Rightarrow 24 = 4C \Rightarrow \boxed{C = 6}$$

Comparing coefficient of x^2 on both sides of (2), we get

$$0 = A + B \Rightarrow B = -A \Rightarrow \boxed{B = \frac{1}{2}}$$

$$\therefore I = \int \left(\frac{-1/2}{x+1} + \frac{1/2}{x-3} + \frac{6}{(x-3)^2} \right) dx = \int \left[\frac{-1/2}{x+1} + \frac{1/2}{x-3} + 6(x-3)^{-2} \right] dx$$

$$= \frac{-1}{2} \log|x+1| + \frac{1}{2} \log|x-3| + 6\frac{(x-3)^{-1}}{-1} + c$$

$$= -\frac{1}{2} \log|x+1| + \frac{1}{2} \log|x-3| - \frac{6}{x-3} + c$$

(iv) Let $I = \int \frac{x^2 + x + 2}{(x+2)(x+1)^3} dx$

Let us first resolve into partial fractions

Let $\frac{x^2 + x + 2}{(x+2)(x+1)^3} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$

Multiplying on both sides by $(x+2)(x+1)^3$, we get

$$x^2 + x + 2 = A(x+1)^3 + B(x+2)(x+1)^2 + C(x+2)(x+1) + D(x+2) \ldots (2)$$

Putting $x = -2$ in (2), we get $[\because x+2 = 0 \text{ gives } x = -2]$

$$(-2)^2 + (-2) + 2 = A(-2+1)^2 + B(0) + C(0) + D(0)$$

$$\Rightarrow 4 = A \Rightarrow \boxed{A = 4}$$

Putting $x = -1$ in (2), we get $[\because x+1 = 0 \text{ gives } x = -1]$

$$(-1)^2 + (-1) + 2 = A(0) + B(0) + C(0) + D(-1+2)$$

$$\Rightarrow 2 = D \Rightarrow \boxed{D = 2}$$

Comparing coefficients of x^3 and constant terms on both sides of (2), we get

$$0 = A + B \Rightarrow B = -A \Rightarrow \boxed{B = -4}$$

$$2 = A + 2B + 2C + 2D \Rightarrow 2C = 2 - A - 2B - 2D$$
$$\Rightarrow 2C = 2 - 4 + 8 - 4$$
$$\Rightarrow 2C = 2$$
$$\Rightarrow \boxed{C = 1}$$

$$\therefore I = \int \left[\frac{4}{x+2} + \frac{-4}{x+1} + \frac{1}{(x+1)^2} + \frac{2}{(x+1)^3} \right] dx$$

$$= 4\log|x+2| - 4\log|x+1| + \frac{(x+1)^{-1}}{-1} + \frac{2(x+1)^{-2}}{-2} + c$$

$$= 4\log|x+2| - 4\log|x+1| - \frac{1}{x+1} - \frac{1}{(x+1)^2} + c$$

(v) Let $I = \int \frac{x^2+1}{x^2-1}dx = \int \left[1 + \frac{2}{x^2-1}\right]dx$

$$\begin{array}{r|l} & 1 \\ x^2-1 & x^2+1 \\ & x^2-1 \\ & - \quad + \\ \hline & 2 \end{array}$$

$$= \int \left[1 + \frac{2}{(x-1)(x+1)}\right]dx$$

$$= \int \left[1 + 2\left(\frac{1/2}{x-1} + \frac{-1/2}{x+1}\right)\right]dx \quad \left[\begin{array}{l}\text{Using partial fractions} \\ \text{as discussed in type 1}\end{array}\right]$$

$$= x + \log|x-1| - \log|x+1| + c$$

$$= x + \log\left|\frac{x-1}{x+1}\right| + c \qquad [\because \log m - \log n = \log\frac{m}{n}]$$

Now, you can try following exercise.

E 4) Evaluate the following integrals:

(i) $\int \frac{3x+2}{(x-1)(x-2)(x-3)}dx$ (ii) $\int \frac{x^3+5x+1}{x^2-4}dx$

7.6 INTERGRATION BY PARTS

If u and v are any two functions of a single variable x such that first derivates of u and v w.r.t. x exist, then by product rule, we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating on both sides, we have

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$\Rightarrow \int u\frac{dv}{dx}dx = uv - \int \frac{du}{dx}v\,dx \qquad \dots (1)$$

Let $u = f(x)$ and $\frac{dv}{dx} = g(x)$ … (2)

$$\Rightarrow \frac{du}{dx} = f'(x) \text{ and } v = \int g(x)dx \qquad \dots (3)$$

Using (2) and (3) in (1), we get

$$\int f(x)\,g(x)dx = f(x)\int g(x)dx - \int \left[f'(x)\int g(x)\,dx\right]dx$$

Or $\int I\ II\ dx = I\int II\ dx - \int\left[\frac{d}{dx}(I)\int II\,dx\right]dx$ … (4)

where I = first function = f(x)

II = second function = g(x)

R.H.S. of equation (4) is known as integration by parts of L.H.S. of equation (4), where I, and II just indicate our choice between the product of two functions taking as first and second functions.

Remark 3:

(i) In case of integration by parts, choice of first and second function is important as explained in part (i) of Example 6 given below.

(ii) If in the product of two functions, one is polynomial function then we take polynomial function as first function.

(iii) Integration by parts is one of the methods (techniques) of integration. It does not mean that integration of product of any two functions exists.

Example 6: Evaluate the following integrals:

(i) $\int xe^x dx$ (ii) $\int x^2 e^{3x} dx$ (iii) $\int x^3 a^x dx$ (iv) $\int \log x \, dx$

Solution:

(i) Let $I = \int \underset{\mathrm{I}}{x} \underset{\mathrm{II}}{e^x} dx$

Integrating by parts (taking x as first and e^x as second function)

$$I = x\int e^x dx - \int\left[\left(\frac{d}{dx}(x)\right)\left(\int e^x dx\right)\right]dx + c_1$$

where c_1 is constant of integration

$$= xe^x - \int (1)(e^x)dx + c_1 = xe^x - \int e^x dx + c_1 = xe^x - \left(e^x + c_2\right) + c_1$$

where c_2 is constant of integration

$$= xe^x - e^x + c, \quad \text{where } c = c_1 - c_2$$

Let us see what happens if we integrate by parts by taking x as second and e^x as first function:

$$I = e^x \int x\,dx - \int\left[\left(\frac{d}{dx}(e^x)\right)\left(\int x\,dx\right)\right]dx + c_1$$

$$= e^x \frac{x^2}{2} - \int\left[e^x \frac{x^2}{2}\right]dx + c_1 = \frac{x^2 e^x}{2} - \frac{1}{2}\int x^2 e^x\,dx + c_1$$

We see that integration becomes more complicated. So choice of first and second function is important.

Note: In future we will add c as constant of integration only once.

(ii) Let $I = \int \underset{\mathrm{I}}{x^2} \underset{\mathrm{II}}{e^{3x}} dx$

Integrating by parts (taking x^2 as first and e^{3x} as second function)

$$I = x^2\left(\frac{e^{3x}}{3}\right) - \int (2x)\left(\frac{e^{3x}}{3}\right)dx + c$$

where c is constant of integration

$$= \frac{x^2 e^{3x}}{3} - \frac{2}{3}\int \underset{\mathrm{I}}{x} \underset{\mathrm{II}}{e^{3x}}\,dx + c$$

Again integrating by parts (taking x as first and e^{3x} as second function)

$$I = \frac{x^2e^{3x}}{3} - \frac{2}{3}\left[x\left(\frac{e^{3x}}{3}\right) - \int (1)\left(\frac{e^{3x}}{3}\right)dx\right] + c$$

$$= \frac{x^2e^{3x}}{3} - \frac{2}{3}\left[\frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x}dx\right] + c$$

$$= \frac{x^2e^{3x}}{3} - \frac{2}{3}\left(\frac{xe^{3x}}{3} - \frac{e^{3x}}{9}\right) + c = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + c$$

(iii) Let I = $\int \underset{\text{I}}{x^3}\,\underset{\text{II}}{a^x}\,dx$

Integrating by parts (taking x^3 as first and a^x as second function)

$$I = \left(x^3\right)\left(\frac{a^x}{\log a}\right) - \int \left(3x^2\right)\left(\frac{a^x}{\log a}\right)dx + c = \frac{x^3a^x}{\log a} - \frac{3}{\log a}\int x^2a^x dx + c$$

Again integrating by parts (taking x^2 as first and a^x as second function)

$$I = \frac{x^3a^x}{\log a} - \frac{3}{\log a}\left[(x^2)\left(\frac{a^x}{\log a}\right) - \int (2x)\left(\frac{a^x}{\log a}\right)dx\right] + c$$

$$= \frac{x^3a^x}{\log a} - \frac{3x^2a^x}{(\log a)^2} + \frac{6}{(\log a)^2}\int \underset{\text{I}}{x}\,\underset{\text{II}}{a^x}\,dx + c$$

Again integrating by parts (taking x as first and a^x as second function)

$$= \frac{x^3a^x}{\log a} - \frac{3x^2a^x}{(\log a)^2} + \frac{6}{(\log a)^2}\left[(x)\left(\frac{a^x}{\log a}\right) - \int (1)\left(\frac{a^x}{\log a}\right)dx\right] + c$$

$$= \frac{x^3a^x}{\log a} - \frac{3x^2a^x}{(\log a)^2} + \frac{6}{(\log a)^2}\left[\frac{xa^x}{\log a} - \frac{a^x}{(\log a)^2}\right] + c$$

$$= \frac{x^3a^x}{\log a} - \frac{3x^2a^x}{(\log a)^2} + \frac{6xa^x}{(\log a)^3} - \frac{6a^x}{(\log a)^4} + c$$

(iv) Let I = $\int \log x\, dx = \int \underset{\text{II}}{1} \times \underset{\text{I}}{\log x}\, dx$

Integrating by parts (taking $\log x$ as first and 1 as second function)

$$I = (\log x)(x) - \int \left(\frac{1}{x}\right)(x)dx + c = x\log x - \int 1dx + c = x\log x - x + c$$

Here, is an exercise for you.

E 5) Evaluate the following integrals:

(i) $\int x^2e^{-x}dx$ (ii) $\int x^3e^{x^2}dx$

7.7 SUMMARY

Let us summarise the topics that we have covered in this unit.

1) Integral of some functions like constant, $x^n, \frac{1}{x^n}$, polynomial, $(ax+b)^n, \frac{1}{ax+b}$, exponential whose integral are directly available.
2) Integration by method of substitution.
3) Integration by use of partial fractions.
4) Integration by parts.

7.8 SOLUTIONS/ANSWERS

E 1) (i) $$\int\left(x^2+\frac{1}{x^2}\right)^2 dx = \int\left(x^4+\frac{1}{x^4}+2x^2.\frac{1}{x^2}\right)dx = \int\left(x^4+\frac{1}{x^4}+2\right)dx$$

$$= \int(x^4+x^{-4}+2)dx$$

$$= \frac{x^5}{5}+\frac{x^{-3}}{-3}+2x+c \qquad \left[\because \int x^n dx = \frac{x^{n+1}}{n+1}+c \text{ and } \int k dx = kx+c\right]$$

$$= \frac{x^5}{5}-\frac{1}{3x^3}+2x+c$$

(ii) $$\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^3 dx = \int\left[x^{3/2}+\frac{1}{x^{3/2}}+3.\sqrt{x}.\frac{1}{\sqrt{x}}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\right]dx$$

$$= \int\left(x^{3/2}+x^{-3/2}+3x^{1/2}+3x^{-1/2}\right)dx$$

$$= \frac{x^{5/2}}{5/2}+\frac{x^{-1/2}}{-1/2}+\frac{3x^{3/2}}{3/2}+\frac{3x^{1/2}}{1/2}+c$$

$$\left[\because \int x^n dx = \frac{x^{n+1}}{n+1}+c\right]$$

$$= \frac{2}{5}x^{5/2}-\frac{2}{\sqrt{x}}+2x^{3/2}+6\sqrt{x}+c$$

(iii) $$\int(\alpha-3)dx = (\alpha-3)x+c \qquad [\because \alpha-3 \text{ is a constant}]$$

(iv) $$\int\sqrt{x}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx = \int(x+1)dx = \frac{x^2}{2}+x+c$$

(v) $$\int(x^a+1)(x^b+1)dx = \int(x^{a+b}+x^a+x^b+1)dx+c$$

$$= \frac{x^{a+b+1}}{a+b+1}+\frac{x^{a+1}}{a+1}+\frac{x^{b+1}}{b+1}+x+c,$$

$$\left[\because \int x^n dx = \frac{x^{n+1}}{n+1} \text{ and } \int k dx = kx, \text{ where } k \text{ is constant}\right]$$

where $a \neq -1, b \neq -1, a+b \neq -1$

(vi) $\int \frac{x^m}{x^n} dx = \int x^{m-n} dx = \frac{x^{m-n+1}}{m-n+1} + c, \; m-n \neq -1 \quad \left[\because \int x^n dx = \frac{x^{n+1}}{n+1}\right]$

(vii) $\int \left(x+\frac{1}{x}\right)\left(x^3+\frac{1}{x^3}\right) dx = \int \left(x^4 + \frac{1}{x^2} + x^2 + \frac{1}{x^4}\right) dx$

$$= \int (x^4 + x^{-2} + x^2 + x^{-4}) dx$$

$$= \frac{x^5}{5} + \frac{x^{-1}}{-1} + \frac{x^3}{3} + \frac{x^{-3}}{-3} + c \left[\because \int x^n dx = \frac{x^{n+1}}{n+1}\right]$$

$$= \frac{x^5}{5} - \frac{1}{x} + \frac{x^3}{3} - \frac{1}{3x^3} + c$$

E 2) (i) $\int \left(a^x + e^x a^x + \frac{x}{a}\right) dx = \int \left(a^x + (ea)^x + \frac{x}{a}\right) dx = \frac{a^x}{\log a} + \frac{(ea)^x}{\log ea} + \frac{x^2}{2a} + c$

(ii) $\int \left(3^{2\log_3 x} + 3^{x\log_3 a} + a^{a\log_a x} + a^{a\log_a a}\right) dx$

$$= \int \left(3^{\log_3 x^2} + 3^{\log_3 a^x} + a^{\log_a x^a} + a^{\log_a a^a}\right) dx$$

$$= \int \left(x^2 + a^x + x^a + a^a\right) dx \qquad [\because a^{\log_a f(x)} = f(x)]$$

$$= \frac{x^3}{3} + \frac{a^x}{\log|a|} + \frac{x^{a+1}}{a+1} + a^a x + c$$

E 3) (i) Let $I = \int \frac{2x+1}{(x^2+x+7)^5} dx$ … (1)

Putting $x^2 + x + 7 = t$

Differentiating

$(2x+1)dx = dt$

$\therefore$ (1) becomes

$$I = \int \frac{dt}{t^5} = \int t^{-5} dt = \frac{t^{-4}}{-4} + c = \frac{-1}{4t^4} + c = -\frac{1}{4(x^2+x+7)^4} + c$$

(ii) Let $I = \int x\sqrt{x+a}\, dx$ … (1)

Putting $\sqrt{x+a} = t \Rightarrow x + a = t^2$

Differentiating

$dx = 2t\,dt$

$\therefore$ (1) becomes

$$I = \int (t^2 - a)t\,(2t)dt = 2\int (t^4 - at^2)dt = 2\left(\frac{t^5}{5} - \frac{at^3}{3}\right) + c$$

$$= 2\left[\frac{(x+a)^{5/2}}{5} - \frac{a}{3}(x+a)^{3/2}\right] + c$$

(iii) Let $I = \int \frac{x}{\sqrt{x+a}} dx$ … (1)

Putting $\sqrt{x+a} = t \Rightarrow x + a = t^2$

Differentiating

$dx = 2t\,dt$

$\therefore$ (1) becomes

$$I = \int \frac{(t^2 - a)}{t} 2t\,dt = 2\int (t^2 - a)\,dt = 2\left(\frac{t^3}{3} - at\right) + c$$

$$= 2\left(\frac{(x+a)^{3/2}}{3} - a\sqrt{x+a}\right) + c$$

(iv) Let $I = \int \frac{1}{(1+x)\log(1+x)} dx$ … (1)

Putting $\log(1+x) = t$

Differentiating

$$\frac{1}{1+x} dx = dt$$

$\therefore$ (1) becomes

$$I = \int \frac{dt}{t} = \log|t| + c = \log|\log(1+x)| + c$$

E 4) (i) Let $I = \int \frac{3x+2}{(x-1)(x-2)(x-3)} dx = \int \left[\frac{5/2}{x-1} + \frac{-8}{x-2} + \frac{11/2}{x-3}\right] dx$

$$\left[\begin{array}{l}\text{Using partial fractions as discussed in type 1, we get} \\ A = \frac{3.1+2}{(1-2)(1-3)} = \frac{5}{2}, B = \frac{3.2+2}{(2-1)(2-3)} = -8, C = \frac{3.3+2}{(3-1)(3-2)} = \frac{11}{2}\end{array}\right]$$

$$= \frac{5}{2}\log|x-1| - 8\log|x-2| + \frac{11}{2}\log|x-3| + c$$

(ii) Let $I = \int \frac{x^3 + 5x + 1}{x^2 - 4} dx$ … (1)

Dividing numerator by denominator, we can write (1) as

$$I = \int \left[x + \frac{9x+1}{x^2-4}\right] dx = \int \left[x + \frac{9x+1}{(x-2)(x+2)}\right] dx$$

$$\begin{array}{r|l} x^2 - 4 & x^3 + 5x + 1 \;\; (x \\ & x^3 - 4x \\ & - \quad + \\ \hline & 9x + 1 \end{array}$$

$$I = \int \left[x + \frac{19/4}{x-2} + \frac{17/4}{x+2}\right] dx \qquad \left[\begin{array}{l}\text{Using partial fractions} \\ \text{as discussed in type 1}\end{array}\right]$$

$$= \frac{x^2}{2} + \frac{19}{4}\log|x-2| + \frac{17}{4}\log|x+2| + c \qquad \left[\because \int \frac{1}{x+a} dx = \log|x+a|\right]$$

$$= \frac{x^2}{2} + \frac{1}{4}(19\log|x-2| + 17\log|x+2|) + c$$

$$= \frac{x^2}{2} + \frac{1}{4}(\log|x-2|^{19} + \log|x+2|^{17}) + c \; [\because n\log m = \log m^n]$$

$$= \frac{x^2}{2} + \frac{1}{4}\log\left|(x-2)^{19}(x+2)^{17}\right| + c \qquad [\because \log m + \log n = \log mn]$$

E 5) (i) Let $I = \int \underset{\text{I}}{x^2}\,\underset{\text{II}}{e^{-x}}\,dx$

Integrating by parts (taking x^2 as first and e^{-x} as second function)

$$I = (x^2)\left(\frac{e^{-x}}{-1}\right) - \int (2x)\left(\frac{e^{-x}}{-1}\right)dx + c$$

where c is constant of integration

$$= -x^2e^{-x} + 2\int \underset{\text{I}}{x}\,\underset{\text{II}}{e^{-x}}\,dx + c$$

Integrating by parts (taking x as first and e^{-x} as second function)

$$I = -x^2e^{-x} + 2\left[(x)\left(\frac{e^{-x}}{-1}\right) - \int (1)\left(\frac{e^{-x}}{-1}\right)dx\right] + c$$

$$= -x^2e^{-x} + 2[-xe^{-x} + \int e^{-x}dx\,] + c$$

$$= -x^2e^{-x} + 2\left[-xe^{-x} + \left(\frac{e^{-x}}{-1}\right)\right] + c$$

$$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + c$$

(ii) Let $I = \int xx^2e^{x^2}dx \quad \dots (1)$

Putting $x^2 = t$

Differentiating

$$2xdx = dt \Rightarrow xdx = \frac{dt}{2}$$

$\therefore$ (1) becomes

$$I = \frac{1}{2}\int \underset{\text{I}}{t}\,\underset{\text{II}}{e^t}\,dt$$

Integrating by parts (taking t as first and e^t as second function)

$$I = \frac{1}{2}\left[(t)(e^t) - \int (1)(e^t)dt\right] + c = \frac{1}{2}(te^t - e^t) + c = \frac{1}{2}(x^2e^{x^2} - e^{x^2}) + c$$

$$= \frac{1}{2}\left(x^2 - 1\right)e^{x^2} + c$$