UNIT 6 DIFFERENTIATION

Structure

6.1 INTRODUCTION

In the preceding unit, we have discussed concept of limit and continuity. In fact, the definition of derivative involves these concepts. So, learner must go through the previous unit before starting this unit. Derivatives have large number of applications in the fields of mathematics, statistics, economics, insurance, industrial, health sector, etc.

In this unit, we will present this concept from a very simple and elementary point of view, keeping in mind that learner knows nothing about derivatives. In this unit, we have discussed some examples basically based on the formulae for derivatives of a constant, polynomial, exponential, logarithmic, parametric and implicit functions. Product rule, quotient rule, chain rule have also been discussed. Finally, we close this unit by giving a touch to higher order derivatives and maxima and minima of functions.

Objectives

After completing this unit, you should be able to:

- find derivative of a function at a particular point and at a general point;
- find derivative by first principle;
- find derivative of some commonly used functions;
- apply the chain rule;
- find derivative of exponential, logarithmic, parametric and implicit functions;
- find higher order derivatives; and
- find maxima and minima of a function.

6.2 DEFINITION OF DERIVATIVE, ITS MEANING AND GEOMETRICAL INTERPRETATION

Definition

Let $f : D \to R$ be a function, where $D \subseteq R$,
i.e. f is a real valued function defined on D.
Let $a \in D$ then derivative of f at x = a is denoted by $f'(a)$ and is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \text{ provided limit exists} \qquad \ldots (1)$$

From definition (1), we see that $f'(a)$ measures the rate at which the function f(x) changes at x = a. This is clear from the figure 6.1 given below.

Geometrical Interpretation

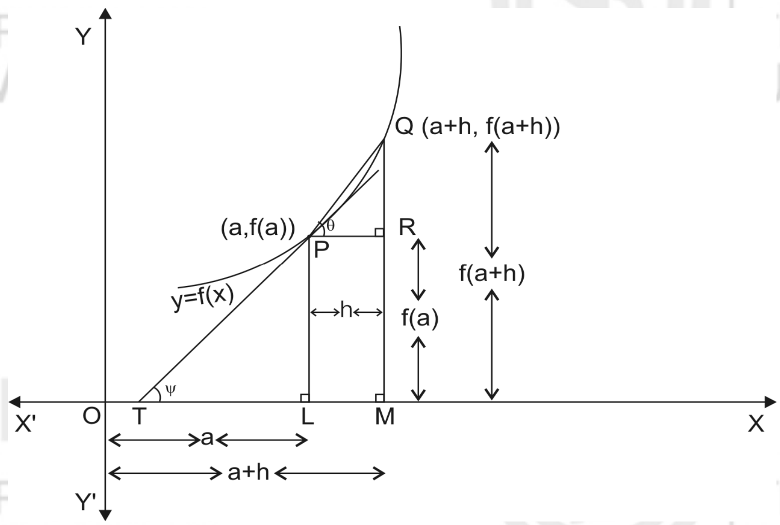

Fig. 6.1

Let PT be the tangent at point P of the curve of the function y = f(x).
Draw $PL \perp OX,\ QM \perp OX, PR \perp QM$
Let OL = a, OM = a + h
$\therefore PR = LM = OM - OL = a + h - a = h$
and $RQ = MQ - MR = MQ - LP = f(a+h) - f(a)$
$\therefore (1) \Rightarrow$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{RQ}{PR}$$

$$= \lim_{h \to 0} \tan\theta \qquad \left[\because \text{ in } \Delta PQR, \tan\theta = \frac{\text{perpendicular}}{\text{Base}} = \frac{RQ}{PR}\right]$$

Now as $h \to 0$, chord PQ tends to coincide with the tangent at point P,
i.e. as $h \to 0 \Rightarrow \theta \to \psi$

$\therefore f'(a) = \lim_{h \to 0} \tan\theta = \tan\psi$

i.e. $f'(a) = \tan\psi$

i.e. (derivative at point x = a) = (tangent of the angle which the tangent line at x = a makes with +ve direction of x-axis)

In fact, if a line makes an angle θ with position direction of x-axis, then value of $\tan\theta$ is known as slope of the line.
Thus in mathematical language we can say

$$(\text{Derivative at a point})_{x=a} \text{ is the slope of the tangent at that point.} \quad \ldots (2)$$

i.e. we can say that derivative measures the rate at which the tangent to the curve at point x = a is changing

Meaning

Rewriting (1)

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \qquad \ldots (3)$$

From the knowledge of previous unit, we know that limit in R.H.S. of (1) or (3) exists if

$$\lim_{h\to 0^-} \frac{f(a+h) - f(a)}{h} \text{ and } \lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} \quad \text{both exist and are equal.}$$

In view of (2), we have, limit in (1) exists if

$$\begin{pmatrix}\text{Slope of the tangent to the left} \\ \text{of the point } x = a\end{pmatrix} = \begin{pmatrix}\text{Slope of the tangent to the right} \\ \text{of the point } x = a\end{pmatrix}$$

i.e. limit in (1) exists if x = a is not a corner point.
i.e. $f'(a)$ does not exists at corner points. … (4)

For example, consider the function
$f(x) = |x|$

See the graph of this function in Fig. 6.2 .We observe that x = 0 is a corner point in its graph.

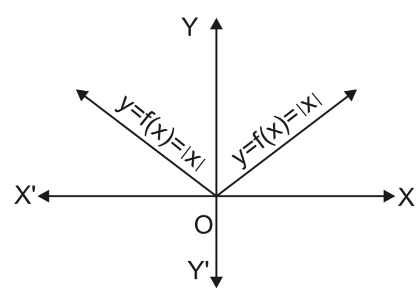

Fig . 6.2

So $f(x) = |x|$ is not differentiable at x = 0. but there is no other corner point in its graph, so it is differentiable at all points of the domain except x = 0.

Remark1:

(i) The last paragraph is very useful to understand the concept of derivative for those learners, who do not have mathematical background.

(ii) However, the units have been written keeping in mind that the learner has no mathematical background after 10^{th} standard.

(iii) In the definition of derivative of a function at a point given by (1), we see that in order to find the derivative at the said point, we have to evaluate the limit in R.H.S. But sometimes functions may have different values for $h \to 0^-$ and $h \to 0^+$. In such cases like modulus function or where there is break for function in order to evaluate the limit we have to find out the L.H.L. and R.H.L. as we have done in the previous unit. But in the definition of derivative these L.H.L. and R.H.L. are known as left hand derivative (L.H.D.) and right hand derivative (R.H.D.) respectively.

A function is said to have derivative at a point if L.H.D. and R.H.D. both exist and are equal at that point, i.e. L.H.D. = R.H.D.
We denote L.H.D. of the function $f(x)$ at $x = a$ by $L(f'(a))$ and R.H.D. of the function $f(x)$ at $x = a$ by $R(f'(a))$. See Example 2 of this unit for more clarity.

6.3 DERIVATIVE AT A POINT

Here, we give some examples which will illustrate the idea as to how we calculate derivative of a function at a point.

Example 1: Find the derivative of the following functions at the indicated points:

(i) $f(x) = a$, at $x = 5$, where a is a real constant

(ii) $f(x) = ax + b$, at $x = 2$, $a \neq 0$

(iii) $f(x) = ax^2 + bx + c$, at $x = 3$, $a \neq 0$

(iv) $f(x) = \dfrac{1}{x}$, at $x = 1$

Solution:

(i) $f(x) = a$, where a is real constant

By definition

$$f'(5) = \lim_{h\to 0}\frac{f(5+h) - f(5)}{h} = \lim_{h\to 0}\frac{a-a}{h} = \lim_{h\to 0}\frac{0}{h} = \lim_{h\to 0} 0 = 0$$

(ii) $f(x) = ax + b$, $a \neq 0$

By definition

$$f'(2) = \lim_{h\to 0}\frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0}\frac{a(2+h) + b - (2a+b)}{h} = \lim_{h\to 0}\frac{ah}{h} = \lim_{h\to 0} a = a$$

(iii) $f(x) = ax^2 + bx + c$, $a \neq 0$

By definition

$$f'(3) = \lim_{h\to 0}\frac{f(3+h) - f(3)}{h} = \lim_{h\to 0}\frac{a(3+h)^2 + b(3+h) + c - (9a + 3b + c)}{h}$$

$$= \lim_{h\to 0}\frac{ah^2 + 6ah + bh}{h} = \lim_{h\to 0}(ah + 6a + b) = 6a + b$$

(iv) $f(x) = \dfrac{1}{x}$

By definition

$$f'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h} = \lim_{h\to 0}\frac{\frac{1}{1+h} - \frac{1}{1}}{h}$$

$$= \lim_{h\to 0}\frac{1 - (1+h)}{h(1+h)} = \lim_{h\to 0}\frac{-h}{h(1+h)} = -\lim_{h\to 0}\frac{1}{1+h} = -\frac{1}{1+0} = -1$$

Here are some exercises for you.

E 1) Find the derivative of the following functions at the indicated points

(i) $f(x) = x^3 + x + 1,\quad$ at $x = -1$

(ii) $f(x) = 2 - 3x^2,\quad$ at $\quad x = 1/2$

E 2) Find the value of a, if $f'(-2) = 3$, where $f(x) = 2x^2 - 3ax + 5$

Example 2: Find the derivative (if exists) of the following functions at the indicated points.

(i) $f(x) = |x|\quad$ at $x = 0$

(ii) $f(x) = \begin{cases} 5 + 2x, & x \ge 1 \\ 9 - 2x, & x < 1 \end{cases}\qquad$ at x = 1

Solution:

(i) By definition

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{|0+h|-|0|}{h} = \lim_{h\to 0}\frac{|h|-0}{h} = \lim_{h\to 0}\frac{|h|}{h}$$

We note that to deal with $|h|$ we must know the sign of h in advance. So We must have to calculate L.H.D. and R.H.D. separately.

$$\text{L.H.D.} = \lim_{h\to 0^-}\frac{|h|}{h}$$

Putting h = 0 – k as $h \to 0^- \Rightarrow k \to 0^+$

$$\text{L.H.D.} = \lim_{k\to 0^+}\frac{|0-k|}{0-k} = \lim_{k\to 0^+}\frac{|-k|}{-k} = \lim_{k\to 0^+}\frac{|-1|\times|k|}{-k} = \lim_{k\to 0^+}\frac{k}{-k}$$

$$= \lim_{k\to 0^+}(-1) = -1 \qquad \dots (1)$$

$$\text{R.H.D.} = \lim_{h\to 0^+}\frac{|h|}{h} = \lim_{h\to 0^+}\frac{h}{h} = \lim_{h\to 0^+}(1) = 1 \qquad \dots (2)$$

From (1) and (2), we have

L.H.D. $\ne$ R.H.D.

$\Rightarrow f'(0)$ does not exists.

(ii) By definition

$$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{f(1+h)-(5+2\times 1)}{h} = \lim_{h\to 0}\frac{f(1+h)-7}{h}$$

We note that function have different values for x<1 and x >1, so we must have to calculate L.H.D. and R.H.D. separately.

$$L(f'(1)) = \text{L.H.D.} = \lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-}\frac{9-2(1+h)-7}{h}$$

$$= \lim_{h\to 0^-}\frac{-2h}{h} = \lim_{h\to 0^-}(-2) = -2 \qquad \dots (1)$$

$$R(f'(1)) = \text{R.H.D.} = \lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+}\frac{5+2(1+h)-7}{h}$$

$$= \lim_{h\to 0^+}\frac{2h}{h} = \lim_{h\to 0^+}(2) = 2 \qquad \dots (2)$$

From (1) and (2)

$L(f'(1)) \ne R(f'(1))$

$\Rightarrow f'(1)$ does not exists.

Remark 2: In part (i) x =0 is a corner point (see Fig. 6.2) that is why its derivative did not exist at this point, which was indicated in equation (4) in Sec. 6.2. Same is the case in part (ii).

6.4 DERIVATIVE BY FIRST PRINCIPLE

In section 6.3 of this unit, we have discussed as to how we calculate the derivative of a function at a given point x = a (say). Suppose we want to calculate the derivative at 10 points, then using the definition 10 times is a very time consuming and lengthy procedure. To get rid of this difficulty, we will introduce a procedure in this section which will provide us the derivative of the function at a general point. After calculating the derivative at the general point we can replace this point by any number of points very quickly (provided derivative at the required point exists). Let us first describe the procedure as to how we calculate the derivative at a general point. After this we shall give some results to get a good understanding of the procedure. This process of finding derivative is known as derivative by **first principle** or by **definition** or by **delta** method or **ab-intio**.

Let us explain the procedure of first principle for the function

$y = f(x)$ … (1)

in the following steps.

Step I Let δx be the small increment (+ve or –ve) in the value of x and δy be the corresponding increment in the value of y.

$\therefore$ (1) becomes

$$y + \delta y = f(x + \delta x) \quad \dots (2)$$

Step II (2) – (1) gives

$$\delta y = f(x + \delta x) - f(x) \quad \dots (3)$$

Step III First we simplify the expression in the R.H.S. of (3). After simplifying the expression, we divide both sides by δx and get

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Step IV Proceeding limit as $\delta x \to 0$ on both sides

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \quad \dots (4)$$

Step V The term in L.H.S. of (4) is denoted by $\frac{dy}{dx}$ and limit in R.H.S. of (4) is evaluated using suitable formula discussed in the previous unit

i.e. $$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \quad \dots (5)$$

The expression obtained after simplification of the R.H.S. of (5), is derivative of y w.r.t. x at a general point x.

If we want the derivative of the function y = f(x) at a particular point x = a (say), then replace x by a in the result.

Some Results

Result 1: Find the derivative of the constant function given by
f (x) = k, where k is a real constant
by using first principle.

Solution: Let y = f(x) = k … (1)

Step I Let δx be the small increment in the value of x and δy be the corresponding increment in the value of y.

$\therefore$ (1) becomes

$$y + \delta y = k \qquad \ldots (2)$$

Step II (2) – (1) gives

$$y + \delta y - y = k - k$$

Or $\delta y = 0$ … (3)

Step III Dividing on both sides of (3) by δx

$$\frac{\delta y}{\delta x} = \frac{0}{\delta x} = 0 \Rightarrow \frac{\delta y}{\delta x} = 0$$

Step IV Proceeding limit as $\delta x \to 0$, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} 0$$

Step V $\frac{dy}{dx} = 0$

i.e. $\frac{d}{dx}(k) = 0$.

So, derivative of a constant function is zero.

Result 2: Find the derivatives of the functions given by

(i) $f(x) = x^2$ (ii) $f(x) = x^3$

by using first principle.

Solution: (i) Let $y = f(x) = x^2$ … (1)

Step I Let δx be the small increment in the value of x and δy be the corresponding increment in the value of y.

$\therefore$ (1) becomes

$$y + \delta y = (x + \delta x)^2 = x^2 + (\delta x)^2 + 2x \times \delta x \qquad \ldots (2)$$

Step II (2) – (1) gives

$$\delta y = x^2 + (\delta x)^2 + 2x \times \delta x - x^2 = (\delta x)^2 + 2x \times \delta x$$

Step III Dividing on both sides by δx, we get

$$\frac{\delta y}{\delta x} = \frac{(\delta x)^2 + 2x \times \delta x}{\delta x} = \delta x + 2x$$

Step IV Proceeding limit as $\delta x \to 0$, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} (\delta x + 2x)$$

Step V $\frac{dy}{dx} = \lim_{\delta x \to 0} \delta x + \lim_{\delta x \to 0} 2x = 0 + 2x = 2x$

i.e. $\frac{d}{dx}(x^2) = 2x = 2x^{2-1}$

Second Method

Let $y = f(x) = x^2$ … (1)

Step I Let δx be the small increment in the value of x and δy be the corresponding increment in the value of y.

$\therefore$ (1) becomes

$y + \delta y = (x + \delta x)^2$ … (2)

Step II (2) – (1) gives

$\delta y = (x + \delta x)^2 - x^2$

Step III Dividing on both sides by δx, we get

$$\frac{\delta y}{\delta x} = \frac{(x + \delta x)^2 - x^2}{\delta x} = \frac{(x + \delta x)^2 - x^2}{(x + \delta x) - x}$$

Step IV Proceeding limit as $\delta x \to 0$, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{(x + \delta x)^2 - x^2}{(x + \delta x) - x}$$

Step V $\dfrac{dy}{dx} = \lim\limits_{x + \delta x \to x} \dfrac{(x + \delta x)^2 - x^2}{(x + \delta x) - x}$ as $\delta x \to 0 \Rightarrow x + \delta x \to x$

i.e. $\dfrac{d}{dx}\left(x^2\right) = 2x^{2-1} = 2x$ $\left[\because \lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}\right]$

(ii) Let $y = f(x) = x^3$ … (1)

Step I Let δx be the small increment in the value of x and δy be the corresponding increment in the value of y.

$\therefore$ (1) becomes

$y + \delta y = (x + \delta x)^3 = x^3 + 3x^2 \times \delta x + 3x \times (\delta x)^2 + (\delta x)^3$ … (2)

Step II (2) – (1) gives

$$\delta y = x^3 + 3x^2 \times \delta x + 3x \times (\delta x)^2 + (\delta x)^3 - x^3$$
$$= 3x^2 \times \delta x + 3x \times (\delta x)^2 + (\delta x)^3$$

Step III Dividing on both sides by δx, we get

$$\frac{\delta y}{\delta x} = \frac{3x^2 \times \delta x + 3x \times (\delta x)^2 + (\delta x)^3}{\delta x} = 3x^2 + 3x \times \delta x + (\delta x)^2$$

Step IV Proceeding limit as $\delta x \to 0$, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left(3x^2 + 3x \times \delta x + (\delta x)^2\right)$$

Step V $\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} 3x^2 + \lim\limits_{\delta x \to 0} 3x \times \delta x + \lim\limits_{\delta x \to 0} (\delta x)^2 = 3x^2 + 0 + 0 = 3x^2$

i.e. $\dfrac{d}{dx}(x^3) = 3x^2 = 3x^{3-1}$

Second Method

Let $y = f(x) = x^3$ … (1)

Step I Let δx be the small increment in the value of x and δy be the corresponding increment in the value of y.

$\therefore$ (1) becomes

$y + \delta y = (x + \delta x)^3$ … (2)

Step II (2) – (1) gives

$\delta y = (x + \delta x)^3 - x^3$

Step III Dividing on both sides by δx, we get

$$\frac{\delta y}{\delta x} = \frac{(x+\delta x)^3 - x^3}{\delta x} = \frac{(x+\delta x)^3 - x^3}{(x+\delta x) - x}$$

Step IV Proceeding limit as $\delta x \to 0$, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{(x+\delta x)^3 - x^3}{(x+\delta x) - x}$$

Step V $\frac{dy}{dx} = \lim_{x+\delta x \to x} \frac{(x+\delta x)^3 - x^3}{(x+\delta x) - x}$ as $\delta x \to 0 \Rightarrow x + \delta x \to x$

i.e. $\frac{d}{dx}(x^3) = 3x^{3-1}$ $\left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$

$$\boxed{\textbf{Similarly, } \frac{d}{dx}(x^n) = nx^{n-1}}$$

Result 3: Find the derivative of the function given by
$f(x) = (ax+b)^2$, where a, b are real constants and $a \neq 0$
by using first principle.

Solution: Let us use second method here.

Let $y = f(x) = (ax+b)^2$ … (1)

Step I Let δx be the small increment in the value of x and δy be the corresponding increment in the value of y.

$\therefore$ (1) becomes

$y + \delta y = [a(x+\delta x)+b]^2 = (ax + a\delta x + b)^2$ … (2)

Step II (2) – (1) gives

$\delta y = (ax + a\delta x + b)^2 - (ax+b)^2$

Step III Dividing on both sides by δx, we get

$$\frac{\delta y}{\delta x} = \frac{(ax + a\delta x + b)^2 - (ax+b)^2}{\delta x} = a\left[\frac{(ax + a\delta x + b)^2 - (ax+b)^2}{a\delta x}\right]$$

$$= a\left[\frac{(ax + a\delta x + b)^2 - (ax+b)^2}{(ax + a\delta x + b) - (ax+b)}\right]$$

Step IV Proceeding limit as $\delta x \to 0$, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = a \lim_{\delta x \to 0}\left[\frac{(ax + a\delta x + b)^2 - (ax+b)^2}{(ax + a\delta x + b) - (ax+b)}\right]$$

$$= a \lim_{a\delta x \to 0}\left[\frac{(ax + a\delta x + b)^2 - (ax+b)^2}{(ax + a\delta x + b) - (ax+b)}\right] \quad \text{as } \delta x \to 0 \Rightarrow a\delta x \to 0$$

Step V $\frac{dy}{dx} = a \lim_{ax + a\delta x + b \to ax+b}\left[\frac{(ax + a\delta x + b)^2 - (ax+b)^2}{(ax + a\delta x + b) - (ax+b)}\right]$

$= a \times 2(ax+b)^{2-1}$ $\left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$

i.e. $\frac{d}{dx}(ax+b)^2 = 2a(ax+b)^{2-1}$

Similarly, we can easily obtain
$\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}$

Result 4: Find the derivative of the exponential function $f(x) = e^{ax}$ by using first principle.

Solution: Let $y = f(x) = e^{ax}$ … (1)

Step I Let δx be the small increment in the value of x and δy be the corresponding increment in the value of y.

$\therefore$ (1) becomes

$y + \delta y = e^{a(x+\delta x)} = e^{ax+a\delta x}$ … (2)

Step II (2) – (1) gives

$\delta y = e^{ax+a\delta x} - e^{ax} = e^{ax} e^{a\delta x} - e^{ax}$ $[\because a^{m+n} = a^m a^n]$

$= e^{ax}(e^{a\delta x} - 1)$

Step III Dividing on both sides by δx, we get

$$\frac{\delta y}{\delta x} = e^{ax}\frac{(e^{a\delta x}-1)}{\delta x} = ae^{ax}\left(\frac{e^{a\delta x}-1}{a\delta x}\right)$$

Step IV Proceeding limit as $\delta x \to 0$, we get

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} ae^{ax}\left(\frac{e^{a\delta x}-1}{a\delta x}\right)$$

Step V $\frac{dy}{dx} = ae^{ax}\lim_{a\delta x\to 0}\frac{e^{a\delta x}-1}{a\delta x}$ as $\delta x \to 0 \Rightarrow a\delta x \to 0$

$= ae^{ax}$ (1) $\left[\because \lim_{x\to 0}\frac{e^x-1}{x} = 1\right]$

i.e. $\frac{d}{dx}(e^{ax}) = ae^{ax}$

Result 5: Find the derivative of the logarithm function $f(x) = \log_a x$ by using first principle.

Solution: Let $y = f(x) = \log_a x$ … (1)

Step I Let δx be the small increment in the values of x and δy be the corresponding increment in the value of y

$\therefore$ (1) becomes

$y + \delta y = \log_a(x+\delta x)$ … (2)

Step II (2) – (1) gives

$\delta y = \log_a(x+\delta x) - \log_a x = \log_a\left(\frac{x+\delta x}{x}\right)$ $\left[\because \log m - \log n = \log\frac{m}{n}\right]$

$= \log_a\left(1+\frac{\delta x}{x}\right)$

Step III Dividing on both sides by δx, we get

$$\frac{\delta y}{\delta x} = \frac{\log_a\left(1+\frac{\delta x}{x}\right)}{\delta x} = \frac{1}{x}\,\frac{x}{\delta x}\log_a\left(1+\frac{\delta x}{x)}\right)$$

$$= \frac{1}{x}\,\log_a\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}} \qquad [\because n\log m = \log m^n]$$

Step IV Proceeding limit as $\delta x \to 0$, we get

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{1}{x}\log_a\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

Step V $\frac{dy}{dx} = \frac{1}{x}\lim_{\frac{\delta x}{x}\to 0}\log_a\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$ as $\delta x \to 0 \Rightarrow \frac{\delta x}{x} \to 0$

$$= \frac{1}{x}\log_a\left[\lim_{\frac{\delta x}{x}\to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right] \qquad \left[\because \text{logarithm is a continuous function on its domain}\right]$$

$$= \frac{1}{x}\log_a e \qquad \left[\because \lim_{x\to 0}(1+x)^{1/x} = e\right]$$

If a function is **continuous** then it **respects limit** i.e. if a function f is continuous and a is point of its domain, then $\lim_{x\to a} f(x) = f(\lim_{x\to a} x) = f(a)$ i.e. limit can be taken inside the function.
i.e. Role of limit and function can be interchanged.

$$\boxed{\text{i.e. } \frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e}$$

In particular, if base of the logarithmic is e in place of a, then

$$\boxed{\frac{d}{dx}(\log_e x) = \frac{1}{x}\log_e e = \frac{1}{x}} \qquad \text{as } \log_e e = 1$$

Remark 3: Keep all these formulae put in the rectangular boxes always in mind, as we will use these formulae later on as standard results.

Some more Formulae of Finding Derivatives:

If u and v are functions of x, then

(i) $\frac{d}{dx}(cu) = c\frac{du}{dx}$, where c is a real constant

(ii) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

(iii) $\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$ (Known as **Product Rule**)

(iv) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (Known as **Quotient Rule**)

Remark 4: Aim of this unit from learners point of view is not to focus on the derivations of the formulae. But main aim of this unit is able to make the learners user friendly as to how these results can be used whenever we encounter a situation where derivative is involved. That is why we will not provide the derivations of more formulae.

Various formulae which are used in many practical situations are listed below:

S. No	Function	Derivative of the function
1	k (constant)	$\frac{d}{dx}(k) = 0$
2	x^n	$\frac{d}{dx}(x^n) = nx^{n-1}$
3	$(ax+b)^n$	$\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}$
4	Exponential function (i) a^{bx} (ii) e^{bx}	(i) $\frac{d}{dx}(a^{bx}) = ba^{bx}\log a$ (ii) $\frac{d}{dx}(e^{bx}) = be^{bx}$
5	Logarithmic function (i) $\log_a x$ (ii) $\log_e x$	(i) $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$ (ii) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$
6	cu, where c is constant and u is a function of x	$\frac{d}{dx}(cu) = c\frac{d}{dx}(u)$
7	(i) $u \pm v$ (ii) uv (iii) $\frac{u}{v}$ where, u, v are functions of x.	(i) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$ (ii) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ (Product Rule) (iii) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (Quotient Rule)
8	$[f(x)]^n$, n is +ve or –ve real number	$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}\frac{d}{dx}(f(x))$
9	$\frac{1}{f(x)}$	$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{d}{dx}(f(x))^{-1} = \frac{-1}{[f(x)]^2}\frac{d}{dx}(f(x))$
10	y = f(u) u = g(w) w = h(x)	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dw}\frac{dw}{dx}$ (Chain Rule)
11	Parametric functions x = f(t) y = g(t)	$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$
12	Polynomial function $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$	$\frac{d}{dx}(f(x)) = na_0x^{n-1} + (n-1)a_1x^{n-2} + \ldots + a_{n-1}$

Now we take some examples. We will write "Diff. w.r.t x" in place of "differentiating with respect to x".

Example 3: Find derivative of the following functions:

(i) 5 (ii) $\alpha+\beta$ (iii) $\frac{1}{\sqrt{11}}$ (iv) $\frac{-\pi}{\sqrt{17}}$ (v) x^{11}

(vi) $x^{5/2}$ (vii) $\frac{1}{\sqrt[3]{x}}$ (viii) $\frac{1}{x^7}$ (ix) $\frac{1}{x}$ (x) $\sqrt{x}$

(xi) $(2x+5)^3$ (xii) $(4-3x)^8$ (xiii) $\left(5-\frac{3}{2}x\right)^{4/9}$ (xiv) $\sqrt{2+3x}$

(xv) x^5+x^2+1 (xvi) $\frac{x^3+x^2+1}{x}$ (xvii) $\left(x^2+\frac{1}{x^2}\right)^2$ (xviii) $\left(x+\frac{1}{x}\right)^{100}$

(xix) $(x^2+1)(x-1)$ (xx) $x^{-3}(1+x^2+x^5+x^8)$

(xxi) $(x^2+1)(x^3+x^2+1)$ (xxii) $(4x+1)^3(7x+1)^4$

(xxiii) $(x+2)^2(x+3)^4(x+1)^5$

Solution:

(i) Let $y = 5$

Diff. w.r.t. x

$$\frac{dy}{dx} = 0 \qquad \left[\begin{array}{l}\because 5 \text{ is a constant and derivative}\\ \text{of a constant function is zero.}\end{array}\right]$$

(ii) Let $y = \alpha+\beta$

Diff. w.r.t. x

$$\frac{dy}{dx} = 0 \qquad \left[\begin{array}{l}\because \alpha \text{ and } \beta \text{ both are constants} \Rightarrow \alpha+\beta \text{ is constant}\\ \text{and derivative of a constant function is zero.}\end{array}\right]$$

(iii) Let $y = \frac{1}{\sqrt{11}}$

Diff. w.r.t. x

$$\frac{dy}{dx} = 0 \qquad \left[\begin{array}{l}\because \frac{1}{\sqrt{\pi}} \text{ is a constant and derivative}\\ \text{of a constant function is zero.}\end{array}\right]$$

(iv) Let $y = \frac{-\pi}{\sqrt{17}}$

Diff. w.r.t. x

$$\frac{dy}{dx} = 0 \qquad \left[\begin{array}{l}\because \frac{-\pi}{\sqrt{17}} \text{ is a constant and derivative}\\ \text{of a constant function is zero.}\end{array}\right]$$

(v) Let $y = x^{11}$

Diff. w.r.t. x

$$\frac{dy}{dx} = 11x^{11-1} = 11x^{10} \qquad \left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$$

(vi) Let $y = x^{5/2}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{5}{2}x^{\frac{5}{2}-1} = \frac{5}{2}x^{\frac{3}{2}} \qquad \left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$$

(vii) Let $y = \frac{1}{\sqrt[3]{x}} = \frac{1}{x^{1/3}} = x^{-1/3}$

Diff. w.r.t. x

$\frac{dy}{dx} = -\frac{1}{3}x^{-4/3}$ $\left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$

(viii) Let $y = \frac{1}{x^7} = x^{-7}$

Diff. w.r.t. x

$\frac{dy}{dx} = -7x^{-8} = -\frac{7}{x^8}$ $\left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$

(ix) Let $y = \frac{1}{x} = x^{-1}$

Diff. w.r.t. x

$\frac{dy}{dx} = (-1)x^{-2} = -\frac{1}{x^2}$ $\left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$

(x) Let $y = \sqrt{x} = x^{1/2}$

Diff. w.r.t.x

$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ $\left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$

(xi) Let $y = (2x+5)^3$

Diff. w.r.t. x

$\frac{dy}{dx} = 3(2x+5)^2(2) = 6(2x+5)^2$ $\left[\because \frac{d}{dx}\left((ax+b)^n\right) = n(ax+b)^{n-1}a\right]$

(xii) Let $y = (4-3x)^8$

Diff. w.r.t. x

$\frac{dy}{dx} = 8(4-3x)^7(-3) = -24(4-3x)^7$ $\left[\because \frac{d}{dx}\left((ax+b)^n\right) = n(ax+b)^{n-1}a\right]$

(xiii) Let $y = \left(5-\frac{3}{2}x\right)^{4/9}$

Diff. w.r.t. x

$\frac{dy}{dx} = \frac{4}{9}\left(5-\frac{3}{2}x\right)^{-5/9}\left(-\frac{3}{2}\right)$ $\left[\because \frac{d}{dx}\left((ax+b)^n\right) = n(ax+b)^{n-1}a\right]$

$= -\frac{2}{3}\left(5-\frac{3}{2}x\right)^{-5/9}$

(xiv) Let $y = \sqrt{2+3x} = (2+3x)^{1/2}$

Diff. w.r.t. x

$\frac{dy}{dx} = \frac{1}{2}(2+3x)^{-1/2}(3)$ $\left[\because \frac{d}{dx}\left((ax+b)^n\right) = n(ax+b)^{n-1}a\right]$

$= \frac{3}{2\sqrt{2+3x}}$

(xv) Let $y = x^5 + x^2 + 1$

Diff. w.r.t. x

$$\frac{dy}{dx} = 5x^4 + 2x + 0 = 5x^4 + 2x \quad \left[\begin{array}{l}\text{Using formula written at serial} \\ \text{number 12 of the table of formulae}\end{array}\right]$$

(xvi) Let $y = \frac{x^3 + x^2 + 1}{x} = \frac{x^3}{x} + \frac{x^2}{x} + \frac{1}{x} = x^2 + x + \frac{1}{x}$

$$y = x^2 + x + \frac{1}{x}$$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{1}{x}\right) = 2x + 1 - \frac{1}{x^2} \quad \left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$$

(xvii) Let $y = \left(x^2 + \frac{1}{x^2}\right)^2$

Diff. w.r.t. x

$$\frac{dy}{dx} = 2\left(x^2 + \frac{1}{x^2}\right)\frac{d}{dx}\left(x^2 + \frac{1}{x^2}\right) \quad \left[\because \frac{d}{dx}\left((f(x))^n\right) = n(f(x))^{n-1}\frac{d}{dx}(f(x))\right]$$

$$= 2\left(x^2 + \frac{1}{x^2}\right)\left(2x - \frac{2}{x^3}\right) = 4\left(x^3 - \frac{1}{x} + \frac{1}{x} - \frac{1}{x^5}\right) = 4\left(x^3 - \frac{1}{x^5}\right)$$

(xviii) Let $y = \left(x + \frac{1}{x}\right)^{100}$

Diff. w.r.t. x

$$\frac{dy}{dx} = 100\left(x + \frac{1}{x}\right)^{99} \frac{d}{dx}\left(x + \frac{1}{x}\right) \quad \text{[Same reason as given in (xvii)]}$$

$$= 100\left(x + \frac{1}{x}\right)^{99}\left(1 - \frac{1}{x^2}\right)$$

(xix) Let $y = (x^2 + 1)(x - 1) = x^3 - x^2 + x - 1$

Diff. w.r.t. x

$$\frac{dy}{dx} = 3x^2 - 2x + 1 - 0 \quad \left[\begin{array}{l}\text{Using formula written at serial} \\ \text{number 12 of the table of formulae.}\end{array}\right]$$

$$= 3x^2 - 2x + 1$$

Alternatively: Using Product Rule

$$\frac{dy}{dx} = (x^2 + 1)\frac{d}{dx}(x - 1) + (x - 1)\frac{d}{dx}(x^2 + 1)$$

$$= (x^2 + 1)(1 - 0) + (x - 1)(2x + 0)$$

$$= x^2 + 1 + 2x^2 - 2x = 3x^2 - 2x + 1$$

(xx) Let $y = x^{-3}(1 + x^2 + x^5 + x^8)$

$$y = x^{-3} + x^{-1} + x^2 + x^5$$

Diff. w.r.t. x

$$\frac{dy}{dx} = -3x^{-4} - 1x^{-2} + 2x + 5x^4 \quad \left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$$

$$= -3x^{-4} - x^{-2} + 2x + 5x^4$$

(xxi) Let $y = (x^2 + 1)(x^3 + x^2 + 1)$

Diff. w.r.t. x [Using Product Rule]

$$\frac{dy}{dx} = (x^2+1)\frac{d}{dx}(x^3+x^2+1) + (x^3+x^2+1)\frac{d}{dx}(x^2+1)$$
$$= (x^2+1)(3x^2+2x) + (x^3+x^2+1)(2x)$$
$$= 3x^4+2x^3+3x^2+2x+2x^4+2x^3+2x$$
$$= 5x^4+4x^3+3x^2+4x$$

(xxii) Let $y = (4x+1)^3(7x+1)^4$

Diff. w.r.t. x [Using Product rule]

$$\frac{dy}{dx} = (4x+1)^3\frac{d}{dx}(7x+1)^4 + (7x+1)^4\frac{d}{dx}(4x+1)^3$$
$$= (4x+1)^3 4(7x+1)^3 7 + (7x+1)^4 3(4x+1)^2 4$$
$$= 4(4x+1)^2)(7x+1)^3[7(4x+1)+3(7x+1)]$$
$$= 4(4x+1)^2(7x+1)^3(49x+10)$$

(xxiii) Let $y = (x+2)^2(x+3)^4(x+1)^5$

Diff. w.r.t. x

$$\frac{dy}{dx} = (x+2)^2(x+3)^4\frac{d}{dx}(x+1)^5 + (x+2)^2(x+1)^5\frac{d}{dx}(x+3)^4 + (x+3)^4(x+1)^5\frac{d}{dx}(x+2)^2$$

$$\left[\begin{array}{l}\because \text{ if } u, v, w, \text{ are functions of } x, \text{ then} \\ \frac{d}{dx}(uvw) = uv\frac{d}{dx}(w) + uw\frac{d}{dx}(v) + vw\frac{d}{dx}(u)\end{array}\right]$$

$$= (x+2)^2(x+3)^4 5(x+1)^4 + (x+2)^2(x+1)^5 4(x+3)^3 + (x+3)^4(x+1)^5 2(x+2)$$
$$= (x+2)(x+3)^3(x+1)^4[5(x+2)(x+3)+4(x+2)(x+1)+2(x+3)(x+1)]$$
$$= (x+2)(x+3)^3(x+1)^4[5(x^2+5x+6)+4(x^2+3x+2)+2(x^2+4x+3)]$$
$$= (x+2)(x+3)^2(x+1)^4(11x^2+45x+44)$$

Example 4: Find the derivative of the following functions:

(i) $\frac{x+1}{x-1}$ (ii) $\frac{8x+3}{6-5x}$ (iii) $\frac{a^2}{x^2+a^2}$ (iv) $\sqrt{\frac{x^2+1}{x+1}}$

Solution:

(i) Let $y = \frac{x+1}{x-1}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{(x-1)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x-1)}{(x-1)^2} \quad \text{[Using Quotient Rule]}$$

$$= \frac{(x-1).1-(x+1).1}{(x-1)^2} = \frac{x-1-x-1}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

(ii) Let $y = \frac{8x+3}{6-5x}$

Diff. w.r.t. x

$$\frac{dy}{dx}=\frac{(6-5x)\frac{d}{dx}(8x+3)-(8x+3)\frac{d}{dx}(6-5x)}{(6-5x)^2} \quad \text{[Using Quotient Rule]}$$

$$=\frac{(6-5x).8-(8x+3)(-5)}{(6-5x)^2}=\frac{63}{(6-5x)^2}$$

(iii) Let $y=\dfrac{a^2}{x^2+a^2}=a^2(x^2+a^2)^{-1}$

Diff. w.r.t. x

$$\frac{dy}{dx}=a^2(-1)(x^2+a^2)^{-2}\frac{d}{dx}(x^2+a^2)$$

$$\left[\begin{array}{l}\text{Don't use quoient rule here because}\\ \text{there is no function of } x \text{ in numerator.}\end{array}\right]$$

$$=-\frac{a^2}{(x^2+a^2)^2}.2x=\frac{-2a^2x}{(x^2+a^2)^2}$$

(iv) Let $y=\sqrt{\dfrac{x^2+1}{x+1}}$

Diff. w.r.t. x

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\frac{x^2+1}{x+1}}}\frac{d}{dx}\left(\frac{x^2+1}{x+1}\right) \quad \left[\because \frac{d}{dx}\left((f(x))^n\right)=n(f(x))^{n-1}\frac{d}{dx}(f(x))\right]$$

$$=\frac{1}{2}\sqrt{\frac{x+1}{x^2+1}}\left[\frac{(x+1)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x+1)}{(x+1)^2}\right] \text{[Using Quotient Rule]}$$

$$=\frac{(x+1)(2x-1)-(x^2+1)(1)}{2\sqrt{x^2+1}(x+1)^{3/2}}=\frac{2x^2+2x-x^2-1}{2\sqrt{x^2+1}\,(x+1)^{3/2}}$$

$$=\frac{x^2+2x-1}{2(x+1)^{3/2}\times\sqrt{x^2+1}}=\frac{(x-1)^2}{2(x+1)^{3/2}\sqrt{x^2+1}}$$

Here, are some exercises for you.

E 3) Differentiate the following functions w.r.t. x

(i) πe (ii) $\dfrac{1}{x^7}$ (iii) $\sqrt{x}$ (iv) $(4-3x)^8$ (v) $\left(x^3+\dfrac{1}{x^3}\right)^2$

E 4) Find the derivative of the following functions:

(i) $\left(x+\dfrac{1}{x}\right)\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)$ (ii) $\left(x^3+\dfrac{1}{x^3}\right)\left(x+\dfrac{1}{x}\right)$ (iii) $\dfrac{x^2}{x^3+1}$ (iv) $\dfrac{x^2+x}{a}$

6.5 CHAIN RULE

Sometimes variables y and x are connected by the relations of the form
$y = f(u)$, $u = g(w)$, $w = h(x)$
and we want to differentiate y w.r.t. x. then chain rule is used, which gives

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dw}\,\frac{dw}{dx}$$

Following example will illustrate the rule more clearly.

Example 5: Find $\frac{dy}{dx}$ in the following cases.

(i) $y = 3u, \quad u = v^2, \quad v = 4x^2 + 5$

(ii) $y = u^2, \quad u = 3v, \quad v = \frac{x}{x+1}$

Solution:

(i) $y = 3u$, $\quad u = v^2$, $\quad v = 4x^2 + 5$

Diff. w.r.t. u $\quad$ Diff. w.r.t. v $\quad$ Diff. w.r.t. x

$\frac{dy}{du} = 3 \quad \frac{du}{dv} = 2v \quad \frac{dv}{dx} = 8x$

$\therefore$ by chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dv}\,\frac{dv}{dx} = 3(2v)(8x)$$

$= 48xv = 48x(4x^2 + 5)$ [Replacing the value of v in terms of x]

(ii) $y = u^2$, $\quad u = 3v$, $\quad v = \frac{x}{x+1}$

Diff. w.r.t. u $\quad$ Diff. w.r.t. v $\quad$ Diff. w.r.t. x

$\frac{dy}{du} = 2u \quad \frac{du}{dv} = 3 \quad \frac{dv}{dx} = \frac{(x+1).1 - x.(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$

$\therefore$ by chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dv}\,\frac{dv}{dx} = (2u)(3)\left(\frac{1}{(1+x)^2}\right)$$

$$= \frac{6u}{(1+x)^2} = \frac{6(3v)}{(1+x)^2} = \frac{18v}{(1+x)^2} = \frac{18x}{(1+x)^3}$$

6.6 DERIVATIVES OF EXPONENTIAL, LOGRITHMIC, PARAMETRIC AND IMPLICIT FUNCTIONS

Let us first take up some examples on derivatives of exponential and logarithmic functions as given in example 12.

Example 6: Find the derivative of the following functions:

(i) 2^x $\quad$ (ii) 5^{ax} $\quad$ (iii) e^{3x} $\quad$ (iv) e^{9x^2} $\quad$ (v) $\log(1 + x^2)$

(vi) $\log_x x^2$ $\quad$ (vii) $\log_2 x$ $\quad$ (viii) $\log_x 2$ $\quad$ (ix) $\frac{1}{\log(1+x)}$

Solution:

(i) Let $y = 2^x$

Diff. w.r.t. x

$$\frac{dy}{dx} = 2^x \log 2 \qquad \left[\because \frac{d}{dx}(a^x) = a^x \log a\right]$$

(ii) Let $y = 5^{ax}$

Diff. w.r.t. x

$$\frac{dy}{dx} = 5^{ax} \log 5 \frac{d}{dx}(ax) \qquad \left[\because \frac{d}{dx}(a^{bx}) = ba^{bx} \log a\right]$$

$$= 5^{ax}\ a \log 5$$

(iii) Let $y = e^{3x}$

Diff. w.r.t. x

$$\frac{dy}{dx} = e^{3x}(3) = 3e^{3x} \qquad \left[\because \frac{d}{dx}(e^{ax}) = ae^{ax}\right]$$

(iv) Let $y = e^{9x^2}$

Diff. w.r.t. x

$$\frac{dy}{dx} = e^{9x^2} \frac{d}{dx}(9x^2) = e^{9x^2}(18x) \qquad \left[\text{Using } \frac{d}{dx}(e^{f(x)}) = e^{f(x)} \frac{d}{dx}(f(x))\right]$$

$$= 18x\, e^{9x^2}$$

(v) Let $y = \log(1 + x^2)$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{1}{1+x^2} \frac{d}{dx}(1+x^2) = \frac{2x}{1+x^2} \qquad \left[\because \frac{d}{dx}(\log f(x)) = \frac{1}{f(x)} \frac{d}{dx}(f(x))\right]$$

(vi) Let $y = \log_x x^2 = \dfrac{\log x^2}{\log x}$ [Using base change formula]

$$= \frac{2 \log_x}{\log_x} = 2 \qquad [\because \log m^n = n \log m]$$

Diff. w.r.t. x

$$\frac{dy}{dx} = 0$$

(vii) Let $y = \log_2 x$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{1}{x} \log_2 e \qquad \left[\because \frac{d}{dx}(\log_a x) = \frac{1}{x} \log_a e\right]$$

(viii) Let $y = \log_x 2 = \dfrac{\log 2}{\log x}$ [Using base change formula]

$$y = (\log 2)(\log x)^{-1}$$

Diff. w.r.t. x

$$\frac{dy}{dx} = -(\log 2)(\log x)^{-2} \frac{d}{dx}(\log x) = -\frac{(\log 2)}{(\log x)^2} \times \frac{1}{x} = -\frac{\log 2}{x(\log x)^2}$$

(ix) Let $y = \dfrac{1}{\log(1+x)}$ $y = [\log(1+x)]^{-1}$

Diff. w.r.t. x

$$\frac{dy}{dx} = -1[\log(1+x)]^{-2} \frac{d}{dx}(\log(1+x))$$

$$= \frac{-1}{[\log(1+x)]^2} \times \frac{1}{1+x} = -\frac{1}{(1+x)[\log(1+x)]^2}$$

Here is an exercise for you.

E 5) Find the derivative of the following functions:

(i) $a^{\log_a 2^x}$ (ii) $3^{\log_3 x^2}$

Following is an example on derivatives of parametric functions.

Example 7: Find $\frac{dy}{dx}$ for the following parametric functions:

(i) $x = 1 + t,\ y = 2 + t^2$ (ii) $x = \frac{1+3t^2}{1-t},\quad y = \frac{1+t^2}{1-t}$

Solution:

(i) $x = 1 + t$, $y = 2 + t^2$

Diff. w.r.t. t Diff. w.r.t. t

$\frac{dx}{dy} = 1$, $\frac{dy}{dt} = 2t$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{2t}{1} = 2t$$

(ii) $x = \frac{1+3t^2}{1-t}$

Diff. w.r.t. t

$$\frac{dx}{dt} = \frac{(1-t)\frac{d}{dt}(1+3t^2) - (1+3t^2)\frac{d}{dt}(1-t)}{(1-t)^2}$$

$$= \frac{(1-t)(6t) - (1+3t^2)(-1)}{(1-6^2} = \frac{6t - 6t^2 + 1 + 3t^2}{(1-t)^2} = \frac{1 + 6t - 3t^2}{(1-t)^2}$$

Now, $y = \frac{1+t^2}{1-t}$

Diff. w.r.t. t

$$\frac{dy}{dt} = \frac{(1-t)\frac{d}{dt}(1+t^2) - (1+t^2)\frac{d}{dt}(1-t)}{(1-t)^2} = \frac{(1-t)(2t) - (1+t^2)(-1)}{(1-t)^2}$$

$$= \frac{2t - 2t^2 + 1 + t^2}{(1-t)^2} = \frac{1 + 2t - t^2}{(1-t)^2}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{1 + 2t - t^2}{1 + 6t - 3t^2}$$

Now, you can try the following the exercise.

E 6) Find $\frac{dy}{dx}$ if $x = 2 + 4t^2,\ y = 9t^2 + 3t + 1$.

Implicit Function

A function defined by y = f(x) is known as explicit function. But sometimes y cannot be easily expressed in terms of x. A function of the form

$f(x, y) = c$, where c is a constant

is known as implicit function.

Procedure

In case of implicit function, differentiate the given relation w.r.t. x and collect all the terms of $\frac{dy}{dx}$ to the left hand side and finally dividing both sides by a term attached with $\frac{dy}{dx}$, we get the value of $\frac{dy}{dx}$.

Following example will explain the procedure more clearly:

Example 8: Find $\frac{dy}{dx}$ in the following cases:

(i) $x^2 + y^2 = c^2$ (ii) $(x-a)^2 + (y-b)^2 = r^2$ (iii) $x^3 + y^3 + xy = 5$

Solution:

(i) $x^2 + y^2 = c^2$

Diff. w.r.t. x

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

(ii) $(x-a)^2 + (y-b)^2 = r^2$

Diff. w.r.t. x

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x-a}{y-b}$$

(iii) $x^3 + y^3 + xy = 5$

Diff. w.r.t. x

$$3x^2 + 3y^2\frac{dy}{dx} + x\frac{dy}{dx} + y.1 = 0$$ [Using Product Rule in the term xy]

$$\Rightarrow (3y^2 + x)\frac{dy}{dx} = -(y + 3x^2)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{3x^2 + y}{x + 3y^2}$$

Now, you can try the following exercise.

E 7) Find $\frac{dy}{dx}$, if $xy^3 + xe^x + xe^{-y} = 3$

6.7 DERIVATIVE OF HIGHER ORDERS

Sometimes we are to differentiate the function more than once.

Derivative of y w.r.t x twice is denoted by $\frac{d^2y}{dx^2}$,

Derivative of y w.r.t x thrice is denoted by $\frac{d^3y}{dx^3}$,

…
…
…

n times differentiation of y w.r.t x is denoted by $\frac{d^ny}{dx^n}$.

Following example illustrate the idea of higher order derivatives.

Example 9: Find the indicated derivatives for the following functions:

(i) $\frac{d^4y}{dx^4}$ for $y = \frac{1}{x}$ (ii) $\frac{d^3y}{dx^3}$ for $y = e^{ax}$

(iii) $\frac{d^3y}{dx^3}$ for $y = (ax+b)^7$ (iv) $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$ for $y = x^2 + x + 1$

Solution:

(i) $y = \frac{1}{x}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{-1}{x^2}$$

Again diff. w.r.t. x

$$\frac{d^2y}{dx^2} = \frac{(-1)(-2)}{x^3} = \frac{2}{x^3}$$

Again diff. w.r.t. x

$$\frac{d^3y}{dx^3} = \frac{2(-3)}{x^4} = -\frac{6}{x^4}$$

Again diff. w.r.t. x

$$\frac{d^4y}{dx^4} = \frac{-6(-4)}{x^5} = \frac{24}{x^5}$$

(ii) $y = e^{ax}$

Diff. w.r.t. x

$$\frac{dy}{dx} = e^{ax}\frac{d}{dx}(ax) \qquad \left[\because \frac{d}{dx}(e^{f(x)}) = e^{f(x)}\frac{d}{dx}(f(x))\right]$$

$$= e^{ax}(a) = ae^{ax}$$

Again diff. w.r.t. x

$$\frac{d^2y}{dx^2} = a^2e^{ax}$$

Again diff. w.r.t. x

$$\frac{d^3y}{dx^3} = a^3e^{ax}$$

(iii) $y = (ax+b)^7$

Diff. w.r.t. x

$$\frac{dy}{dx} = 7(ax+b)^6 a \qquad \left[\because \frac{d}{dx}\left((ax+b)^n\right) = n(ax+b)^{n-1}a\right]$$

Again diff. w.r.t .x

$$\frac{d^2y}{dx^2} = 42(ax+b)^5 a^2$$

Again diff. w.r.t. x

$$\frac{d^3y}{dx^3} = 210(ax+b)^4 a^3$$

(iv) $y = x^2 + x + 1$

Diff. w.r.t. x

$$\frac{dy}{dx} = 2x + 1 \qquad \begin{bmatrix} \text{Using formula written at serial} \\ \text{number 12 of the table of formulae} \end{bmatrix}$$

Again diff. w.r.t. x

$$\frac{d^2y}{dx^2} = 2$$

Again diff. w.r.t. x

$$\frac{d^3y}{dx^3} = 0$$

Again diff. w.r.t. x

$$\frac{d^4y}{dx^4} = 0$$

6.8 CONCEPT OF MAXIMA AND MINIMA

Differentiation has a large number of applications in different fields such as mathematics, statistics, economics, actuarial science, etc. Concept of differentiation is also useful to obtain maxima or minima point(s) and their corresponding value(s) of a given function. Actually sometimes we are interested only to find maximum or minimum value(s) of a function. The aim of this section is to meet this interest. Without going into theoretical details, let us discuss the concept geometrically.

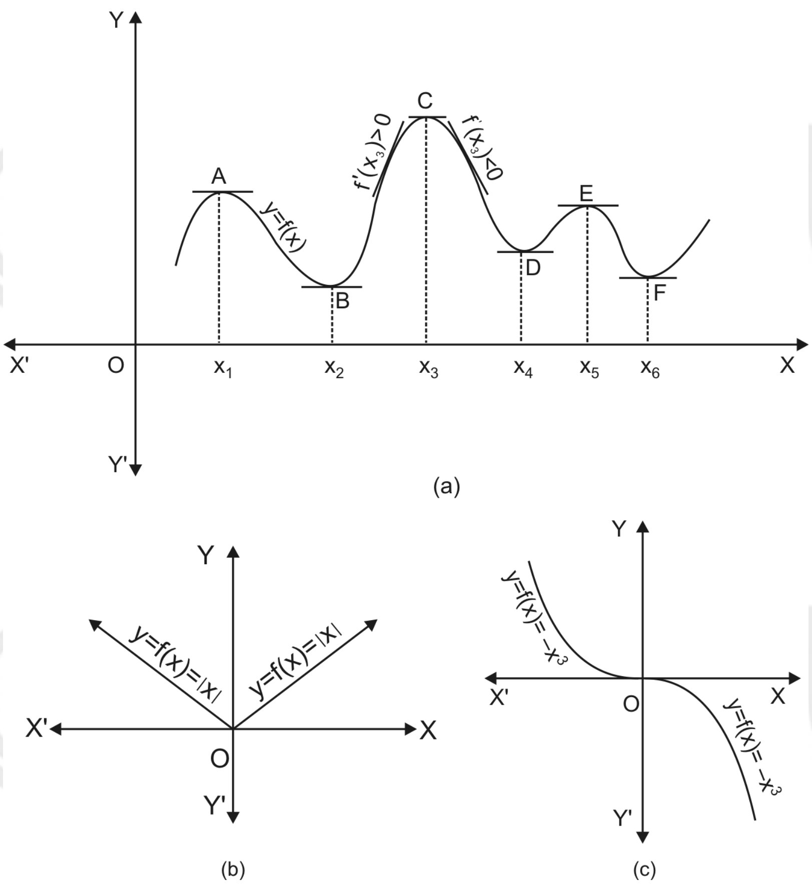

Fig. 6.3

In equation (2) of Sec. 6.2 of this unit we have seen that

$$(\text{Derivative at a point})_{x=a} \text{ is the slope of the tangent at that point.} \quad \ldots (1)$$

In Fig 6.3 a we see that x_1 is a point where the function y = f(x) takes maximum value (local) compare to all points which are very close to x_1

i.e. $f(x_1) > f(x)$ … (2)

for all points x which are very close to x_1

The point x_1 is known as local maxima point of the function y = f(x) (local maxima means that it satisfy the equation (2) as given above, i.e. there may be points in the domain of the function where value of the function f is greater than $f(x_1)$)

Similarly, points x_3 and x_5 in the same figure are points of maxima (local). On the other hand in the same figure x_2 is a point where the function y = f(x) takes minimum value (local) compare to all points which are very close to x_2

i.e. $f(x_2) < f(x)$

for all points x which are very close to x_2

The point x_2 is known as local minima point of the function y = f(x).
Similarly x_4 and x_6 are points of local minima.

But one interesting point to be noted here is that tangent lines at all the points whether it is point of maxima or minima is parallel to x-axis.
i.e. slope of the tangent at point of maxima or minima = tan0 = 0 … (3)

$$\left[\begin{array}{l}\because \text{ if a line makes an angle } \theta \text{ with } +\text{ve direction of} \\ x-\text{axis then slope of the line is defined as } \tan\theta, \text{ in} \\ \text{this case } \theta = 0 \text{ as tangent line is parallel to } x-\text{axis.}\end{array}\right]$$

In view of equation (1) and (3), we have

Derivative of the function at a point of maxima or minima = 0 … (4)
, provided that derivative at that point exists

Also note that converse of (4) does not hold.
For example, take graph of the function $y = f(x) = -x^3$ shown in Fig. 6.3 c.
we see that

$$\left.\frac{dy}{dx}\right|_{x=0} = f'(0) = -3(0)^2 = 0 \qquad \left[\because \frac{dy}{dx} = f'(x) = -3x^2\right]$$

But x = 0 is neither point of maxima nor minima.
Now consider the function $y = f(x) = |x|$, whose graph is shown in Fig. 6.3 b.
We see that x = 0 is a point of minima, in fact absolute minima or global minima (absolute minima or global minima means that function assumes smallest value at x = 0 in whole domain of the function not only at those points which are very close to x = 0)

But we have seen in part (i) of Example 2 of this unit that derivative of the function $y = f(x) = |x|$ does not exists at x = 0.

Equation (4) implies points obtained by putting first derivative equal to zero may be points of maxima or minima. Second derivative test differentiate between points of maxima and minima which is stated below:

Second Derivative Test: It states that if the function f is twice differentiable at a point 'c', where c is point of the domain of the function f, then

(i) c is point of local minima if $f'(c) = 0$ and $f''(c) > 0$.

(ii) c is point of local maxima if $f'(c) = 0$ and $f''(c) < 0$.

(iii) test fails if $f''(c) = 0$. In this case we use first order derivative test, which can be concluded as:

- if $f'(c)$ changes its sign from positive to negative as we cross the point x = c, then x = c is point of maxima (see Fig. 6.3 a at point x_3)
- if $f'(c)$ changes its sign from negative to positive as we cross the point x = c, then x = c is point of minima. This can be noted at points x_2, x_4, x_6) in Fig. 6.3 a.
- if $f'(c)$ does not change its sign as we cross the point x = c, then x = c is neither point of minima nor maxima. See Fig. 6.3 c in which x = 0 is such a point. Point of this nature is called point of inflection. Normal curve has two such points at $x = \mu + \sigma$ and $x = \mu - \sigma$. You can observe it by differentiating normal density twice and putting double derivative equal to zero. Normal distribution is discussed in Unit 13 and Unit 14 of MST-003.

With the following two examples followed by an exercise, let us close this Sec.

Example 10: Find local maximum and minimum values of the function $f(x) = 2x^3 - 15x^2 + 36x + 9$.

Solution: Given function is

$f(x) = 2x^3 - 15x^2 + 36x + 9$

Dif. w.r.t.x

$f'(x) = 6x^2 - 30x + 36 \qquad \ldots (1)$

For maxima or minima

$f'(x) = 0$

$\Rightarrow 6x^2 - 30x + 36 = 0$

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow (x-2)(x-3) = 0$

$\Rightarrow x = 2, 3$

Diff. (1) w.r.t. x

$f''(x) = 12x - 30$

At x = 2, $f''(2) = 24 - 30 = -6 < 0$

$\therefore$ by second order derivative test, x = 2 is point of maxima and maximum value is given by

$f(2) = 2(2)^3 - 15(2)^2 + 36(2) + 9 = 16 - 60 + 72 + 9 = 27$

At x = 3, $f''(3) = 36 - 30 = 6 > 0$

$\therefore$ by second order derivative test, x = 3 is point of minima and minimum value of the function is given by

$$f(3) = 2(3)^3 - 15(3)^2 + 36(3) + 9 = 54 - 135 + 108 + 9 = 36$$

Example 11: Find local maximum and minimum values of the function $f(x) = \frac{x^4}{4} + \frac{x^3}{3} - 2x^2 - 4x + 5.$

Solution: Given function is

$$f(x) = \frac{x^4}{4} + \frac{x^3}{3} - 2x^2 - 4x + 5$$

Diff. w.r.t. x

$$f'(x) = \frac{4x^3}{4} + \frac{3x^2}{3} - 4x - 4 = x^3 + x^2 - 4x - 4 \qquad \dots (1)$$

For maxima or minima

$$f'(x) = 0$$

$$\Rightarrow x^3 + x^2 - 4x - 4 = 0 \qquad \dots (2)$$

By inspection x = –1 is a root of equation (2)

$\Rightarrow (x+1)$ is a factor of $x^3 + x^2 - 4x - 4$

$\therefore$ (2) can be written as

$$(x+1)(x^2 - 4) = 0$$

$$\Rightarrow (x+1)(x-2)(x+2) = 0$$

$$\Rightarrow x = -1, 2, -2$$

Diff. (1) w.r.t. x

$$f''(x) = 3x^2 + 2x - 4$$

At x = –1, $f''(-1) = 3(-1)^2 + 2(-1) - 4 = 3 - 2 - 4 = -3 < 0$

At x = 2, $f''(2) = 3(2)^2 + 2(2) - 4 = 12 > 0$ and

At x = –2, $f''(-2) = 3(-2)^2 + 2(-2) - 4 = 4 > 0$

$\therefore$ by second order derivative test x = 2, –2 are points of minima and x = –1 is point of maxima.

Local minimum value at x = 2 is f(2) = $\frac{(2)^4}{4} + \frac{(2)^3}{3} - 2(2)^2 - 4(2) + 5 = -\frac{13}{3}$ and

Local minimum value at x = –2 is given by

$$f(-2) = \frac{(-2)^4}{4} + \frac{(-2)^3}{3} - 2(-2)^2 - 4(-2) + 5 = \frac{19}{3}$$

Local maximum value at x = – 1 is given by

$$f(-1) = \frac{(-1)^4}{4} + \frac{(-1)^3}{3} - 2(-1)^2 - 4(-1) + 5 = \frac{83}{12}$$

Now, you can try the following exercise.

E 8) Find local maximum and minimum values of the function $f(x) = 4x^3 - 21x^2 + 18x + 9.$

6.9 SUMMARY

Let us summarise the topics that we have covered in this unit:

1) Derivative at a point.
2) Derivatives of constant, polynomials and some others commonly used functions such as $x^n, (ax+b)^n$, etc. Product Rule and Quotient Rule.
3) Chain rule.
4) Derivatives of exponential, logarithmic, parametric and implicit functions.
5) Derivatives of higher orders.
6) Concept of maxima and minima.

6.10 SOLUTIONS/ANSWERS

E 1) (i) $f(x) = x^3 + x + 1, \quad \text{at } x = -1$

By definition

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h\to 0} \frac{(-1+h)^3 + (-1+h) + 1 - [(-1)^3 + (-1) + 1]}{h}$$

$$= \lim_{h\to 0} \frac{-1 + h^3 + 3h - 3h^2 - 1 + h + 1 - (-1 - 1 + 1)}{h}$$

$$= \lim_{h\to 0} \frac{h^3 - 3h^2 + 4h}{h} = \lim_{h\to 0}(h^2 - 3h + 4) = 0 - 0 + 4 = 4$$

(ii) By definition

$$f'\left(\frac{1}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{1}{2} + h\right) - f\left(\frac{1}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{2 - 3\left(\frac{1}{2} + h\right)^2 - \left[2 - 3\left(\frac{1}{2}\right)^2\right]}{h}$$

$$= \lim_{h\to 0} \frac{2 - 3\left(\frac{1}{4} + h + h^2\right) - \left(2 - \frac{3}{4}\right)}{h}$$

$$= \lim_{h\to 0} \frac{-3h - 3h^2}{h} = \lim_{h\to 0}(-3 - 3h) = -3 - 0 = -3$$

E 2) $f(x) = 2x^2 - 3ax + 5$

By definition

$$f'(-2) = \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h}$$

$$= \lim_{h\to 0} \frac{2(-2+h)^2 - 3a(-2+h) + 5 - [2(-2)^2 - 3a(-2) + 5]}{h}$$

$$= \lim_{h\to 0} \frac{2(4 + h^2 - 4h) + 6a - 3ah + 5 - (8 + 6a + 5)}{h}$$

$$= \lim_{h\to 0} \frac{2h^2 - 8h - 3ah}{h} = \lim_{h\to}(2h - 8 - 3a) = -8 - 3a$$

But according to problem

$$f'(-2) = 3 \Rightarrow -8 - 3a = 3 \Rightarrow -3a = 11 \Rightarrow a = -\frac{11}{3}$$

E 3) (i) Let $y = \pi e$

Diff. w.r.t. x

$$\frac{dy}{dx} = 0 \qquad \left[\begin{array}{l}\because \pi \text{ and } e \text{ both are constant} \Rightarrow \pi e \text{ is a constant} \\ \text{and derivative of a constant function is zero.}\end{array}\right]$$

(ii) Let $y = \frac{1}{x^7} = x^{-7}$

Diff. w.r.t. x

$$\frac{dy}{dx} = -7x^{-8} = -\frac{7}{x^8} \qquad \left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$$

(iii) Let $y = \sqrt{x} = x^{1/2}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \qquad \left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$$

(iv) Let $y = (4 - 3x)^8$

Diff. w.r.t. x

$$\frac{dy}{dx} = 8(4 - 3x)^7(-3) \qquad \left[\because \frac{d}{dx}\left((ax+b)^n\right) = n(ax+b)^{n-1}a\right]$$

$$= -24(4 - 3x)^7$$

(v) Let $y = \left(x^3 + \frac{1}{x^3}\right)^2$

Diff. w.r.t. x

$$\frac{dy}{dx} = 2\left(x^3 + \frac{1}{x^3}\right)\frac{d}{dx}\left(x^3 + \frac{1}{x^3}\right)$$

$$\left[\because \frac{d}{dx}\left((f(x))^n\right) = n(f(x))^{n-1}\frac{d}{dx}(f(x))\right]$$

$$= 2\left(x^3 + \frac{1}{x^3}\right)\left(3x^2 - \frac{3}{x^4}\right)$$

$$= 6\left(x^5 - \frac{1}{x} + \frac{1}{x} - \frac{1}{x^7}\right) = 4\left(x^5 - \frac{1}{x^7}\right)$$

E 4) (i) $y = \left(x + \frac{1}{x}\right)\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) = x^{3/2} + x^{1/2} + \frac{1}{x^{1/2}} + \frac{1}{x^{3/2}}$

$$y = x^{3/2} + x^{1/2} + x^{-1/2} + x^{-3/2}$$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2} + \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} - \frac{3}{2}x^{-5/2}$$

$$= \frac{3}{2}\sqrt{x} + \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}} - \frac{3}{2x^{5/2}}$$

(ii) Let $y = \left(x^3 + \frac{1}{x^3}\right)\left(x + \frac{1}{x}\right) = x^4 + x^2 + \frac{1}{x^2} + \frac{1}{x^4}$

Diff. w.r.t. x

$$\frac{dy}{dx} = 4x^3 + 2x - \frac{2}{x^3} - \frac{4}{x^5}$$

(iii) Let $y = \frac{x^2}{x^3+1}$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{(x^3+1)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(x^3+1)}{(x^3+1)^2} \quad \text{[Using Quotient Rule]}$$

$$= \frac{(x^3+1)(2x) - x^2(3x^2)}{(x^3+1)^2}$$

$$= \frac{2x^4 + 2x - 3x^4}{(x^3+1)^2} = \frac{2x - x^4}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}$$

(iv) Let $y = \frac{x^2+x}{a} = \frac{1}{a}(x^2+x)$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{1}{a}\frac{d}{dx}(x^2+x) = \frac{1}{a}(2x+1)\left[\begin{array}{l}\text{Do not use quoient rule}\\ \text{because in the denominator}\\ \text{there is no function of } x.\end{array}\right]$$

E5) (i) Let $y = a^{\log_a 2^x}$ $y = 2^x$ $[\because a^{\log_a f(x)} = f(x)]$

Diff. w.r.t. x

$$\frac{dy}{dx} = 2^x \log 2 \qquad \left[\because \frac{d}{dx}\left(a^x\right) = a^x \log a\right]$$

(ii) Let $y = 3^{\log_3 x^2} = x^2$ $[\because a^{\log_a f(x)} = f(x)]$

Diff. w.r.t. x

$$\frac{dy}{dx} = 2x \qquad \left[\because \frac{d}{dx}\left(x^n\right) = nx^{n-1}\right]$$

E 6) $x = 2 + 4t^2$, $y = 9t^2 + 3t + 1$

Diff. w.r.t. t Diff. w.r.t. t

$$\frac{dx}{dt} = 8t \qquad \frac{dy}{dt} = 18t + 3$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{18t+3}{8t}$$

E 7) $xy^3 + xe^x + xe^{-y} = 3$

Diff. w.r.t. x

$$x\left(3y^2\frac{dy}{dx}\right) + (1)y^3 + xe^x + (1)e^x + xe^{-y}\left(-\frac{dy}{dx}\right) + (1)e^{-y} = 0$$

$$\Rightarrow \left(3xy^2 - xe^{-y}\right)\frac{dy}{dx} = -y^3 - xe^x - e^x - e^{-y}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^3 + xe^x + e^x + +e^{-y}}{3xy^2 - xe^{-y}} = \frac{y^3 + xe^x + e^x + e^{-y}}{xe^{-y} - 3xy^2}$$

E 8) Given function is

$f(x) = 4x^3 - 21x^2 + 18x + 9$

Diff. w.r.t. x

$f'(x) = 12x^2 - 42x + 18$ … (1)

For maxima or manima

$f'(x) = 0$

$\Rightarrow 12x^2 - 42x + 18 = 0$

$\Rightarrow 2x^2 - 7x + 3 = 0$

$\Rightarrow 2x^2 - 6x - x + 3 = 0$

$\Rightarrow 2x(x-3) - 1(x-3) = 0$

$\Rightarrow (x-3)(2x-1) = 0$

$\Rightarrow x = 3, 1/2$

Diff. (1) w.r.t. x

$f''(x) = 24x - 42$

At $x = 3$, $f''(3) = 24\times 3 - 42 = 72 - 42 = 30 > 0$

At $x = 1/2$, $f''(1/2) = 24\times\frac{1}{2} - 42 = 12 - 42 = -30 < 0$

$\therefore$ by second order derivative test x = 3 is point of minima and x = 1/2is point of maxima.

Local minimum value at x = 3 is given by

$f(3) = 4(3)^3 - 21(3)^2 + 18(3) + 9 = 108 - 189 + 54 + 9 = -18$

Local maximum value at $x = 1/2$ is given by

$$f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 21\left(\frac{1}{2}\right)^2 + 18\left(\frac{1}{2}\right) + 9$$

$$= \frac{1}{2} - \frac{21}{4} + 9 + 9 = \frac{21 - 21 + 72}{4} = \frac{53}{4}$$