UNIT 5 LIMIT AND CONTINUITY

Structure

5.1 INTRODUCTION

In Unit 2 of this course, i.e. MST-001 we have discussed, in detail, the concept of functions and various types of functions. In that unit we have also obtained the value of the function at certain points. That is, the value of a function f(x) has been obtained at certain value of x in its domain.

Here, in this unit, we are going to introduce the concept of limit as well as continuity. That is, we are going to find the limiting value of the function f(x) when x approaches to certain value. That is, we are interested in finding that value to which f(x) approaches to as x approaches to the certain value. Also, this limiting value and the value of the function at certain value of x are compared to define continuity.

Objectives

After completing this unit, you should be able to:

- get an idea of limit;
- evaluate the limits of different functions;
- evaluate the infinite limit of some functions;
- check the continuity of a function at a point; and
- check the continuity of a function at a general point.

5.2 CONCEPT OF LIMIT

In Unit 2 of this course, we have discussed concept of function, consider a function

$y = f(x) = 3x + 2$

The following table shows the values of y for different values of x which are very close to 2.

x	1.9	1.98	1.998	1.9998	…	1.99999998	…
y = f(x)	7.7	7.94	7.994	7.9994	…	7.99999994	…

x	2.1	2.01	2.001	2.0001	…	2.00000001	…
y = f(x)	8.3	8.03	8.003	8.0003	…	8.00000003	…

We note that as x approaches to 2 either from left (means x comes nearer and nearer to 2 but remains < 2) or from right (means x comes nearer and nearer to 2 but remains > 2), then y = f(x) approaches to 8 in the same manner.
i.e. as $x \to 2$ then $f(x) \to 8$ and we write it as $\lim\limits_{x\to 2} f(x) = 8$.

In general if $f(x) \to l$ as $x \to a$ then we write it as $\lim\limits_{x\to a} f(x) = l$.

In this unit, we discuss how to evaluate $\lim\limits_{x\to a} f(x)$ in different situations. In this unit we shall also discuss the concept of infinite limit, some standard limits, left hand limit (L.H.L.) and right hand limit (R.H.L.). Finally we shall conclude the unit after introducing the concept of continuity.

5.3 DIRECT SUBSTITUTION METHOD

Suppose we want to evaluate $\lim\limits_{x\to a} f(x)$. This method is applied when limiting value of f(x) remains same irrespective of this fact whether x approaches to a from left hand side (L.H.S.) or right hand side (R.H.S.). As the name of this method itself suggests, in this method we directly substitute a in place of x.

Before we take some examples based on the direct substitution method we list some results (algebra of limits) without proof.
If f and g are real valued functions (real valued function means range of the function is subset of R, set of real numbers) defined on the domain D such that $\lim\limits_{x\to a} f(x)$, $\lim\limits_{x\to a} g(x)$ both exist, then the following results hold good.

1. $\lim\limits_{x\to a}(f(x)+g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$
2. $\lim\limits_{x\to a}\left(f(x)-g(x)\right) = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$
3. $\lim\limits_{x\to a} f(x)g(x) = \left(\lim\limits_{x\to a} f(x)\right)\left(\lim\limits_{x\to a} g(x)\right)$
4. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$, provided $\lim\limits_{x\to a} g(x) \neq 0$
5. $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$
6. $\lim\limits_{x\to a} \dfrac{1}{g(x)} = \dfrac{1}{\lim\limits_{x\to a} g(x)}$
7. $\lim\limits_{x\to a} \log f(x) = \log\left(\lim\limits_{x\to a} f(x)\right)$
8. $\lim\limits_{x\to a} e^{f(x)} = e^{\lim\limits_{x\to a} f(x)}$
9. $\lim\limits_{x\to a} f(x)^{g(x)} = \left(\lim\limits_{x\to a} f(x)\right)^{\lim\limits_{x\to a} g(x)}$
10. $\lim\limits_{x\to a} kf(x) = k\lim\limits_{x\to a} f(x)$, where k is a constant
11. $\lim\limits_{x\to a} k = k$, where k is a constant

Remark 1:

(i) These results are used so frequently; that we have no need to mention these results each time.

(ii) Hereafter, we will use D.S.M. for Direct Substitution Method.

Now we are in position to discuss some examples based on (D.S.M.).

Example 1: Evaluate the following limits:

(i) $\lim_{x\to 3}(x^2-2x+4)$ (ii) $\lim_{x\to 2} x(x^2-4)$ (iii) $\lim_{x\to -1}(1+x+x^2+x^3+...+x^{100})$

(iv) $\lim_{x\to\sqrt{2}}\frac{x^2+3}{3-x^4}$ (v) $\lim_{x\to 3}\sqrt{25-x^2}$

(vi) $\lim_{x\to 0} f(x)$, where f(x) = $\begin{cases} 2-x^2, & x\neq 0 \\ 2, & x=0 \end{cases}$

Solution:

(i) $\lim_{x\to 3}(x^2-2x+4) = (3)^2-2\times 3+4$ [By D. S. M.]

$= 9 - 6 + 4 = 7$

(ii) $\lim_{x\to 2} x(x^2-4) = 2[(2)^2-4] = 2(4-4) = 0$

(iii) $\lim_{x\to -1}(1+x+x^2+x^3+...+x^{100}) = 1+(-1)^1+(-1)^2+(-1)^3+...+(-1)^{100}$

$=1+(-1)+(1)+(-1)+(1)+...+(-1)+(1)$

= 1 + (50 terms each containing 1)

+ (50 terms each containing (–1))

= 1 + 50 + (– 50) = 1

(iv) $\lim_{x\to\sqrt{2}}\frac{x^3+3}{3-x^4} = \frac{\lim_{x\to\sqrt{2}}(x^3+3)}{\lim_{x-\sqrt{2}}(3-x^4)} = \frac{(\sqrt{2})^3+3}{3-\left(\sqrt{2}\right)^4} = \frac{2\sqrt{2}+3}{3-4} = -(2\sqrt{2}+3)$

(v) $\lim_{x\to 3}\sqrt{25-x^2} = \sqrt{\lim_{x\to 3}(25-x^2)} = \sqrt{25-(3)^2} = \sqrt{25-9} = \sqrt{16} = 4$

(vi) $f(x) = \begin{cases} 2-x^2, & x\neq 0 \\ 2, & x=0 \end{cases}$

$\lim_{x\to 0} f(x) = \lim_{x\to 0}(2-x^2)$ $[\because x\to 0 \Rightarrow x\neq 0,\ \text{so } f(x)=2-x^2]$

$= 2 - (0)^2 = 2 - 0 = 2$

Remark 2: Limit of polynomial functions is always evaluated by D.S.M.

Now, you can try the following exercise.

E 1) Evaluate the following limits:

(i) $\lim_{x\to 2}(x^2-2x+3)^{x^2+1}$ (ii) $\lim_{x\to 1}\log(x^4+x^2+1)$ (iii) $\lim_{x\to 5} 3$

(iv) $\lim_{x\to 3} 4f(x)$, where $f(x)=(x-5)^2$

D.S.M. discussed above does not always work, in some situations it may fail. In next section we shall see when it fails and what are the alternate methods in such situations?

5.4 FAILURE OF DIRECT SUBSTITUATION METHOD

In mathematics following seven forms are known as indeterminate form, i.e. as such these forms are not defined.

(i) $\frac{0}{0}$ (ii) $\frac{\infty}{\infty}$ (iii) $0\times\infty$ (iv) $\infty-\infty$ (v) 0^0 (vi) 1^∞ (vii) ∞^0

So, if by direct substitution any of the above mentioned forms take place then D.S.M. fails and we need some alternate methods. Some of them are listed below:

I Factorisation Method

II Least Common Multiplier Method

III Rationalisation Method

IV Use of some Standard Results

Let us discuss these methods one by one:

5.4.1 Factorisation Method

This method is useful, when we get $\frac{0}{0}$ form by direct substitution in the given expression of the type $\lim_{x\to a}\frac{f(x)}{g(x)}$. This will happen if f(x) and g(x) both becomes zero on direct substitution.

$\Rightarrow$ both have at least one common factor (x – a).

In this case express f(x) = (x – a) (some factor) and g(x) = (x – a) (some factor) either by long division method or by any other method known to you. Then cancel out the common factor and again try D.S.M. If D.S.M. works, we get the required limit.

If D.S.M. fails again, repeat the same procedure. Ultimately, after a finite number of steps, you will get the result as the numerator and dominator both are of finite degrees.

Let us explain the method with the help of the following example.

Example 2: Evaluate the following limits:

(i) $\lim_{x\to 2}\frac{x^2-4}{x-2}$ (ii) $\lim_{x\to -1}\frac{x^3+1}{x+1}$

(iii) $\lim_{x\to 1}\frac{x^2+x-2}{x^2-5x+4}$ (iv) $\lim_{x\to 3}\frac{x^3-3x^2+x}{x^3-10x^2+27x-18}$

Solution:

(i) $\lim_{x\to 2}\frac{x^2-4}{x-2}$ $\left[\frac{0}{0}\text{ form, so D.S.M. fails}\right]$

Using factorisation method, we have

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}\quad [\because a^2-b^2=(a-b)(a+b)]$$

$$=\lim_{x\to 2}(x+2)\quad \left[\begin{array}{l}\because x\to 2\Rightarrow x-2\neq 0,\ \text{ so dividing}\\ \text{numerator and denominator by } x-2.\end{array}\right]$$

$= 2 + 2 = 4$ [By D.S.M.]

Remember, we cannot cancel 0 by 0. But here x – 2 is not equal to zero because x is approaching to 2 and not equal to 2 and hence x – 2 is approaching to zero and not equal to zero.

(ii) $\lim_{x\to -1} \frac{x^3+1}{x+1}$ $\left[\frac{0}{0} \text{ form, so D.S.M. fails}\right]$

Using factorisation method, we have

$$\lim_{x\to -1} \frac{x^3+1}{x+1} = \lim_{x\to -1} \frac{x^3+1^3}{x+1}$$

$$= \lim_{x\to -1} \frac{(x+1)(x^2-x+1)}{x+1} \quad [\because a^3+b^3=(a+b)(a^2-ab+b^2)]$$

$$= \lim_{x\to -1}(x^2-x+1) \quad \left[\begin{array}{l}\because x\to -1 \Rightarrow x+1\neq 0,\\ \text{so dividing numerator and}\\ \text{denominator by } x+1.\end{array}\right]$$

$$= (-1)^2-(-1)+1 = 1+1+1 = 3$$

(iii) $\lim_{x\to 1} \frac{x^2+x-2}{x^2-5x+4}$ $\left[\frac{0}{0} \text{ form, so D.S.M. fails}\right]$

Using factorisation method, we have

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-5x+4} = \lim_{x\to 1} \frac{x^2+2x-x-2}{x^2-4x-x+4}$$

$$= \lim_{x\to 1} \frac{x(x+2)-1(x+2)}{x(x-4)-1(x-4)} = \lim_{x\to 1} \frac{(x+2)(x-1)}{(x-4)(x-1)}$$

$$= \lim_{x\to 1} \frac{x+2}{x-4} \quad \left[\begin{array}{l}\because x\to 1 \Rightarrow x-1\neq 0,\\ \text{so dividing numerator and}\\ \text{denometor by } x-1.\end{array}\right]$$

$$= \frac{1+2}{1-4} = \frac{3}{-3} = -1 \quad \text{[By D.S.M.]}$$

(iv) Let $I = \lim_{x\to 3} \frac{x^3-3x^2+x}{x^3-10x^2+27x-18}$ $\left[\frac{0}{0} \text{ form, so D.S.M. fails}\right]$

As on putting x = 3, the numerator and as well denominator both becomes zero, therefore x – 3 is a factor of x^3-3x^2+x as well as of $x^3-10x^2+27x-18$. Dividing x^3-3x^2+x by x – 3, we get x^2+1 as the quotient and 0 as the remainder and on dividing $x^3-10x^2+27x-18$, we get x^2-7x+6 as the quotient and 0 as the remainder.

$$\therefore I = \lim_{x\to 3} \frac{(x^2+1)(x-3)}{(x^2-7x+6)(x-3)} = \lim_{x\to 3} \frac{x^2+1}{x^2-7x+6} \quad \left[\begin{array}{l}\text{Cancelling out the}\\ \text{factor } x-3\end{array}\right]$$

$$= \frac{(3)^2+1}{(3)^2-7\times 3+6} \quad \text{[By D.S.M.]}$$

$$= \frac{9+1}{9-21+6} = \frac{10}{-6} = -\frac{5}{3}$$

Here is an exercise for you.

E 2) Evaluate the following limits:

(i) $\lim_{x\to 2} \frac{x^3-7x^2+16x-12}{x^4-6x^3-3x^2+52x-60}$ (ii) $\lim_{x\to 2} \frac{x^3-4x^2+5x-2}{x^3-2x-4}$

5.4.2 Least Common Multiplier Method

This method is useful in $\infty - \infty$ form.

Procedure: Take L.C.M. of the given expression and simplify it. Most of the times after simplification it reduces to $\frac{0}{0}$ form then solve it as explained in factorisation method.

Let us take an example based on this method.

Example 3: Evaluate $\lim_{x\to 3}\left(\frac{1}{x-3} - \frac{3}{x^2-3x}\right)$

Solution: $\lim_{x\to 3}\left(\frac{1}{x-3} - \frac{3}{x^2-3x}\right)$ [$\infty - \infty$ form, so D.S.M. fails]

Using LCM method, we have

$$\lim_{x\to 3}\left(\frac{1}{x-3} - \frac{3}{x^2-3x}\right) = \lim_{x\to 3}\left(\frac{1}{x-3} - \frac{3}{x(x-3)}\right) = \lim_{x\to 3}\left(\frac{x-3}{x(x-3)}\right) = \lim_{x\to 3}\frac{1}{x} = \frac{1}{3}$$

Here is an exercise for you.

E 3) Evaluate $\lim_{x\to -2}\left(\frac{1}{x+2} - \frac{6}{x^3+x^2-2x}\right)$

5.4.3 Rationalisation Method

This method is explained in the following example.

Example 4: Evaluate the following limits:

(i) $\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}$ (ii) $\lim_{x\to 3}\frac{\sqrt{5x-6}-\sqrt{x+6}}{x^2-9}$

Solution:

(i) $\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}$ $\left[\frac{0}{0} \text{ form, so D.S.M. fails}\right]$

Rationalising the numerator, we have

$$\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x} = \lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}\times\frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}$$

$$= \lim_{x\to 0}\frac{\left(\sqrt{4+x}\right)^2-(2)^2}{x\left(\sqrt{4+x}+2\right)} \quad [\because a^2-b^2 = (a-b)(a+b)]$$

$$= \lim_{x\to 0}\frac{4+x-4}{x(\sqrt{4+x}+2)} = \lim_{x\to 0}\frac{x}{x\left(\sqrt{4+x}+2\right)}$$

$$= \lim_{x\to 0}\frac{1}{\sqrt{4+x}+2} = \frac{1}{\sqrt{4+0}+2} = \frac{1}{2+2} = \frac{1}{4}$$

(ii) $\lim_{x\to 3}\frac{\sqrt{5x-6}-\sqrt{x+6}}{x^2-9}$ $\left[\frac{0}{0} \text{ form, so D.S.M. fails}\right]$

Rationalising the numerator, we have

$$\lim_{x\to3}\frac{\sqrt{5x-6}-\sqrt{x+6}}{x^2-9} = \lim_{x\to3}\frac{\sqrt{5x-6}-\sqrt{x+6}}{x^2-9}\times\frac{\sqrt{5x-6}+\sqrt{x+6}}{\sqrt{5x-6}+\sqrt{x+6}}$$

$$= \lim_{x\to3}\frac{(\sqrt{5x-6})^2-\left(\sqrt{x+6}\right)^2}{(x^2-9)(\sqrt{5x-6}+\sqrt{x+6})}$$

$$= \lim_{x\to3}\frac{5x-6-(x+6)}{(x^2-9)(\sqrt{5x-6}+\sqrt{x+6})}$$

$$= \lim_{x\to3}\frac{4x-12}{(x^2-3^2)(\sqrt{5x-6}+\sqrt{x+6})}$$

$$= \lim_{x\to3}\frac{4(x-3)}{(x-3)(x+3)(\sqrt{5x-6}+\sqrt{x+6})}$$

$$= \lim_{x\to3}\frac{4}{(x+3)(\sqrt{5x-6}+\sqrt{x+6})}$$

$$= \frac{4}{(3+3)(\sqrt{15-6}+\sqrt{3+6})}$$

$$= \frac{4}{6(\sqrt{9}+\sqrt{9})} = \frac{4}{6(3+3)} = \frac{4}{36} = \frac{1}{9}$$

Here is an exercise for you.

E 4) Evaluate $\lim\limits_{x\to2}\dfrac{\sqrt{3+x}-\sqrt{5}}{x-2}$.

5.4.4 Use of some Standard Results

Here, we list without proof some very useful standard results which hold in limits.

1. $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a} = na^{n-1}$ [a and n are any real numbers, provided a^n, a^{n-1} exist]

2. $\lim\limits_{\theta\to0}\dfrac{\sin\theta}{\theta} = \lim\limits_{\theta\to0}\dfrac{\theta}{\sin\theta} = 1$

3. $\lim\limits_{\theta\to0}\cos\theta = 1$

4. $\lim\limits_{\theta\to0}\dfrac{\tan\theta}{\theta} = \lim\limits_{\theta\to0}\dfrac{\theta}{\tan\theta} = 1$

5. $\lim\limits_{x\to0}\dfrac{a^x-1}{x} = \log_e|a|$, in particular, $\lim\limits_{x\to0}\dfrac{e^x-1}{x} = \log_e e = 1$

6. $\lim\limits_{x\to0}\dfrac{\log(1+x)}{x} = 1$

7. $\lim\limits_{x\to0}(1+x)^{1/x} = e$

8. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = e$

Let us consider an example based on these standard results.

Example 5: Evaluate the following limits:

(i) $\lim\limits_{x\to 3}\dfrac{x^5-243}{x^4-81}$ (ii) $\lim\limits_{x\to 2}\dfrac{x^{10/3}-2^{10/3}}{x^{4/3}-2^{4/3}}$ (iii) $\lim\limits_{x\to 0}\dfrac{\sin 4x}{3x}$ (iv) $\lim\limits_{x\to 0}\cos 5x$

(v) $\lim\limits_{x\to 0}\dfrac{\tan 3x}{\sin 2x}$ (vi) $\lim\limits_{x\to 0}\dfrac{2^{5x}-1}{x}$ (vii) $\lim\limits_{x\to 0}\dfrac{e^{ax}-1}{x}$ (viii) $\lim\limits_{x\to 0}\dfrac{\log(1+5x)}{x}$

(ix) $\lim\limits_{x\to 0}\dfrac{\log(1+3x)}{e^{2x}-1}$ (x) $\lim\limits_{x\to 0}(1+8x)^{1/x}$

Solution:

(i) Let I = $\lim\limits_{x\to 3}\dfrac{x^5-243}{x^4-81} = \lim\limits_{x\to 3}\dfrac{x^5-3^5}{x^4-3^4}$

Dividing numerator and denominator by x – 3, we get

$$I = \lim_{x\to 3}\dfrac{\dfrac{x^5-3^5}{x-3}}{\dfrac{x^4-3^4}{x-3}} = \dfrac{\lim\limits_{x\to 3}\dfrac{x^5-3^5}{x-3}}{\lim\limits_{x\to 3}\dfrac{x^4-3^4}{x-3}}$$

$$= \frac{5(3)^{5-1}}{4(3)^{4-1}} = \frac{5}{4}\times\frac{3^4}{3^3} = \frac{5}{4}\times 3 = \frac{15}{4} \qquad \left[\text{Using } \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

(ii) $\lim\limits_{x\to 2}\dfrac{x^{10/3}-2^{10/3}}{x^{4/3}-2^{4/3}}$

Dividing numerator and denominator by x – 2, we get

$$\lim_{x\to 2}\frac{x^{10/3}-2^{10/3}}{x^{4/3}-2^{4/3}} = \lim_{x\to 2}\dfrac{\dfrac{x^{10/3}-2^{10/3}}{x-2}}{\dfrac{x^{4/3}-2^{4/3}}{x-2}} = \dfrac{\lim\limits_{x\to 2}\dfrac{x^{10/3}-2^{10/3}}{x-2}}{\lim\limits_{x\to 2}\dfrac{x^{4/3}-2^{4/3}}{x-2}}$$

$$= \dfrac{\dfrac{10}{3}(2)^{\frac{10}{3}-1}}{\dfrac{4}{3}(2)^{\frac{4}{3}-1}} = \frac{10}{3}\times\frac{3}{4}\times\frac{2^{\frac{7}{3}}}{2^{\frac{1}{3}}} = \frac{5}{2}\times 2^{\frac{7}{3}-\frac{1}{3}} = \frac{5}{2}.2^2 = 10$$

(iii) $\lim\limits_{x\to 0}\dfrac{\sin 4x}{3x} = \lim\limits_{x\to 0}\dfrac{\sin 4x}{4x}\times\dfrac{4x}{3x}$ $\qquad\left[\begin{array}{l}\text{Dividing and multipling}\\ \text{by } 4x\end{array}\right]$

$= \lim\limits_{x\to 0}\dfrac{\sin 4x}{4x}\,\dfrac{4}{3} = \dfrac{4}{3}\lim\limits_{x\to 0}\dfrac{\sin 4x}{4x}$

$= \dfrac{4}{3}\lim\limits_{4x\to 0}\dfrac{\sin 4x}{4x}$ $\qquad[\text{As } x\to 0 \Rightarrow 4x\to 0]$

$= \dfrac{4}{3}\times 1 = \dfrac{4}{3}$ $\qquad\left[\because \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1\right]$

(iv) $\lim\limits_{x\to 0}\cos 5x = \lim\limits_{5x\to 0}\cos 5x = 1$ $\qquad\left[\begin{array}{l}\text{As } x\to 0 \Rightarrow 5x\to 0 \text{ and}\\ \lim\limits_{\theta\to 0}\cos\theta = 1\end{array}\right]$

(v) $\lim\limits_{x\to 0}\dfrac{\tan 3x}{\sin 2x} = \lim\limits_{x\to 0}\dfrac{\tan 3x}{3x}\times\dfrac{2x}{\sin 2x}\times\dfrac{3}{2}$

$$= \frac{3}{2}\left(\lim_{3x\to 0}\frac{\tan 3x}{3x}\right)\left(\lim_{2x\to 0}\frac{2x}{\sin 2x}\right) = \frac{3}{2}(1)(1) = \frac{3}{2}$$

$$\left[\because \lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1 \text{ and } \lim_{x\to 0}\frac{\theta}{\sin\theta} = 1\right]$$

(vi) $\lim\limits_{x\to 0}\frac{2^{5x}-1}{x} = \lim\limits_{x\to 0}\frac{2^{5x}-1}{5x}\times 5 = 5\lim\limits_{5x\to 0}\frac{2^{5x}-1}{5x}$ $[\because x\to 0 \Rightarrow 5x\to 0]$

$= 5\ \log_e 2$ $\left[\because \lim\limits_{x\to 0}\frac{a^x-1}{x} = \log_e |a|\right]$

(vii) $\lim\limits_{x\to 0}\frac{e^{ax}-1}{x} = \lim\limits_{x\to 0}\frac{e^{ax}-1}{ax}\times a = a\lim\limits_{ax\to 0}\frac{e^{ax}-1}{ax}$ $[\because x\to 0 \Rightarrow ax\to 0]$

$= a(1)$ $\left[\because \lim\limits_{x\to 0}\frac{e^x-1}{x} = 1\right]$

$= a$

(viii) $\lim\limits_{x\to 0}\frac{\log(1+5x)}{x} = \lim\limits_{x\to 0} 5\frac{\log(1+5x)}{5x}$

$= 5\lim\limits_{5x\to 0}\frac{\log(1+5x)}{5x}$ $[\because x\to 0 \Rightarrow 5x\to 0]$

$= 5(1)$ $\left[\because \lim\limits_{x\to 0}\frac{\log(1+x)}{x} = 1\right]$

$= 5$

(ix) $\lim\limits_{x\to 0}\frac{\log(1+3x)}{e^{2x}-1} = \lim\limits_{x\to 0}\frac{\log(1+3x)}{3x}\times\frac{2x}{e^{2x}-1}\times\frac{3}{2}$

$= \frac{3}{2}\left(\lim\limits_{3x\to 0}\frac{\log(1+3x)}{3x}\right)\left(\lim\limits_{2x\to 0}\frac{2x}{e^{2x}-1}\right)$

$= \frac{3}{2}(1)(1) = \frac{3}{2}$ as $\lim\limits_{x\to 0}\frac{\log(1+x)}{x} = 1$ and $\lim\limits_{x\to 0}\frac{x}{e^x-1} = 1$

(x) $\lim\limits_{x\to 0}(1+8x)^{1/x} = \lim\limits_{x\to 0}\left[(1+8x)^{\frac{1}{8x}}\right]^8 = \left[\lim\limits_{8x\to 0}(1+8x)^{\frac{1}{8x}}\right]^8$ as $x\to 0 \Rightarrow 8x\to 0$

$= (e)^8 = e^8$ $\left[\because \lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e\right]$

Here, is an exercise for you.

E 5) Evaluate the following limits:

(i) $\lim\limits_{x\to\sqrt{2}}\frac{x^{10}-32}{x-\sqrt{2}}$ (ii) $\lim\limits_{x\to 0}\frac{(ab)^{3x}-1}{x}$ (iii) $\lim\limits_{x\to 0}\frac{e^{\sin x}-1}{\tan x}$

(iv) $\lim\limits_{x\to 0}\frac{\log(1+8x^2)}{e^{x^2}-1}$ (v) $\lim\limits_{x\to 0}\frac{a^x-e^x}{x}$ (vi) $\lim\limits_{x\to 0}\frac{e^x-(1+2x)}{2^x-1}$

(vii) $\lim\limits_{x\to 0}\frac{x(1+2x)^{1/x}-(e^{2x}-1)}{x}$

5.5 CONCEPT OF INFINTE LIMIT

Consider the following cases

$\frac{1}{10} = 0.1$

$\frac{1}{100} = 0.01$

$\frac{1}{1000} = 0.001$

$\frac{1}{10000} = 0.0001$

…

…

…

$\frac{1}{10^n} = 0.\underbrace{000...1}_{\text{n times}}$

We see that as x (denominator) becomes larger and larger than $\frac{1}{x}$ becomes smaller and smaller and approaches to zero.

$\therefore$ we write $\lim\limits_{x\to\infty} \frac{1}{x} = 0$

Or $\lim\limits_{x\to\infty} \frac{1}{x^n} = 0,\quad$ where $n > 0$

Remark 3: $x \to \infty$ means that whatever large real number K (say) we take then x > K, i.e. no real number can be greater than x.

Let us consider an example, which involve infinite limit.

Example 6: Evaluate the following limits:

(i) $\lim\limits_{x\to\infty} \frac{3x^2 + 5x - 1}{4x^2 - 3x + 9}$ (ii) $\lim\limits_{x\to\infty} \frac{5x^5 + x + 1}{x^3 + 5}$ (iii) $\lim\limits_{x\to\infty} \frac{x^5 + 1}{4x^7 + 3x^2 + 7}$

Solution:

(i) $\lim\limits_{x\to\infty} \frac{3x^2 + 5x - 1}{4x^2 - 3x + 9}$

Here degree of numerator = Degree of denominator = 2

$\therefore$ dividing numerator and denominator by x^2, we get

$$\lim_{x\to\infty} \frac{3x^2 + 5x - 1}{4x^2 - 3x + 9} = \lim_{x\to\infty} \frac{3 + \frac{5}{x} - \frac{1}{x^2}}{4 - \frac{3}{x} + \frac{9}{x^2}} = \frac{3 + 0 - 0}{4 - 0 + 0} = \frac{3}{4} \quad \left[\begin{array}{l}\because \lim\limits_{x\to\infty} \frac{1}{x^n} = 0 \\ \text{for } n > 0\end{array}\right]$$

(ii) $\lim\limits_{x\to\infty} \frac{5x^5 + x + 1}{x^3 + 5}$

Here degree of numerator > degree of denominator.

$\therefore$ dividing numerator and denominator by x^3 [Least of degrees], we get

$$\lim_{x\to\infty} \frac{5x^5 + x + 1}{x^3 + 5} = \lim_{x\to\infty} \frac{5x^2 + \frac{1}{x^2} + \frac{1}{x^3}}{1 + \frac{5}{x^3}} = \frac{\infty + 0 + 0}{1 + 0} = \infty$$

(iii) $\lim_{x\to\infty} \frac{x^5 + 1}{4x^7 + 3x^2 + 7}$

Here degree of numerator < degree of denominator.

$\therefore$ dividing numerator and denominator by x^5 [Least of degrees], we get

$$\lim_{x\to\infty} \frac{x^5 + 1}{4x^7 + 3x^2 + 7} = \lim_{x\to\infty} \frac{1 + \frac{1}{x^5}}{4x^2 + \frac{3}{x^3} + \frac{7}{x^5}} = \frac{1 + 0}{\infty + 0 + 0} = \frac{1}{\infty} = 0$$

In general, without calculating actual limit we can know the answer in advance of rational functions, in the cases when as $x \to \infty$ see the following result without proof.

$$\lim_{x\to\infty} \frac{a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots + a_{m-1} x + a_m}{b_0 x^n + b_1 x^{n-1} + b_2 x^{n-2} + \ldots + b_{n-1} x + b_n} = \begin{cases} \frac{a_0}{b_0}, & \text{if } m = n \\ 0, & \text{if } m < n \\ \infty, & \text{if } m > n \end{cases}$$

Now, you can try the following exercise.

E 6) Evaluate $\lim_{x\to\infty} \frac{\sqrt{x^2 + 3} + \sqrt[4]{x^4 + 2}}{\sqrt[7]{x^7 + x^2} - \sqrt[3]{x^2 + 5}}$.

5.6 CONCEPT OF LEFT HAND AND RIGHT HAND LIMITS

We note that on the real line, we can approach any real number 2(say) either from left or from right. Consider the example $y = f(x) = 3x + 2$. We see that as x takes the values 1.9, 1.98, 1.998, 1.9998, ... then corresponding values taken by y are 7.7, 7.94, 7.994, 7.9994, … respectively as shown below.

x	1.9	1.98	1.998	1.9998	…	1.99999998	…
y = f(x)	7.7	7.94	7.994	7.9994	…	7.99999994	…

x	2.1	2.01	2.001	2.0001	…	2.0000001	…
y = f(x)	8.3	8.03	8.003	8.0003	…	8.0000003	…

i.e. as x is coming nearer and nearer to 2 from left then y is also coming nearer and nearer to 8 from left. If x approaches like this from left (see Fig. 5.1), then we say that x is approaching form left to 2 and is denoted by putting a –ve sign as a right superscript of 2 i.e. 2^-

i.e. we write the limit of the function as

$\lim_{x\to 2^-} f(x)$ … (1)

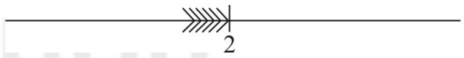

Fig. 5.1

If limit (1) exists, then we call it left hand limit (L.H.L.) of the function f(x) as x tends to 2.

Similarly we see that as x takes the values 2.1, 2.01, 2.001, 2.0001, … then corresponding values taken by y are 8.3, 8.03, 8.003, 8.0003, … respectively. i.e. as x is coming nearer and nearer to 2 from right then y is also coming nearer and nearer to 8 from right. If x approaches like this from right (see Fig. 5.2) then we say that x is approaching from right to 2 and is denoted by putting +ve sign as a superscript of 2 i.e. 2^+ and the limit of the function as

$$\lim_{x\to 2^+} f(x) \qquad \dots (2)$$

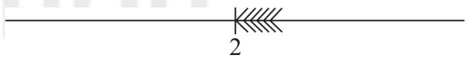

Fig. 5.2

If limit (2) exists, then we call it right hand limit (R.H.L.) of the function f(x) as x tends to 2.

Remark 4:

(i) L.H. and R.H. limits are used when functions have different values for $x \to 2^-$ and $x \to 2^+$.

For example, in case of

(a) modules functions,

(b) functions having different values just below or above the value to which x is tending, i.e. there is break in function.

(ii) Limit exists if L.H.L. and R.H.L. both exist and are equal.

Following example illustrates the idea of L.H.L. and R.H.L.

Example 7: Evaluate the following limits:

(i) $\lim_{x\to 0}|x|$ (ii) $\lim_{x\to 3}|x-3|$ (iii) $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} x^2+1, & x \le 1 \\ 1-x^2, & x > 1 \end{cases}$

(iv) $\lim_{x\to 4} f(x)$, where $f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \ne 4 \\ 0, & x = 4 \end{cases}$

Solution:

(i) $\lim_{x\to 0}|x|$

Here we have to use the concept of L.H.L. and R.H.L., because of the presence of the modulus function.

L.H.L. = $\lim_{x\to 0^-}|x|$

Here, as x is approaching to zero from its left and hence x is having little bit lesser value than 0.

Let us put x = 0 – h, where h is + ve real and is very small.

As $x \to 0^- \Rightarrow h \to 0^+$

L.H.L. $= \lim_{h\to 0^+}|0-h| = \lim_{h\to 0^+}|-h| = \lim_{h\to 0^+}|-1|\times|h| = \lim_{h\to 0^+}|h| \text{ as } |-1| = -(-1) = 1$

$= \lim_{h\to 0^+} h \qquad [\because h \to 0^+ \Rightarrow h > 0 \Rightarrow |h| = h\ ]$

$= 0 \qquad \ldots (1)$

R.H.L. $= \lim_{x\to 0^+}|x|$

Here, as x is approaching to zero from its right and hence x is having slightly greater value than 0.

Let us put x = 0 + h, where h is +ve real and is very small.

As $x \to 0^+ \Rightarrow h \to 0^+$

$\therefore$ R.H.L. $= \lim_{h\to 0^+}|0+h| = \lim_{h\to 0^+}|h| = \lim_{h\to 0^+} h = 0 \qquad \ldots (2)$

From (1) and (2)

L.H.L. = R.H.L.

$\therefore \lim_{x\to 0}|x|$ exists and equal to 0.

(ii) $\lim_{x\to 3}|x-3|$

L.H.L. $= \lim_{x\to 3^-}|x-3|$

Putting x = 3 – h, where h is +ve real and very small.

As $x \to 3^- \Rightarrow h \to 0^+$

L.H.L. $= \lim_{h\to 0^+}|3-h-3| = \lim_{h\to 0^+}|-h| = \lim_{h\to 0^+}|h| = \lim_{x\to 0^+} h = 0 \quad \ldots (1)$

R.H.L.$= \lim_{x\to 3^+}|x-3|$

Putting x = 3 + h as $x \to 3^+ \Rightarrow h \to 0^+$

R.H.L.$= \lim_{h\to 0^+}|3+h-3| = \lim_{h\to 0^+}|h| = \lim_{h\to 0^+} h = 0 \qquad \ldots (2)$

From (1) and (2)

L.H.L. = R.H.L.

$\therefore \lim_{x\to 3}|x-3|$ exists and equal to 0.

(iii) $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} x^2+1, & x \le 1 \\ 1-x^2, & x > 1 \end{cases}$

L.H.L. $= \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x^2+1) \left[\begin{array}{l}\because x \to 1^- \text{ means } x \text{ is slightly less than} \\ 1 \text{ and hence in this case } f(x) = x^2+1\end{array}\right]$

$= (1)^2 + 1 = 1 + 1 = 2 \qquad \ldots (1)$

R.H.L. $= \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(1-x^2) \left[\begin{array}{l}\because x \to 1^+ \text{ means } x \text{ is slightly greater than} \\ 1 \text{ and hence in this case } f(x) = 1-x^2\end{array}\right]$

$= 1-(1)^2 = 1-1 = 0 \qquad \ldots (2)$

From (1) and (2)

LH.L. $\neq$ R.H.L.

$\therefore \lim_{x\to 1} f(x)$ does not exist.

(iv) $\lim_{x\to 4} f(x)$, where $f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4 \\ 0, & x = 4 \end{cases}$

$$\text{L.H.L.} = \lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \frac{|x-4|}{x-4} \quad \left[\begin{array}{l}\because x \to 4^- \Rightarrow x \text{ is slightly less than } 4 \\ \text{i.e } x \neq 4, \text{ so in this case } f(x) = \dfrac{|x-4|}{x-4}\end{array}\right]$$

Putting x = 4 – h, where h is +ve real and very small.

As $x \to 4^- \Rightarrow h \to 0^+$

$$\text{L.H.L.} = \lim_{h\to 0^+} \frac{|4-h-4|}{4-h-4} = \lim_{h\to 0^+} \frac{|-h|}{-h} = \lim_{h\to 0^+} \frac{|-1|\times|h|}{-h}$$

$$= \lim_{h\to 0^+} \frac{(1)(h)}{-h} = \lim_{h\to 0^+} (-1) = -1 \qquad \dots (1)$$

$$\text{R.H.L.} = \lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{|x-4|}{x-4}$$

Putting x = 4 + h as $x \to 4^+ \Rightarrow h \to 0^+$

$$\therefore \text{R.H.L.} = \lim_{h\to 0^+} \frac{|4+h-4|}{4+h-4} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1 \qquad \dots (2)$$

From (1) and (2)

L.H.L. $\neq$ R.H.L.

$\therefore \lim\limits_{x\to 4} f(x)$ does not exist.

Example 8: If $\lim\limits_{x\to 0} f(x)$ exists, then find the value of k for

$$f(x) = \begin{cases} x-|x|, & x \le 0 \\ k, & x > 0 \end{cases}$$

Solution: $f(x) = \begin{cases} x-|x|, & x \le 0 \\ k, & x > 0 \end{cases}$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left(x-|x|\right)$$

Putting x = 0 – h as $x \to 0^- \Rightarrow h \to 0^+$

$$\text{L.H.L.} = \lim_{h\to 0^+} (0-h-|0-h|) = \lim_{h\to 0^+} (-h-|-h|)$$

$$= \lim_{h\to 0^+} (-h-|-1|\times|h|) = \lim_{h\to 0^+} (-h-|h|) = \lim_{h\to 0^+} (-h-h)$$

$$= \lim_{x\to 0^+} (-2h) = -2(0) = 0 \qquad \dots (1)$$

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} k = k \qquad \dots (2)$$

Since, it is given that $\lim\limits_{x\to 0} f(x)$ exists.

$\therefore$ we must have

L.H.L. = R.H.L. $\Rightarrow 0 = k$ or $k = 0$

Here are some exercises for you.

E 7) Evaluate the following limits:

(i) $\lim\limits_{x\to 0^-} \dfrac{5x+|x|}{3|x|-7x}$ (ii) $\lim\limits_{x\to 5^+} (3-|x|)$ (iii) $\lim\limits_{x\to 0} \dfrac{x}{|x|}$

E 8) If $\lim\limits_{x\to 3} f(x)$ exists then find a, for $f(x) = \begin{cases} ax+3, & x \le 3 \\ 2(x+1), & x > 3 \end{cases}$

5.7 CONTINUITY OF A FUNCTION AT A POINT

In Sec. 5.6, we have discussed the concept of L.H.L. and R.H.L. Adding one more-step, we can define continuity at a point.
A function f(x) is said to be continuous at x = a if

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$, i.e. for continuity at a point x = a, we must have

i.e. $\text{L.H.L.}_{\text{at } x=a} = \text{R.H.L.}_{\text{at } x=a} =$ value of the function at x = a

Diagrammatically, continuity at x = a means graph of the function f(x) from a value slightly less than ‘a’ to a value slightly greater than ‘a’ has no gap, i.e. if we draw the graph with pencil then we don’t have to pick up the pencil as we cross the point where x = a. Look at the Fig. 5.3 to 5.5.
In Fig. 5.3 f(x) is not continuous at x = a.
In Fig. 5.4 f(x) is not continuous at x = a.
In Fig. 5.5 f(x) is continuous at x = a.

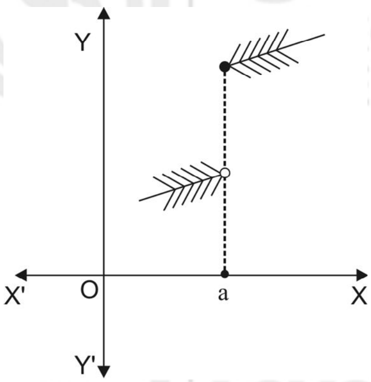

Fig. 5.3

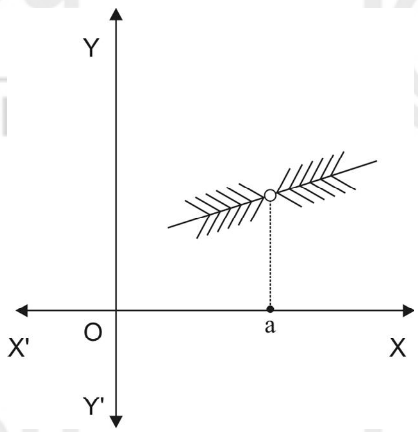

Fig. 5.4

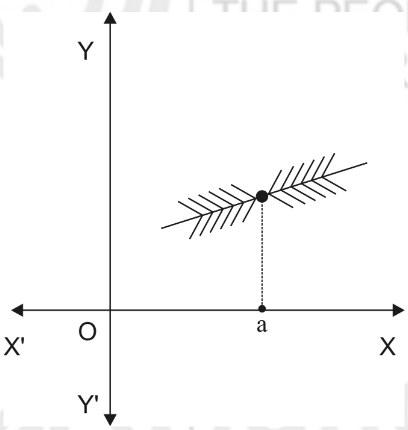

Fig. 5.5

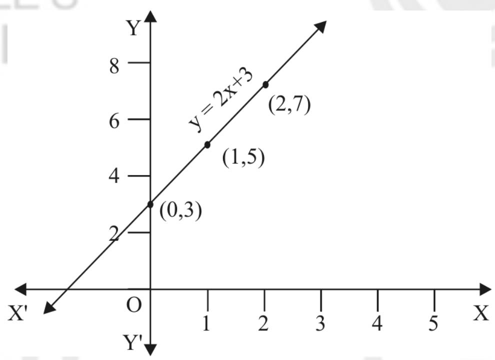

Fig. 5.6

Functions whose graphs are given in Fig. 5.6 and Fig. 5.7 are discussed below.

(i) Consider the function f: $R \to R$ defined by
f(x) = 2x + 3

x	0	1	2
y	3	5	7

See the graph of this function in Fig. 5.6. We note that this function is continuous at all points of its domain as there is no gap at any point in its graph.

(ii) Consider the function f: $R \rightarrow R$ defined by

$$f(x) = \begin{cases} 1, & x \geq 1 \\ 2, & x < 1 \end{cases}$$

See the graph of this function in Fig. 5.7.

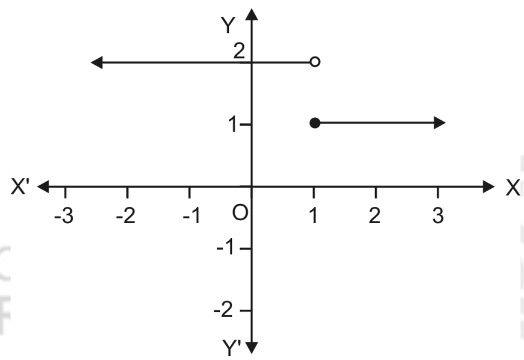

Fig .5.7

We note that, if we draw the graph of this function with pencil, then we will have to pick up the pencil as we cross the point where x = 1. Therefore this function is not continuous at x = 1. Also this function is continuous at all points of its domain except at x = 1.

Now, let us consider some examples on continuity at a point.

Example 9: Discuss the continuity of the following functions at given point:

(i) $f(x) = |x|$ at x = 0

(ii) $f(x) = |x-3|$ at x = 3

(iii) $f(x) = \begin{cases} x^2+1, & x \leq 1 \\ 1-x^2, & x > 1 \end{cases}$ at x = 1

(iv) $f(x) = \begin{cases} \dfrac{|x-4|}{x-4} & , x \neq 4 \\ 0 & , x = 4 \end{cases}$ at x = 4

(v) $f(x) = \dfrac{x}{|x|}$ at x = 0

Solution:

(i) $f(x) = |x|$ at x = 0

L.H.L. = 0, R.H.L. = 0 $\left[\begin{array}{l}\text{Already calculated in} \\ \text{Example 7 of this unit}\end{array}\right]$

Also, at x = 0, f(x) = $|0| = 0$

$\therefore$ L.H.L$_{\text{at } x = 0}$ = R.H.L$_{\text{at } x = 0}$ = f(0)

$\Rightarrow f(x)$ is continuous at x = 0

(ii) $f(x) = |x-3|$, at x = 3

L.H.L. = 0, R.H.L. = 0 $\begin{bmatrix}\text{Already calculated in}\\ \text{Example 7 of this unit}\end{bmatrix}$

Also, $f(3) = |3-3| = |0| = 0$

$\therefore$ L.H.L$_{\text{at } x=3}$ = R.H.L$_{\text{at } x=3}$ = f (0)

$\Rightarrow$ f (x) is continuous at x = 3

(iii) $f(x) = \begin{cases} x^2+1, & x \le 1 \\ 1-x^2, & x > 1 \end{cases}$ at x = 1

L.H.L = 2, but R.H.L. = 0 $\begin{bmatrix}\text{Already calculated in}\\ \text{Example 7 of this unit}\end{bmatrix}$

As L.H.L.$_{\text{at } x=1} \neq$ R.H.L$_{\text{at } x=1}$

$\therefore$ f is not continuous at x = 1

(iv) $f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4 \\ 0, & x = 4 \end{cases}$ at x = 4

$\therefore$ L.H.L.$_{\text{at } x=4} = -1$, but R.H.L.$_{\text{at } x=4} = 1$ $\begin{bmatrix}\text{Already calculated in}\\ \text{Example 7 of this unit}\end{bmatrix}$

As L.H.L$_{\text{at } x=4} \neq$ R.H.L$_{\text{at } x=4}$

$\therefore$ f(x) is not continuous at x = 4.

(v) $f(x) = \dfrac{x}{|x|}$, at x = 0

$$\text{L.H.L.} = \lim_{x \to 0^-} \frac{x}{|x|}$$

Putting x = 0 – h as $x \to 0^- \Rightarrow h \to 0^+$

$$\text{L.H.L.} = \lim_{h \to 0^+} \frac{0-h}{|0-h|} = \lim_{h \to 0^+} \frac{-h}{|-h|} = \lim_{h \to 0^+} \frac{-h}{|-1| \times |h|}$$

$$= \lim_{h \to 0^+} \frac{-h}{h} = \lim_{h \to 0^+} (-1) = -1 \qquad \ldots (1)$$

$$\text{R.H.L.} = \lim_{x \to 0^+} \frac{x}{|x|} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1 \qquad \ldots (2)$$

From (1) and (2)

L.H.L. $\neq$ R.H.L.

$\therefore \lim\limits_{x \to 0} \dfrac{x}{|x|}$ does not exist.

Hence f is not continuous at x = 0.

Example 10: Find the values of a and b, if the function f given below is continuous at x = 2

$$f(x) = \begin{cases} 7, & x < 2 \\ ax+b, & x > 2 \\ a+5, & x = 2 \end{cases}$$

Solution: L.H.L. $= \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 7 = 7$

R.H.L. $= \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (ax + b) = 2a + b$

Also f(2) = a + 5

Since, f is given to be continuous at x = 2, therefore we must have

L.H.L.$_{\text{at } x = 2}$ = R.H.L.$_{\text{at } x = 2}$ = f (2)

$$\Rightarrow \underset{\text{I}}{7} = \underset{\text{II}}{2a + b} = \underset{\text{III}}{a + 5}$$

I & III $\Rightarrow 7 = a + 5 \Rightarrow a = 2$

I & II $\Rightarrow 7 - 2a = b \Rightarrow b = 7 - 4 = 3$

$\therefore a = 2, b = 3$

Here is an exercise for you.

E 9) Find the relation between a and b if the function f is given to be continuous at x = 0, where

$$f(x) = \begin{cases} 2x - a, & x \geq 0 \\ ax + b + 3, & x < 0 \end{cases}$$

5.8 CONTINUOUS FUNCTION

In section 5.7, we have discussed the continuity of a function at a point. In this section, we define what we mean by continuous function.

Continuous Function: A function f is said to be continuous if it is continuous at each point of its domain.

For example, function y =f(x) = 2x + 3 whose graph is given in Fig. 5.6 is a continuous function as we have already discussed that it is continuous at all points of its domain.

Algebra of continuous functions:

If f and g are two continuous functions on a common domain then

(1) f + g is continuous

(2) f – g is continuous

(3) fg is continuous

(4) f/g is continuous, provided g(x) $\neq 0$ $\forall$ points x of its domain.

5.9 SUMMARY

In this unit, we have:

1) Given the concept of limit.
2) Discussed direct substitution method of evaluation of limit.
3) Explained factorisation, L.C.M. rationalisation, and some standard methods to evaluate a given limit.
4) Given the concept of infinite limit.
5) Given the concept of L.H.L. and R.H.L.
6) Discussed the continuity of a function at a point.
7) Discussed what we mean by continuous function.

5.10 SOLUTIONS/ANSWERS

E 1) (i) $\lim_{x\to 2}(x^2-2x+3)^{x^2+1} = \left(2^2-2\times 2+3\right)^{2^2+1} = 3^5 = 243$

(ii) $\lim_{x\to 1}\log(x^4+x^2+1) = \log\left(\lim_{x\to 1}(x^4+x^2+1)\right)$

$$= \log(1^4+1^2+1) = \log 3$$

(iii) $\lim_{x\to 5} 3 = 3$ as f(x) = 3 is a constant function

(iv) $\lim_{x\to 3} 4f(x) = 4\lim_{x\to 3} f(x) = 4\lim_{x\to 3}(x-5)^2$

$$= 4(3-5)^2 = 4(-2)^2 = 4\times 4 = 16$$

E 2) (i) Let $I = \lim_{x\to 2}\frac{x^3-7x^2+16x-12}{x^4-6x^3-3x^2+52x-60}$ $\left[\frac{0}{0}\text{ form, so D.S.M. fails}\right]$

Using factorisation method, we have

$$I = \lim_{x\to 2}\frac{(x-2)(x^2-5x+6)}{(x-2)(x^3-4x^2-11x+30)}$$

Cancelling the common factor $x-2 \neq 0$, we get

$$I = \lim_{x\to 2}\frac{x^2-5x+6}{x^3-4x^2-11x+30} \qquad \left[\text{Again }\frac{0}{0}\text{ form}\right]$$

Again using factorisation method, we have

$$I = \lim_{x\to 2}\frac{(x-2)(x-3)}{(x-2)(x^2-2x-15)} = \lim_{x\to 2}\frac{x-3}{x^2-2x-15}$$

$$= \frac{2-3}{4-4-15}\ [\text{By D.S.M.}]$$

$$= \frac{-1}{-15} = \frac{1}{15}$$

(ii) $\lim_{x\to 2}\frac{x^3-4x^2+5x-2}{x^3-2x-4}$ $\left[\frac{0}{0}\text{ form, so D.S.M. fails}\right]$

Using factorisation method, we have

$$\lim_{x\to 2}\frac{x^3-4x^2+5x-2}{x^3-2x-4} = \lim_{x\to 2}\frac{(x-2)(x^2-2x+1)}{(x-2)(x^2+2x+2)}$$

$$= \lim_{x\to 2}\frac{x^2-2x+1}{x^2+2x+2} = \frac{4-4+1}{4+4+2} = \frac{1}{10}$$

E 3) $\lim_{x\to -2}\left(\frac{1}{x+2}-\frac{6}{x^3+x^2-2x}\right)$ [$\infty-\infty$ form, so D.S.M. fails]

Using LCM method, we have

$$\lim_{x\to -2}\left(\frac{1}{x+2}-\frac{6}{x^3+x^2-2x}\right) = \lim_{x\to -2}\left(\frac{1}{x+2}-\frac{6}{x(x^2+x-2)}\right)$$

$$= \lim_{x\to -2}\left(\frac{1}{x+2}-\frac{6}{x(x+2)(x-1)}\right)$$

$$= \lim_{x\to -2}\left(\frac{x(x-1)-6}{x(x+2)(x-1)}\right) = \lim_{x\to -2}\frac{x^2-x-6}{x(x+2)(x-1)}$$

$$= \lim_{x\to -2}\frac{(x-3)(x+2)}{x(x+2)(x-1)}$$

$$= \lim_{x\to -2}\frac{x-3}{x(x-1)} \quad \left[\begin{array}{l}\because x\to -2 \Rightarrow x\neq -2 \\ \Rightarrow x+2\neq 0 \text{ so, cancelling} \\ \text{out } x+2\end{array}\right]$$

$$= \frac{-2-3}{-2(-2-1)} = \frac{-5}{6} = -\frac{5}{6}$$

E 4) $\lim_{x\to 2}\frac{\sqrt{3+x}-\sqrt{5}}{x-2}$

Rationalising the numerator, we have

$$\lim_{x\to 2}\frac{\sqrt{3+x}-\sqrt{5}}{x-2} = \lim_{x\to 2}\frac{\sqrt{3+x}-\sqrt{5}}{x-2}\times\frac{\sqrt{3+x}+\sqrt{5}}{\sqrt{3+x}+\sqrt{5}}$$

$$= \lim_{x\to 2}\frac{\left(\sqrt{3+x}\right)^2-\left(\sqrt{5}\right)^2}{(x-2)\left(\sqrt{3+x}+\sqrt{5}\right)} \quad [\because (a-b)(a+b)=a^2-b^2]$$

$$= \lim_{x\to 2}\frac{3+x-5}{(x-2)\left(\sqrt{3+x}+\sqrt{5}\right)} = \lim_{x\to 2}\frac{x-2}{(x-2)\left(\sqrt{3+x}+\sqrt{5}\right)}$$

$$= \lim_{x\to 2}\frac{1}{\sqrt{3+x}+\sqrt{5}} \quad \left[\begin{array}{l}\text{Canceling out the common} \\ \text{factor } x-2\neq 0\end{array}\right]$$

$$= \frac{1}{\sqrt{3+2}+\sqrt{5}} = \frac{1}{\sqrt{5}+\sqrt{5}} = \frac{1}{2\sqrt{5}}$$

$$= \frac{1}{2\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{10} \quad \left[\begin{array}{l}\text{Rationalising the} \\ \text{denominator}\end{array}\right]$$

E 5) (i) $\lim_{x\to\sqrt{2}}\frac{x^{10}-32}{x-\sqrt{2}} = \lim_{x\to\sqrt{2}}\frac{x^{10}-\left(\sqrt{2}\right)^{10}}{x-\sqrt{2}}$

$$= 10\left(\left(\sqrt{2}\right)^{10-1}\right) \quad \left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

$$= 10(\sqrt{2})^9 = 10(\sqrt{2})^8(\sqrt{2}) = 10\times 16\times\sqrt{2} = 160\sqrt{2}$$

(ii) $\lim_{x\to 0}\frac{(ab)^{3x}-1}{x} = \lim_{x\to 0}\frac{(ab)^{3x}-1}{3x}\times 3$

$$= 3\lim_{3x\to 0}\frac{(ab)^{3x}-1}{3x} \quad \text{as } x\to 0 \Rightarrow 3x\to 0$$

$$= 3\log|ab| \quad \left[\because \lim_{x\to 0}\frac{a^x-1}{x} = \log|a|\right]$$

(iii) $\lim\limits_{x\to 0}\dfrac{e^{\sin x}-1}{\tan x}=\lim\limits_{x\to 0}\dfrac{e^{\sin x}-1}{\sin x}\times\dfrac{\sin x}{x}\times\dfrac{x}{\tan x}$

$$=\left(\lim_{\sin x\to 0}\frac{e^{\sin x}-1}{\sin x}\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{x}{\tan x}\right)$$

as $x\to 0\Rightarrow \sin x\to 0$

$$=(1)(1)(1)=1 \qquad \left[\because \lim_{x\to 0}\frac{e^x-1}{x}=1,\ \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \text{ and } \lim_{\theta\to 0}\frac{\theta}{\tan\theta}=1\right]$$

(iv) $\lim\limits_{x\to 0}\dfrac{\log(1+8x^2)}{e^{x^2}-1}=\lim\limits_{x\to 0}\dfrac{\log(1+8x^2)}{8x^2}\times\dfrac{x^2}{e^{x^2}-1}\times 8$

$$=8\left(\lim_{8x^2\to 0}\frac{\log(1+8x^2)}{8x^2}\right)\left(\lim_{x^2\to 0}\frac{x^2}{e^{x^2}-1}\right)$$

$$\left[\because \text{as } x\to 0\ \Rightarrow 8x^2\to 0 \text{ and } x^2\to 0\right]$$

$$=8(1)(1)=8 \qquad \left[\because \lim_{x\to 0}\frac{\log(1+x)}{x}=1 \text{ and } \lim_{x\to 0}\frac{x}{e^x-1}=1\right]$$

(v) $\lim\limits_{x\to 0}\dfrac{a^x-e^x}{x}=\lim\limits_{x\to 0}\dfrac{a^x-1-(e^x-1)}{x}=\lim\limits_{x\to 0}\left(\dfrac{a^x-1}{x}-\dfrac{e^x-1}{x}\right)$

$$=\left(\lim_{x\to 0}\frac{a^x-1}{x}\right)-\left(\lim_{x\to 0}\frac{e^x-1}{x}\right)$$

$$=\log|a|-\log|e| \qquad \left[\because \lim_{x\to 0}\frac{a^x-1}{x}=\log|a| \text{ and } \lim_{x\to 0}\frac{e^x-1}{1}=\log|e|\right]$$

$$=\log\left|\frac{a}{e}\right| \qquad \text{as} \quad \log\frac{m}{n}=\log m-\log n$$

or $\lim\limits_{x\to 0}\dfrac{a^x-e^x}{x}=\log|a|-1$ as $\log e=1$

(vi) Let $I=\lim\limits_{x\to 0}\dfrac{e^x-(1+2x)}{2^x-1}=\lim\limits_{x\to 0}\dfrac{(e^x-1)-2x}{2^x-1}$

Dividing numerator and denominator by x, we get

$$I=\lim_{x\to 0}\frac{\dfrac{e^x-1}{x}-2}{\dfrac{2^x-1}{x}}=\frac{\lim\limits_{x\to 0}\dfrac{e^x-1}{x}-2}{\lim\limits_{x\to 0}\dfrac{2^x-1}{x}}=\frac{1-2}{\log 2}=\frac{-1}{\log 2}$$

(vii) $\lim\limits_{x\to 0}\dfrac{x(1+2x)^{1/x}-(e^{2x}-1)}{x}=\lim\limits_{x\to 0}\left((1+2x)^{1/x}-\dfrac{e^{2x}-1}{x}\right)$

$$= \lim_{x\to 0}(1+2x)^{1/x} - \lim_{x\to 0}\frac{e^{2x}-1}{x}$$

$$= \lim_{x\to 0}\left[(1+2x)^{\frac{1}{2x}}\right]^2 - \lim_{x\to 0}\frac{e^{2x}-1}{2x}\times 2$$

$$= \left[\lim_{2x\to 0}(1+2x)^{\frac{1}{2x}}\right]^2 - 2\lim_{2x\to 0}\frac{e^{2x}-1}{2x}$$

as $x \to 0 \Rightarrow 2x \to 0$

$$= (e)^2 - 2\log e \quad \left[\begin{array}{l}\because \lim_{x\to 0}(1+x)^{1/x} = e \\ \text{and} \lim_{x\to 0}\frac{e^x-1}{x} = \log e\end{array}\right]$$

$$= e^2 - 2 \qquad \text{as } \log e = 1$$

Rule to be Remembered:
$\sqrt[n]{f(x)} = [f(x)]^{1/n}$

E 6) $$\lim_{x\to\infty}\frac{\sqrt{x^2+3}+\sqrt[4]{x^4+2}}{\sqrt[7]{x^7+x^2}-\sqrt[3]{x^2+5}} = \lim_{x\to\infty}\frac{x\left(\sqrt{1+\frac{3}{x^2}}\right)+x\left(\sqrt[4]{1+\frac{2}{x^4}}\right)}{x\left(\sqrt[7]{1+\frac{1}{x^5}}\right)-x\left(\sqrt[3]{\frac{1}{x}+\frac{5}{x^3}}\right)}$$

$$= \lim_{x\to\infty}\frac{\sqrt{1+\frac{3}{x^2}}+\sqrt[4]{1+\frac{2}{x^4}}}{\sqrt[7]{1+\frac{1}{x^5}}-\sqrt[3]{\frac{1}{x}+\frac{5}{x^3}}}$$

$$= \frac{\sqrt{1+0}+\sqrt[4]{1+0}}{\sqrt[7]{1+0}-\sqrt[3]{0+0}} = \frac{1+1}{1-0} = \frac{2}{1} = 2$$

E 7) (i) $\lim_{x\to 0^-}\frac{5x+|x|}{3|x|-7x}$

Putting $x = 0-h$ as $x \to 0^- \Rightarrow h \to 0^+$

$$\lim_{x\to 0^-}\frac{5x+|x|}{3|x|-7x} = \lim_{h\to 0^+}\frac{5(0-h)+|0-h|}{3|0-h|-7(0-h)} = \lim_{h\to 0^+}\frac{-5h+|-h|}{3|-h|+7h}$$

$$= \lim_{h\to 0^+}\frac{-5h+|-1||h|}{3|-1||h|+7h} = \lim_{h\to 0^+}\frac{-5h+h}{3h+7h} = \lim_{h\to 0^+}\frac{-4h}{10h}$$

$$= \lim_{h\to 0^+}\frac{-2}{5} = -\frac{2}{5}$$

(ii) $\lim_{x\to 5^+}(3-|x|)$

Putting $x = 5+h$ as $x \to 5^+ \Rightarrow h \to 0^+$

$\therefore \lim_{x\to 5^+}(3-|x|) = \lim_{h\to 0^+}(3-|5+h|$

$$= \lim_{h\to 0^+}(3-(5+h)) \quad \left[\begin{array}{l}\because \text{as } h \to 0^+ \Rightarrow 5+h > 5 > 0 \\ \Rightarrow |5+h| = 5+h\end{array}\right]$$

$$= \lim_{h\to 0^+}(-2-h) = -2-0 = -2$$

(iii) $\lim\limits_{x\to 0}\frac{x}{|x|}$

L.H.L. $= \lim\limits_{x\to 0^-}\frac{x}{|x|}$

Putting $x = 0 - h$ as $x \to 0^- \Rightarrow h \to 0^+$

L.H.L. $= \lim\limits_{h\to 0^+}\frac{0-h}{|0-h|} = \lim\limits_{h\to 0^+}\frac{-h}{|-h|} = \lim\limits_{h\to 0^+}\frac{-h}{|-1||h|}$

$= \lim\limits_{h\to 0^+}\frac{-h}{h} = \lim\limits_{h\to 0^+}(-1) = -1$ … (1)

R.H.L. $= \lim\limits_{x\to 0^+}\frac{x}{|x|} = \lim\limits_{x\to 0^+}\frac{x}{x} = \lim\limits_{x\to 0^+} 1 = 1$ … (2)

$\left[\begin{array}{l}\because x \to 0^+ \Rightarrow x \text{ is sligtly greater} \\ \text{than } 0, \text{ so } |x| = x\end{array}\right]$

From (1) and (2)

L.H.L. $\neq$ R.H.L.

$\therefore \lim\limits_{x\to 0}\frac{x}{|x|}$ does not exist.

E 8) $f(x) = \begin{cases} ax+3, & x \le 3 \\ 2(x+1), & x > 3 \end{cases}$

L.H.L. $= \lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-}(ax+3) = 3a+3$ … (1)

R.H.L. $= \lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} 2(x+1) = 2(3+1) = 8$ … (2)

Since, it is given that $\lim\limits_{x\to 3} f(x)$ exists.

$\therefore$ we must have

L.H.L. = R.H.L.

$\Rightarrow 3a+3 = 8 \Rightarrow 3a = 5 \Rightarrow a = 5/3$

E 9) $f(x) = \begin{cases} 2x-a, & x \ge 0 \\ ax+b+3, & x < 0 \end{cases}$ at $x = 0$

L.H.L. $= \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}(ax+b+3)$

Putting $x = 0 - h$ as $x \to 0^- \Rightarrow h \to 0^+$

L.H.L. $= \lim\limits_{h\to 0^+}(a(0-h)+b+3) = \lim\limits_{h\to 0^+}(-ah+b+3) = b+3$

R.H.L. $= \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}(2x-a) = 2(0) - a = -a$

Also $f(0) = 2(0) - a = 0 - a = -a$

Since f is given to be continuous at $x = 0$, so we must have

$\text{L.H.L.} = \text{R.H.L.} = f(0)$

$\Rightarrow b + 3 = -a \Rightarrow a + b + 3 = 0$

Which is the required relation between a and b.