UNIT 8 DEFINITE INTEGRATION

Structure

8.1 INTRODUCTION

In Units 6 and 7 we have discussed concept of differentiation and concept of indefinite integral. But on many occasions, we are interested in finding out the probability of a continuous random variable in certain limits, this job is done by using the concept of definite integral.

This unit discusses about definite integral, evaluation of definite integral of some commonly used functions with the help of large number of examples. Properties of definite integral and how these are used have been also discussed in this unit with the help of number of examples.

Objectives

After completing this unit, you should be able to:

- define definite integration and give its geometrical meaning;
- evaluate the integration of some commonly used functions;
- explain the properties of the definite integrations; and
- evaluate the integrations using properties of definite integrations.

8.2 MEANING AND GEOMETRICAL INTERPRETATION

Notation and Definition

You have already studied that if

$$\frac{d}{dx}(F(x)) = f(x), \text{ then } \int f(x)dx = F(x) + c$$

where c is arbitrary constant and hence value of $\int f(x)dx$ is indefinite.

Here, we are going to discuss definite integrals.

Definite integral of a function f(x) within the limits

$a < x < b$ or $a \le x < b$ or $a < x \le b$ or $a \le x \le b$ … (1)

is denoted by $\int_a^b f(x)dx$

and is defined as

> **Fundamental theorem of Integral Calculus:**
> If f is integrable on [a,b] and F is such that
> $\frac{d}{dx}\left(F(x)\right)=f(x),\quad x\in[a,b]$
> then
> $\int_a^b f(x)\,dx = F(b)-F(a)$

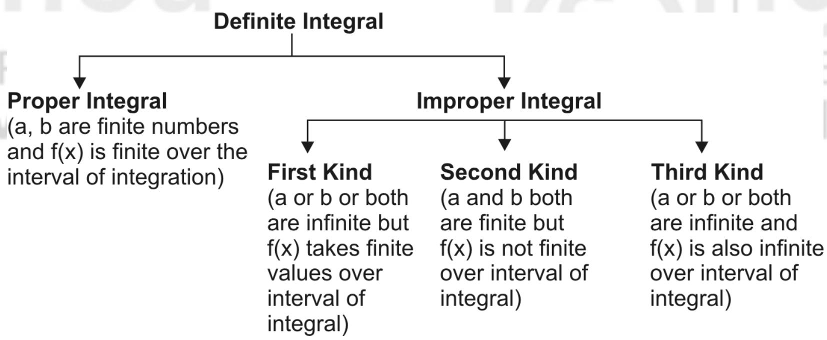
$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a) \qquad \ldots (2) \left[\begin{array}{l}\text{By fundamental theorem}\\ \text{of integral calculus}\end{array}\right]$$

, where a is called lower limit, b is called upper limit and either of the intervals in (1) to be used is known as interval of integration.

For example, $\int_2^5 (x+3)dx = \left[\frac{x^2}{2}+3x+c\right]_2^5 = \frac{(5)^2}{2}+3(5)+c-\left[\frac{2^2}{2}+3\times 2+c\right]$

$$= 25/2+15-2-6 = 39/2$$

The integral $\int_a^b f(x)dx$ being definite integral, so it will have a definite value, but equation (2) suggests that value of L.H.S. of (2) depends on a, b and f(x). Following diagram shows that definite integral have different names based on the role of a, b and f(x).



Here we shall discuss proper integrals and only first kind of improper integrals as these will applicable later on in the subsequent courses.

Note: The notation $[F(x)]_a^b$ means that function F(x) is to be evaluated at top and bottom limits and then subtract. Some authors use American text book notation $|F(x)|_a^b$ instead of $[F(x)]_a^b$. But here we will use square bracket notation.

Geometrical Interpretation

The definite integral $\int_a^b f(x)dx$ represents the area bounded by the function y = f(x), x-axis and between the lines x = a, x = b as shown by the shaded region in the following Fig. 8.1

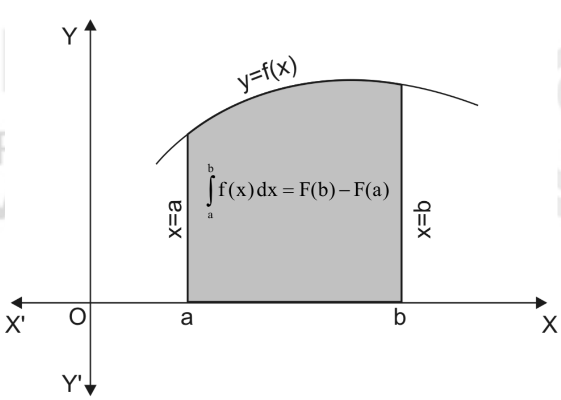


Fig. 8.1

Remark 1: In solving numerical problems, generally c is not used.

For example, if $f(x) = x^2 + x + 6$ then $\int f(x)dx = \frac{x^3}{3} + \frac{x^2}{2} + 6x + c = F(x)$

We will not write it as

$$\int_1^2 f(x)dx = [F(x)]_1^2 = \left[\frac{x^3}{3} + \frac{x^2}{2} + 6x + c\right]_1^2$$

$$= \frac{2^3}{3} + \frac{2^2}{2} + 6\times 2 + c - \left(\frac{1^3}{3} + \frac{1^2}{2} + 6\times 1 + c\right)$$

$$= \frac{8}{3} + 2 + 12 + c - \frac{1}{3} - \frac{1}{2} - 6 - c = \frac{8}{3} + 8 - \frac{1}{3} - \frac{1}{2}$$

$$= \frac{16 + 48 - 2 - 3}{6} = \frac{59}{6}$$

But thoughout the unit, we will write it as

$$\int_1^2 f(x)dx = \int_1^2 (x^2 + x + 6)dx = \left[\frac{x^3}{3} + \frac{x^2}{2} + 6x\right]_1^2$$

$$= \frac{8}{3} + \frac{4}{3} + 12 - \left(\frac{1}{3} + \frac{1}{2} + 6\right) = \frac{59}{6}$$

8.3 DEFINITE INTEGRAL OF SOME COMMONLY USED FUNCTIONS

Let us here consider some examples of definite integrals based on the formulae of indefinite integral already discussed in Unit 7 of this course.

Example 1: Evaluate the following integrals:

(i) $\int_2^6 8dx$ (ii) $\int_0^2 (x^2 + 1)dx$ (iii) $\int_2^3 (3x + 4)dx$

(iv) $\int_2^5 4x^3 dx$ (v) $\int_2^3 x^a dx$ (vi) $\int_2^4 \sqrt{18x - 24}\, dx$

(vii) $\int_1^2 (x+1)(x^2 - 1)dx$ (viii) $\int_2^3 \frac{x^3 + 5}{x^2}dx$ (ix) $\int_2^5 \frac{1}{2x - 3}dx$

(x) $\int_2^3 \frac{4}{x}dx$ (xi) $\int_1^4 2^x dx$ (xii) $\int_0^2 e^{3x} dx$

(xiii) $\int_0^2 3^{4x+1} dx$ (xiv) $\int_1^3 e^{2x+5} dx$

Solution:

(i) $\int_2^6 8dx = [8x]_2^6 = 8\times 6 - 8\times 2 = 48 - 16 = 32$

(ii) $\int_0^2 (x^2 + 1)dx = \left[\frac{x^3}{3} + x\right]_0^2 = \frac{8}{3} + 2 - 0 - 0 = \frac{14}{3}$

(iii) $\int_2^3 (3x+4)dx = \left[\frac{3x^2}{2}+4x\right]_2^3 = \frac{27}{2}+12-6-8 = \frac{27}{2}-2 = \frac{23}{2}$

(iv) $\int_2^5 4x^3 dx = \left[\frac{4x^4}{4}\right]_2^5 = \left[x^4\right]_2^5 = 625-16 = 609$

(v) $\int_2^3 x^a dx = \left[\frac{x^{a+1}}{a+1}\right]_2^3 = \frac{1}{a+1}\left(3^{a+1}-2^{a+1}\right)$

(vi) $\int_2^4 \sqrt{18x-24}\,dx = \left[\frac{(18x-24)^{3/2}}{\frac{3}{2}\times 18}\right]_2^4 \qquad \left[\because \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a\times(n+1)}\right]$

$$= \frac{1}{27}\left[(72-24)^{3/2}-(36-24)^{3/2}\right] = \frac{1}{27}\left[48^{3/2}-(12)^{3/2}\right]$$

$$= \frac{1}{27}\left[48\sqrt{48}-12\sqrt{12}\right] = \frac{1}{27}\left[48\times 4\sqrt{3}-12\times 2\sqrt{3}\right]$$

$$= \frac{24}{27}\left[8\sqrt{3}-\sqrt{3}\right] = \frac{8}{9}(7\sqrt{3}) = \frac{56\sqrt{3}}{9}$$

(vii) $\int_1^2 (x+1)(x^2-1)dx = \int_1^2 (x^3-x+x^2-1)dx$

$$= \left[\frac{x^4}{4}-\frac{x^2}{2}+\frac{x^3}{3}-x\right]_1^2$$

$$= \frac{16}{4}-\frac{4}{2}+\frac{8}{3}-2-\left(\frac{1}{4}-\frac{1}{2}+\frac{1}{3}-1\right)$$

$$= 4-2+\frac{8}{3}-2-\left(\frac{3-6+4-12}{12}\right)$$

$$= \frac{8}{3}+\frac{11}{12} = \frac{32+11}{12} = \frac{43}{12}$$

(viii) $\int_2^3 \frac{x^3+5}{x^2}dx = \int_2^3\left(\frac{x^3}{x^2}+\frac{5}{x^2}\right)dx = \int_2^3 (x+5x^{-2})dx = \left[\frac{x^2}{2}+\frac{5x^{-1}}{-1}\right]_2^3$

$$= \left[\frac{x^2}{2}-\frac{5}{x}\right]_2^3 = \frac{9}{2}-\frac{5}{3}-\left(\frac{4}{2}-\frac{5}{2}\right)$$

$$= \frac{27-10}{6}-\frac{4-5}{2} = \frac{17}{6}+\frac{1}{2} = \frac{17+3}{6} = \frac{20}{6} = \frac{10}{3}$$

(ix) $\int_2^5 \frac{1}{2x-3}dx = \left[\frac{1}{2}\log|2x-3|\right]_2^5 = \frac{1}{2}\left[\log 7-\log 1\right] = \frac{1}{2}\log 7$ as $\log 1 = 0$

(x) $\int_2^3 \frac{4}{x}dx = 4\int_2^3 \frac{1}{x}dx = 4\left[\log|x|\right]_2^3 = 4(\log 3-\log 2) = 4\log\frac{3}{2}$

(xi) $\int_1^4 2^x dx = \left[\frac{2^x}{\log 2}\right]_1^4 = \frac{1}{\log 2}\left[2^4-2^1\right] = \frac{14}{\log 2}$

(xii) $\int_0^2 e^{3x}dx = \left[\frac{e^{3x}}{3}\right]_0^2 = \frac{1}{3}(e^6 - e^0) = \frac{e^6 - 1}{3}$

(xiii) $\int_0^2 3^{4x+1}dx = \left[\frac{3^{4x+1}}{4\log 3}\right]_0^2 = \frac{1}{4\log 3}\left[3^9 - 3^1\right] = \frac{1}{4\log 3}(19683 - 3) = \frac{19680}{4\log 3}$

(xiv) $\int_1^3 e^{2x+5}dx = \left[\frac{e^{2x+5}}{2}\right]_1^3 = \frac{1}{2}(e^{11} - e^7)$

Example 2: Evaluate the following integrals:

(i) $\int_1^6 x\sqrt{x+3}dx$ (ii) $\int_0^1 \frac{2x+3}{x^2+3x+5}dx$

(iii) $\int_0^2 \frac{2x+7}{(x-3)(x+1)(x-4)}dx$ (iv) $\int_0^3 \frac{x-5}{(x+1)(x+2)^2}dx$

Solution:

(i) Let $I = \int_1^6 x\sqrt{x+3}dx$ … (1)

Putting $\sqrt{x+3} = t \Rightarrow x + 3 = t^2$

Differentiating

dx = 2tdt

Also when x = 1, t = 2 and when x = 6, t = 3

$\therefore$ (1) becomes

$$I = \int_2^3 (t^2 - 3)t(2t)dt = 2\int_2^3 (t^4 - 3t^2)dt = 2\left[\frac{t^5}{5} - t^3\right]_2^3$$

$$= 2\left[\frac{243}{5} - 27 - \left(\frac{32}{5} - 8\right)\right] = 2\left(\frac{243}{5} - \frac{32}{5} - 19\right)$$

$$= 2\left(\frac{243 - 32 - 95}{5}\right) = 2\left(\frac{116}{5}\right) = \frac{232}{5}$$

(ii) Let $I = \int_0^1 \frac{2x+3}{x^2+3x+5}dx$ … (1)

Putting $x^2 + 3x + 5 = t$

Differentiating

$(2x+3)dx = dt$

Also when $x = 0,\ t = 5$ and when $x = 1,\ t = 9$

$\therefore$ (1) becomes

$$I = \int_5^9 \frac{dt}{t} = \left[\log|t|\right]_5^9 = \log 9 - \log 5 = \log\frac{9}{5}$$

(iii) Let $I = \int_0^2 \frac{2x+7}{(x-3)(x+1)(x-4)}dx = \int_0^2 \left(\frac{-13/4}{x-3} + \frac{1/4}{x+1} + \frac{3}{x-4}\right)dx$

$$\left[\begin{array}{l}\text{Using partial fractions as discussed in type 1, we get} \\ A = \frac{2\times 3+7}{(3+1)(3-4)} = -\frac{13}{4}, B = \frac{2(-1)+7}{(-1-3)(-1-4)} = \frac{1}{4}, C = \frac{2\times 4+7}{(4-3)(4+1)} = 3\end{array}\right]$$

$$I = -\frac{13}{4}\left[\log|x-3|\right]_0^2 + \frac{1}{4}\left[\log|x+1|\right]_0^2 + 3\left[\log|x-4|\right]_0^2$$

$$= -\frac{13}{4}(\log 1 - \log 3) + \frac{1}{4}(\log 3 - \log 1) + 3(\log 2 - \log 4)$$

$$= -\frac{13}{4}(0 - \log 3) + \frac{1}{4}(\log 3 - 0) + 3(\log 2 - \log 2^2) \quad \text{as } \log 1 = 0$$

$$= \frac{13}{4}\log 3 + \frac{1}{4}\log 3 + 3(\log 2 - 2\log 2) \quad \left[\because \log m^n = n\log m\right]$$

$$= \frac{1}{4}(13\log 3 + \log 3) + 3(-\log 2)$$

$$= \frac{14\log 3}{4} - 3\log 2 = \frac{7}{2}\log 3 - 3\log 2$$

(iv) Let $I = \int_0^3 \frac{x-5}{(x+1)(x+2)^2}\,dx$ … (1)

First we resolve into partial fractions

Let $\frac{x-5}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

Multiply on both sides by $(x+1)(x+2)^2$, we get

$x - 5 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$ … (2)

Putting $x = -1$ in (2), we get $\quad [\because x+1 = 0 \text{ gives } x = -1]$

$-6 = A(-1+2)^2 + B(0) + C(0) \Rightarrow -6 = A \Rightarrow \boxed{A = -6}$

Putting $x = -2$ in (2), we get $\quad [\because x+2 = 0 \text{ gives } x = -2]$

$-7 = A(0) + B(0) + C(-2+1) \Rightarrow -7 = -C \Rightarrow \boxed{C = 7}$

Comparing coefficient of x^2 on both sides of (2), we get

$0 = A + B \Rightarrow B = -A \Rightarrow \boxed{B = 6}$

$$\therefore I = \int_0^3 \left[\frac{-6}{x+1} + \frac{6}{x+2} + \frac{7}{(x+2)^2}\right]dx$$

$$= -6\left[\log|x+1|\right]_0^3 + 6\left[\log|x+2|\right]_0^3 + 7\left[\frac{(x+2)^{-1}}{-1}\right]_0^3$$

$$= -6(\log 4 - \log 1) + 6(\log 5 - \log 2) - 7\left(\frac{1}{5} - \frac{1}{2}\right)$$

$$= -6\log 2^2 + 6\log 5 - 6\log 2 - 7\left(\frac{2-5}{10}\right) \quad \text{as } \log 1 = 0$$

$$= -12\log 2 + 6\log 5 - 6\log 2 + \frac{21}{10} \quad [\because \log m^n = n\log m]$$

$$= 6\log 5 - 18\log 2 + \frac{21}{10}$$

Now, you can try the following exercises.

E 1) Evaluate the following integrals:

(i) $\int_2^4 x^2 dx$ (ii) $\int_0^2 \left(x - \frac{5}{2}\right) dx$

E 2) Evaluate the following integrals:

(i) $\int_2^5 \frac{x}{x^2+3} dx$ (ii) $\int_0^2 \frac{e^x}{3+e^{2x}} dx$ (iii) $\int_0^1 \frac{2x^2+1}{(x-4)(x-2)^3} dx$

8.4 ELEMENTARY PROPERTIES OF DEFINITE INTEGRAL

Here, first we list some properties and then we will use these properties to evaluate some integrals.

P 1 $\int_a^b f(x)dx = \int_a^b f(t)dt$ (Change of variable property)

P 2 $\int_a^b f(x)dx = -\int_b^a f(x)dx$ (Interchange of limits property)

P 3 $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \qquad a < c < b$

In general

We can introduce any number of points between a and b

e.g. $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \ldots + \int_{c_{n-1}}^{c_n} f(x)dx + \int_{c_n}^b f(x)dx$

where, $a < c_1 < c_2 < \ldots < c_{n-1} < c_n < b$

P 4 $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is an even function} \\ 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$

P 5 $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

In particular, $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

P 6 $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

P7 $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

Proof:

P 1 Let $\int f(x)dx = F(x)$, so $\int f(t)dt = F(t)$

Now, by fundamental theorem of integral calculus

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a) \quad \ldots (1)$$

and $$\int_a^b f(t)dt = [F(t)]_a^b = F(b) - F(a) \quad \ldots (2)$$

From (1) and (2)

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

P 2 Let $\int f(x)dx = F(x)$

$\therefore$ by fundamental theorem of integral calculus

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a) \quad \ldots (1)$$

and $$\int_b^a f(x)dx = [F(x)]_b^a = F(a) - F(b) = -[F(b) - F(a)] \quad \ldots (2)$$

From (1) and (2)

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

P 3 Let $\int f(x)dx = F(x)$

$\therefore$ by fundamental theorem of integral calculus

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a) \quad \ldots (1)$$

and $$\int_a^c f(x)dx + \int_c^b f(x)dx = [F(x)]_a^c + [F(x)]_c^b = (F(c) - F(a)) + (F(b) - F(c))$$

$$= F(b) - F(a) \quad \ldots (2)$$

From (1) and (2)

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

P 4 Using property 3, we have

$$\int_{-a}^a f(x) = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx \quad \ldots (1)$$

Let $I = \int_{-a}^0 f(x)dx$

Putting $x = -t$

Differentiating

$dx = -dt$

Also, when $x = -a, \ t = a$ and when $x = 0, \ t = 0$

$$\therefore I = -\int_a^0 f(-t)dt = \int_0^a f(-x)dx \quad \ldots (2) \qquad \text{[Using properties1 and 2]}$$

Using (2) in (1) , we get

$$\int_{-a}^{a} f(x)dx = \int_0^a f(-x)dx + \int_0^a f(x)dx \quad \ldots (3)$$

$$= \begin{cases} \int_0^a f(x)dx + \int_0^a f(x)dx, & \text{if f is even} \\ -\int_0^a f(x)dx + \int_0^a f(x)dx, & \text{if f is odd} \end{cases}$$

$$= \begin{cases} 2\int_0^a f(x)dx, & \text{if f is even function} \\ 0, & \text{if f is an odd function} \end{cases}$$

P 5 Let $I = \int_a^b f(x)dx \quad \ldots (1)$

R.H.S. suggests that we should put

$x = a + b - t$

Differentiating

$dx = -dt$

Also, when x = a $\Rightarrow$ t = b and when x = b $\Rightarrow$ t = a

$\therefore$ (1) becomes

$$I = \int_b^a f(a+b-t)(-dt) = -\int_b^a f(a+b-t)dt$$

$$= \int_a^b f(a+b-t)dt \qquad \text{[Using property 2]}$$

$$= \int_a^b (a+b-x)dx \qquad \text{[Using property 1]}$$

In particular

If we put $a = 0, b = a$ in this result, then

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

P 6 $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx \quad$ [Using property 3]

$$= I_1 + I_2 \quad \ldots (1)$$

$$I_2 = \int_a^{2a} f(x)dx$$

Putting x = 2a – t

Differentiating

$dx = -dt$

Also, when $x = a, t = a$ and when $x = 2a, t = 0$

$$\therefore I_2 = \int_a^0 f(2a-t)(-dt) = \int_0^a f(2a-t)dt \qquad \text{[Using property 2]}$$

$$= \int_0^a (2a-x)dx \qquad \ldots (2) \quad \text{[Using property 1]}$$

Using (2) in (1), we get

$$\int_0^{2a} f(x) = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

P7 From property 6

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

$$= \begin{cases} \int_0^a f(x)dx + \int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ \int_0^a f(x)dx - \int_0^a f(x)dx, & \text{if } f(2a-x) = -f(x) \end{cases}$$

$$= \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

8.5 EXAMPLES BASED ON PROPERTIES OF DEFINITE INTEGRAL

In this section, you will see how the properties of definite integral, discussed in previous Sec. are used and save lot of calculation work.

Example 3: Evaluate the following integrals:

(i) $\int_1^4 |x-2|dx$ (ii) $\int_0^2 |2x-3|dx$ (iii) $\int_0^3 f(x)dx$, where $f(x) = \begin{cases} x+1, & 0 \le x < 1 \\ 2x+3, & 1 \le x < 3 \end{cases}$

(iv) $\int_{-99}^{99} (x^3 + x + e^x - e^{-x})dx$ (v) $\int_{-2}^{2} \log\left(\frac{5-x}{5+x}\right)dx$ (vi) $\int_{-3}^{3} 2^{|2x|}dx$

(vii) $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx$ (viii) $\int_2^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{5-x}}dx$ (ix) $\int_2^7 \frac{\sqrt[5]{x+1}}{\sqrt[5]{x+1}+\sqrt[5]{10-x}}dx$

Solution:

(i) Let $I = \int_1^4 |x-2|dx = \int_1^2 |x-2|dx + \int_2^4 |x-2|dx$ [By P3]

$$I = \int_1^2 -(x-2)dx + \int_2^4 (x-2)dx \qquad \left[\begin{array}{l} \because \text{for } 1 \le x < 2, x-2 < 0 \text{ so} \\ |x-2| = -(x-2) \\ \text{and for } 2 \le x \le 4, x-2 \ge 0 \text{ so} \\ |x-2| = x-2 \end{array}\right]$$

$$= -\left[\frac{x^2}{2} - 2x\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4 = -\left(\frac{4}{2} - 4 - \left(\frac{1}{2} - 2\right)\right) + \left(\frac{16}{2} - 8 - \left(\frac{4}{2} - 4\right)\right)$$

$$= -\left(2 - 4 - \frac{1}{2} + 2\right) + (8 - 8 - 2 + 4) = \frac{1}{2} + 2 = \frac{5}{2}$$

(ii) $\int_0^2 |2x-3|dx = \int_0^{3/2} |2x-3|dx + \int_{3/2}^2 |2x-3|dx$ [By P 3]

$$= \int_0^{3/2} -(2x-3)dx + \int_{3/2}^2 (2x-3)dx$$

$$\left[\begin{array}{l}\because \text{for } 0 \le x < 3/2 \Rightarrow 2x-3 < 0 \text{ so } |2x-3| = -(2x-3) \text{ and} \\ \text{for } 3/2 \le x \le 2 \Rightarrow 2x-3 \ge 0 \text{ so } |2x-3| = 2x-3\end{array}\right]$$

$$= -\left[x^2 - 3x\right]_0^{3/2} + \left[x^2 - 3x\right]_{3/2}^2$$

$$= -\left(\frac{9}{4} - \frac{9}{2} - 0 + 0\right) + \left(4 - 6 - \frac{9}{4} + \frac{9}{2}\right)$$

$$= -\left(-\frac{9}{4}\right) - 2 + \frac{9}{4} = \frac{9}{4} - 2 + \frac{9}{4} = \frac{9-8+9}{4} = \frac{10}{4} = \frac{5}{2}$$

(iii) Let $I = \int_0^3 f(x)dx$, where $f(x) = \begin{cases} x+1, & 0 \le x < 1 \\ 2x+3, & 1 \le x < 3 \end{cases}$ … (1)

Now, $I = \int_0^1 f(x)dx + \int_1^3 f(x)dx$ [Using property 3]

$$= \int_0^1 (x+1)dx + \int_1^3 (2x+3)dx \quad \text{[Using (1)]}$$

$$= \left[\frac{x^2}{2} + x\right]_0^1 + \left[x^2 + 3x\right]_1^3$$

$$= \frac{1}{2} + 1 - 0 - 0 + (9 + 9 - 1 - 3) = \frac{3}{2} + 14 = \frac{31}{2}$$

(iv) Let $I = \int_{-99}^{99} (x^3 + x + e^x - e^{-x})dx$

Let $f(x) = x^3 + x + e^x - e^{-x}$

$\therefore f(-x) = (-x)^3 + (-x) + e^{-x} - e^{-(-x)} = -x^3 - x + e^{-x} - e^x$

$= -(x^3 + x + e^x - e^{-x}) = -f(x)$

$\Rightarrow f(x)$ is an odd function

$\therefore I = \int_{-99}^{99} (x^3 + x + e^x + e^{-x})dx = 0$ [By property 4]

(v) Let $I = \int_{-2}^{2} \log\left(\frac{5-x}{5+x}\right)dx$

Let $f(x) = \log\left(\frac{5-x}{5+x}\right)$

$$\therefore f(-x) = \log\left(\frac{5-(-x)}{5+(-x)}\right) = \log\left(\frac{5+x}{5-x}\right) = \log\left(\frac{5-x}{5+x}\right)^{-1}$$

$$= -\log\left(\frac{5-x}{5+x}\right) = -f(x) \qquad [\because \log m^n = n\log m]$$

$\Rightarrow$ f(x) is an odd function

$$\therefore I = \int_{-2}^{2} \log\left(\frac{5-x}{5+x}\right) = 0 \qquad \text{[By property 4]}$$

(vi) $$\int_{-3}^{3} 2^{|2x|}dx = \int_{-3}^{0} e^{|2x|}dx + \int_{0}^{3} e^{|2x|}dx$$

$$= \int_{-3}^{0} e^{-2x}dx + \int_{0}^{3} e^{2x}dx \begin{bmatrix} \because \text{for } -3 < x < 0, 2x < 0 \text{ so } |2x| = -2x \text{ and} \\ \text{for } 0 < x < 3, 2x > 0 \text{ so } |2x| = 2x \end{bmatrix}$$

$$= \left[\frac{e^{-2x}}{-2}\right]_{-3}^{0} + \left[\frac{e^{2x}}{2}\right]_{0}^{3} = -\frac{1}{2}(1-e^6) + \frac{1}{2}(e^6 - 1)$$

$$= \frac{1}{2}(-1 + e^6 + e^6 - 1) = \frac{2e^6 - 2}{2} = e^6 - 1$$

(vii) Let $$I = \int_a^b \frac{f(x)}{f(x) + f(a+b-x)}dx \qquad \ldots (1)$$

$$= \int_a^b \frac{f(a+b-x)}{f(a+b-x) + f(a+b-(a+b-x))}dx \qquad \text{[Using property 5]}$$

$$I = \int_a^b \frac{f(a+b-x)}{f(a+b-x) + f(x)}dx \qquad \ldots (2)$$

(1) + (2) gives

$$2I = \int_a^b \frac{f(x) + f(a+b-x)}{f(x) + f(a+b-x)}dx = \int_a^b 1dx = [x]_a^b = b - a$$

$$\Rightarrow I = \frac{b-a}{2}$$

(viii) Let $$I = \int_2^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{5-x}}dx \qquad \ldots (1)$$

$$= \int_2^3 \frac{\sqrt{5-x}}{\sqrt{5-x} + \sqrt{5-(5-x)}}dx \qquad \text{[Using property 5]}$$

$$I = \int_2^3 \frac{\sqrt{5-x}}{\sqrt{5-x} + \sqrt{x}}dx \qquad \ldots (2)$$

(1) + (2) gives

$$2I = \int_2^3 \frac{\sqrt{x} + \sqrt{5-x}}{\sqrt{x} + \sqrt{5-x}}dx = \int_2^3 1dx = [x]_2^3 = 3 - 2 = 1$$

$$\Rightarrow I = 1/2$$

(ix) Let $$I = \int_2^7 \frac{\sqrt[5]{x+1}}{\sqrt[5]{x+1} + \sqrt[5]{10-x}}dx \qquad \ldots (1)$$

$$= \int_2^7 \frac{\sqrt[5]{9-x+1}}{\sqrt[5]{9-x+1}+\sqrt[5]{10-(9-x)}}dx \quad \text{[Using property 5]}$$

$$I = \int_2^7 \frac{\sqrt[5]{10-x}}{\sqrt[5]{10-x}+\sqrt[5]{1+x}}dx \quad \ldots (2)$$

(1) + (2) gives

$$2I = \int_2^7 \frac{\sqrt[5]{x+1}+\sqrt[5]{10-x}}{\sqrt[5]{10-x}+\sqrt[5]{x+1}}dx = \int_2^7 1dx = [x]_2^7 = 7-2=5$$

$$\Rightarrow I = 5/2$$

Now, you can try the following exercise.

E 3 Evaluate the following integrals:

(i) $\int_0^5 |x^2-3x+2|dx$ (ii) $\int_1^5 (|x-1|+|x+2|+|x-3|)dx$

(iii) $\int_{-1}^2 f(x)dx$, where $f(x) = \begin{cases} 4-3x, & -1 \le x < 1 \\ 2x+1, & 1 \le x \le 2 \end{cases}$

(iv) $\int_{-77}^{77} x^5\sqrt{a^2-x^2}dx$ (v) $\int_{-1}^{1} \frac{x^5+x^7}{4-x^4}dx$ (vi) $\int_0^2 x(2-x)^{11/2}dx$ (vii) $\int_{-3}^{3} 4x|x|dx$

(viii) $\int_1^5 f(x)dx$, where $f(x) = \begin{cases} 1, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \\ 3, & 3 \le x < 4 \\ 4, & 4 \le x < 5 \\ 5, & x \ge 5 \end{cases}$

8.6 SUMMARY

Let us summarise the topics that we have covered in this unit:

1) Integration of some particular functions like

   $k, x^n (n \ne -1), \frac{1}{x}, (ax+b)^n, \frac{1}{ax+b}$ polynomial and exponential functions.

2) Definite integral by use of substitution and partial fraction.

3) Elementary properties of definite integral.

4) Examples based on elementary properties of the definite integral.

8.7 SOLUTIONS/ANSWERS

E 1) (i) $\int_2^4 x^2dx = \left[\frac{x^3}{3}\right]_2^4 = \frac{1}{3}(64-8) = \frac{56}{3}$

(ii) $\int_0^2 \left(x-\frac{5}{2}\right)dx = \left[\frac{x^2}{2}-\frac{5}{2}x\right]_0^2 = \frac{4}{2}-\frac{5}{2}\times 2-0-0 = 2-5=-3$

E 2) (i) Let $I = \int_2^5 \frac{x}{x^2+3}dx$ … (1)

Putting $x^2 = t$

Differentiating

$2xdx = dt \Rightarrow xdx = dt/2$

Also, when $x = 2$, $t = 4$ and when $x = 5$, $t = 25$

$$\therefore I = \frac{1}{2}\int_4^{25}\frac{dt}{t+3} = \frac{1}{2}\left[\log|t+3|\right]_4^{25} = \frac{1}{2}(\log 28 - \log 7) = \frac{1}{2}\log\left(\frac{28}{7}\right)$$

$$= \frac{1}{2}\log 4 = \frac{1}{2}\log 2^2 = \frac{1}{2}\times 2\log 2 = \log 2$$

(ii) Let $I = \int_0^2 \frac{e^{2x}}{3+e^{2x}}dx$ …(1)

Putting $e^{2x} = t$

Differentiating

$2e^{2x}dx = dt \Rightarrow e^{2x}dx = \frac{dt}{2}$

Also, when $x = 0, t = 1$ and when $x = 2, t = e^4$

$$\therefore I = \frac{1}{2}\int_1^{e^4}\frac{dt}{3+t} = \frac{1}{2}\left[\log|3+t|\right]_1^{e^4} = \frac{1}{2}\left[\log(3+e^4) - \log 4\right] = \frac{1}{2}\log\left(\frac{3+e^4}{4}\right)$$

(iii) $\int_0^1 \frac{2x^2+1}{(x-4)(x-2)^3}dx$

Let us first resolve into partial fractions

Let $\frac{2x^2+1}{(x-4)(x-2)^3} = \frac{A}{x-4} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$

Multiplying on both sides by $(x-4)(x-2)^3$, we get

$$2x^2 + 1 = A(x-2)^3 + B(x-4)(x-2)^2 + C(x-4)(x-2) + D(x-4) \quad ...(2)$$

Putting $= 4$ in (2) we get $[\because x - 4 = 0 \text{ gives } x = 4]$

$33 = A(4-2)^2 + B(0) + C(0) + D(0) \Rightarrow 33 = 4A \Rightarrow \boxed{A = \frac{33}{4}}$

Putting $x = 2$ in (2), we get $[\because x - 2 = 0 \text{ gives } x = 2]$

$9 = A(0) + B(0) + C(0) + D(2-4) \Rightarrow 9 = -2D \Rightarrow \boxed{D = -\frac{9}{2}}$

Comparing coefficients of x^3 and constant terms on both sides of (2), we get

$0 = A + B \Rightarrow B = -A \Rightarrow \boxed{B = -\frac{33}{4}}$

$1 = -8A - 16B + 8C - 4D$ … (3)

Putting values of A, B and D in (3), we get

$$1=-8\times\frac{33}{4}-16\left(-\frac{33}{4}\right)+8C-4\left(-\frac{9}{2}\right)$$

$$1=-66+132+8C+18\Rightarrow 8C=-83\ \Rightarrow \boxed{C=\frac{-83}{8}}$$

$$\therefore I=\int_0^1\left[\frac{33/4}{x-4}+\frac{-33/4}{x-2}+\frac{-83/8}{(x-2)^2}+\frac{-9/2}{(x-2)^3}\right]dx$$

$$=\frac{33}{4}\Big[\log|x-4|\Big]_0^1-\frac{33}{4}\Big[\log|x-2|\Big]_0^1-\frac{83}{8}\left[\frac{(x-2)^{-1}}{-1}\right]_0^1$$

$$-\frac{9}{2}\left[\frac{(x-2)^{-2}}{-2}\right]_0^1$$

$$=\frac{33}{4}\big(\log|1-4|-\log|0-4|\big)-\frac{33}{4}(\log|1-2|-\log|0-2|)$$

$$+\frac{83}{8}\left[\frac{1}{1-2}-\frac{1}{0-2}\right]+\frac{9}{4}\left[\frac{1}{(1-2)^2}-\frac{1}{(0-2)^2}\right]$$

$$=\frac{33}{4}(\log 3-\log 4)-\frac{33}{4}(\log 1-\log 2)+\frac{83}{8}\left(-1+\frac{1}{2}\right)+\frac{9}{4}(1-\frac{1}{4})$$

$$=\frac{33}{4}(\log 3-\log 2^2)-\frac{33}{4}(0-\log 2)-\frac{83}{16}+\frac{27}{16}\quad \text{as } \log 1=0$$

$$=\frac{33}{4}\log 3-\frac{33}{2}\log 2+\frac{33}{4}\log 2-\frac{56}{16}\qquad [\text{as } \log m^n=n\log m]$$

$$=\frac{33}{4}\log 3-\frac{33}{4}\log 2-\frac{56}{16}$$

E 3) (i) Let $I=\int_0^5\left|x^2-3x+2\right|dx\ =\int_0^5|(x-2)(x-1)|dx$

$$=\int_0^1|(x-2)(x-1)|dx+\int_1^2|(x-2)(x-1)|dx+\int_2^5|(x-2)(x-1)|dx$$

$$=\int_0^1(x-2)(x-1)dx+\int_1^2-(x-2)(x-1)dx+\int_2^5(x-2)(x-1)dx$$

$$\left[\begin{array}{l}\because \text{for } 0\le x<1,\ (x-2)(x-1)>0 \text{ so} |(x-2)(x-1)|=(x-2)(x-1)\\ \text{for } 1\le x<2,\ (x-2)(x-1)\le 0 \text{ so} |(x-2)(x-1)|=-(x-1)(x-2) \text{ and}\\ \text{for } 2\le x<5,\ (x-2)(x-1)\ge 0 \text{ so} |(x-2)(x-1)|=(x-2)(x-1)\end{array}\right]$$

$$=\int_0^1(x^2-3x+2)dx-\int_1^2(x^2-3x+2)dx+\int_2^5(x^2-3x+2)dx$$

$$=\left[\frac{x^3}{3}-\frac{3x^2}{2}+2x\right]_0^1-\left[\frac{x^3}{3}-\frac{3x^2}{2}+2x\right]_1^2+\left[\frac{x^3}{3}-\frac{3x^2}{2}+2x\right]_2^5$$

$$= \left[\frac{1}{3} - \frac{3}{2} + 2 - (0 - 0 + 0)\right] - \left[\left(\frac{8}{3} - 6 + 4\right) - \left(\frac{1}{3} - \frac{3}{2} + 2\right)\right]$$

$$+ \left[\left(\frac{125}{3} - \frac{75}{2} + 10\right) - \left(\frac{8}{3} - 6 + 4\right)\right]$$

$$= \frac{2 - 9 + 12}{6} - \left(\frac{8}{3} - 2 - \frac{1}{3} + \frac{3}{2} - 2\right) + \left(\frac{125}{3} - \frac{75}{2} + 10 - \frac{8}{3} + 2\right)$$

$$= \frac{5}{6} - \frac{16 - 12 - 2 + 9 - 12}{6} + \frac{250 - 225 + 60 - 16 + 12}{6}$$

$$= \frac{5}{6} - \frac{-1}{6} + \frac{81}{6} = \frac{5 + 1 + 81}{6} = \frac{87}{6} = \frac{29}{2}$$

(ii) Let $I = \int_1^5 \left(|x-1| + |x+2| + |x-3|\right)dx$

$$= \int_1^5 |x-1|dx + \int_1^5 |x+2|dx + \int_1^5 |x-3|dx$$

$$= \int_1^5 (x-1)dx + \int_1^5 (x+2)dx + \int_1^3 |x-3|dx + \int_3^5 |x-3|dx$$

$$\left[\begin{array}{l} \because \text{for } 1 < x < 5, x - 1 > 0 \\ \text{so } |x-1| = x - 1 \end{array}\right]$$

$$= \left[\frac{x^2}{2} - x\right]_1^5 + \left[\frac{x^2}{2} + 2x\right]_1^5 + \int_1^3 -(x-3)dx + \int_3^5 (x-3)dx$$

$$\left[\begin{array}{l} \because \text{for } 1 < x < 3, x - 3 < 0 \text{ so } |x-3| = -(x-3) \text{ and} \\ \text{for } 3 \le x \le 5, x - 3 \ge 0 \text{ so } |x-3| = x - 3 \end{array}\right]$$

$$= \left(\frac{25}{2} - 5 - \frac{1}{2} + 1\right) + \left(\frac{25}{2} + 10 - \frac{1}{2} - 2\right) - \left[\frac{x^2}{2} - 3x\right]_1^3 + \left[\frac{x^2}{2} - 3x\right]_3^5$$

$$= \frac{25 - 10 - 1 + 2}{2} + \frac{25 + 20 - 1 - 4}{2} - \left(\frac{9}{2} - 9 - \frac{1}{2} + 3\right) + \left(\frac{25}{2} - 15 - \frac{9}{2} + 9\right)$$

$$= \frac{16}{2} + \frac{40}{2} - \left(\frac{9 - 18 - 1 + 6}{2}\right) + \left(\frac{25 - 30 - 9 + 18}{2}\right)$$

$$= 8 + 20 - \frac{-4}{2} + \frac{4}{2} = 28 + 2 + 2 = 32$$

(iii) Let $I = \int_{-1}^{2} f(x)dx,$

where $f(x) = \begin{cases} 4 - 3x, & -1 \le x < 1 \\ 2x + 1, & 1 \le x \le 2 \end{cases}$ … (1)

Now,

$I = \int_{-1}^{1} f(x)dx + \int_{1}^{2} f(x)dx$ [Using property (2)]

$$= \int_{-1}^{1}(4-3x)dx + \int_{1}^{2}(2x+1)dx \quad \text{[Using (1)]}$$

$$= \left[4x - \frac{3x^2}{2}\right]_{-1}^{1} + \left[x^2 + x\right]_1^2 = 4 - \frac{3}{2} - \left(-4 - \frac{3}{2}\right) + 4 + 2 - 1 - 1$$

$$= \frac{8-3}{2} - \left(\frac{-8-3}{2}\right) + 4 = \frac{5}{2} + \frac{11}{2} + 4 = \frac{5+11+8}{2} = \frac{24}{2} = 12$$

(iv) Let $I = \int_{-77}^{77} x^5\sqrt{a^2 - x^2}\,dx$

Let $f(x) = x^5\sqrt{a^2 - x^2}$

$\therefore f(-x) = (-x)^5\sqrt{a^2 - (-x)^2} = -x^5\sqrt{a^2 - x^2} = -f(x)$

$\Rightarrow f(x)$ is an odd function.

$\therefore I = \int_{-77}^{77} x^5\sqrt{a^2 - x^2}\,dx = 0$ [By property 4]

(v) $\int_{-1}^{1}\frac{x^5 + x^7}{4 - x^4}dx$

Let $f(x) = \frac{x^5 + x^7}{4 - x^4}$

$f(-x) = \frac{(-x)^5 + (-x)^7}{4 - (-x)^4} = \frac{-x^5 - x^7}{4 - x^4} = -\frac{x^5 + x^7}{4 - x^4} = -f(x)$

$\Rightarrow$ f(x) is an odd function.

$\therefore \int_{-1}^{1}\frac{x^5 + x^7}{4 - x^4}dx = 0$ [By property 4]

(vi) Let $I = \int_{0}^{2} x(2-x)^{11/2}dx = \int_{0}^{2}(2-x)\left(2-(2-x)\right)^{11/2}dx$ [Using property 5]

$$= \int_{o}^{2}(2-x)(x)^{11/2}dx = \int_{0}^{2}(2x^{11/2} - x^{13/2})dx$$

$$= \left[2\frac{x^{13/2}}{13/2} - \frac{x^{15/2}}{15/2}\right]_0^2 = \frac{4}{13} \times 2^{13/2} - \frac{2}{15} \times (2)^{15/2} - 0 + 0$$

$$= \frac{4}{13} \times 2^6 \times \sqrt{2} - \frac{2}{15} \times 2^7 \times \sqrt{2} = 2^8\left(\frac{1}{13} - \frac{1}{15}\right)\sqrt{2}$$

$$= 2^8\left(\frac{15-13}{13 \times 15}\right)\sqrt{2} = \frac{2^8 \times 2}{195}\sqrt{2} = \frac{512}{195}\sqrt{2}$$

(vii) Let $I = \int_{-3}^{3} 4x|x|dx$

Let $f(x) = 4x|x|$

$f(-x) = 4(-x)|-x| = -4x|(-1)x| = -4x|-1| \times |x| = -4x|x| = -f(x)$

$\Rightarrow f(x)$ is an odd function.

$\therefore I = \int_{-3}^{3} 4x|x|dx = 0$ [Using property 4]

(viii) $\int_{1}^{5} f(x)dx$,

$$\text{where } f(x) = \begin{cases} 1, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \\ 3, & 3 \le x < 4 \\ 4, & 4 \le x < 5 \\ 5, & x \ge 5 \end{cases} \quad \ldots (1)$$

$$\text{Now, } \int_{1}^{5} f(x)dx = \int_{1}^{2} f(x)dx + \int_{2}^{3} f(x)dx + \int_{3}^{4} f(x)dx + \int_{4}^{5} f(x)dx$$

[Using property 2]

$$= \int_{1}^{2} 1dx + \int_{2}^{3} 2dx + \int_{3}^{4} 3dx + \int_{4}^{5} 4dx \text{ [Using (1)]}$$

$$= [x]_1^2 + [2x]_2^3 + [3x]_3^4 + [4x]_4^5$$

$$= (2-1) + (6-4) + (12-9) + (20-16)$$

$$= 1 + 2 + 3 + 4 = 10$$