UNIT 5 NUMBER SYSTEMS, EXPONENTS AND LOGARITHMS

Structure

5.1 INTRODUCTION

Numbers are an essential element of our everyday thought and language. In present day life every activity requires the use of numbers. Prices and weights of commodities, measurement, scores of games, travel time tables, etc., are all expressed in numbers. Man owes much of his success or progress to the art of computing with numbers.

The present course of mathematics in schools introduces the students to the various properties of natural numbers, integers, rational numbers, and real numbers. They are essential for arithmetic, algebra and geometry and their study begins from class I and continues till class VIII. It is thus assumed that the student who enters class IX, i.e., the secondary stage, has a usable knowledge of all these types of number systems.

However, the teaching of number systems and their properties is dominated by the learning of rule-based algorithms or routine processes which is very detrimental to the enjoyment of mathematics and what it is all about. Rule-based instructions are of very little value. They appeal to the dull but may repel the intelligent. This unit introduces the system of real numbers along with its sub-systems. The relationship among these systems is sketched so that there are no gaps in their treatment. The concepts are explained and illustrated with examples drawn from familiar situations. The idea of proof has been interwoven to illustrate the consistency of the systems and to provide reason for arbitrary rules which otherwise may be psychologically damaging. It is expected that teachers will be able to plan their lessons more effectively after going through this unit.

5.2 OBJECTIVES

At the end of the unit, the teacher will be able to :

- understand the system of real numbers and the relationships between its various sub-systems;

- see the relevance of the central position which the system of real numbers occupies in mathematics;
- gain an understanding of algorithms and the routine processes of carrying out computations
- provide reasons for various routine rules, conventions, and techniques used in working with four fundamental operations:
- help students to explore/investigate the structure of the real number system with the help of examples and diagrams,
- illustrate clearly with diagrams (on a number line) the meaning and relationships between the four fundamental operations on real numbers,
- help students to acquire an indepth knowledge of the system of real numbers;
- appreciate the role of numbers in social life and as an essential ingredient of our everyday thought and language.

5.3 NUMBER SYSTEMS

It may be noted that "number" is different from "numeral". Thus, the teaching of the number system incorporates the idea of numeration but is distinct from it. We shall assume the knowledge of the base-ten numeration system while discussing number systems in this unit.

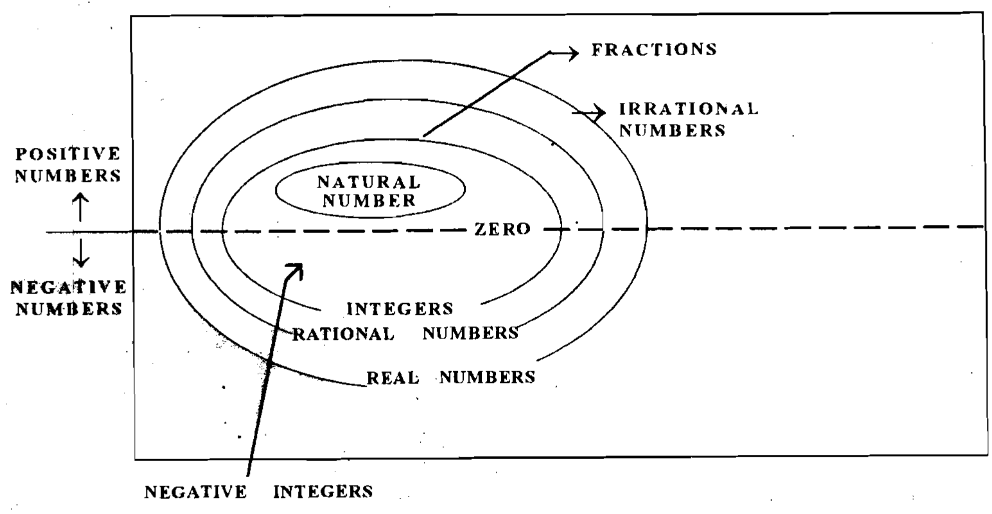
Further, we shall use the idea of "set", an idea which is basic and central to all mathematics The language of sets simplifies and clarifies explanations. It is also assumed that before reaching the secondary stage students have sufficiently worked with numbers and are familiar with basic ideas, at least intuitively The basic knowledge of concepts such as one-to-one matching, counting, addition, subtraction, multiplication division fractions, decimals etc., has been assumed If necessary, teachers may recapitulate these ideas before teaching this unit.

5.3.1 Sets of Numbers

Main Teaching Point : The relationship between different sets of numbers.

Teaching-Learning Process

1. Let pupils recall that "fractions", "negative numbers", "zero", "decimals", etc., are different kinds of numbers. Ask them to give examples/instances to illustrate their use
2. Encourage pupils to observe that these different kinds of numbers overlap and are related to each other. Ask them to arrange these numbers in order starting with the simplest. A diagram such as the following may be useful.



3. Discuss the above diagram with the help of questions such as -
   - Are natural numbers positive?
   - Is it true that {fractions} $\cap$ {integers} = ϕ ?
   - Why is zero on the dotted line separating positive and negative numbers?
   - Is there a one-to-one correspondence between the set of positive integers and the set of negative integers?

Methodology used : The discussion method is used to bring out the relationship between different sets of numbers using diagrams and their analysis. A quiz organised in the class can be an effective technique for recapitulation.

5.3.2 Natural Numbers : Counting Numbers

Main Teaching Point : Properties of natural numbers.

Teaching-Learning Process

The teaching of natural numbers can be approached in two ways, either by following the history of the development of number idea, or by simulating the same through activities in the classroom. In both the cases, the key idea to be exploited **is a one-to-one correspondence between objects in a group and notches or tallymarks to keep a record of the sizes of group.**

To any given finite set we attach a label, called a number. The same label is attached to all those sets which are in one-to-one correspondence with the given set. Through activities — playing with blocks, crayons and other objects — two primary number concepts, cardinal and ordinal, may be developed. A cardinal number tells the size of a group and an ordinal number tells the place in the sequence of numbers (which is used in counting). The natural number is an idea (or a mental tool) which helps in telling "how many objects are in a group" or the size of the group.

Natural Numbers (N) : 1, 2, 3, 4, 5. . . . . are also called counting numbers. They start with the "unit" or "one" and are successively obtained by using the idea of **"one more" or the successor. The set of natural number has no largest number.**

Counting is matching a given finite set and one of the natural number sets - {1}, {1, 2}, {1, 2, 3}, {1, 2, 3, 4} etc. in which every element is matched from the given set. The process of counting the objects in any set will result in the same number, no matter how we arrange the objects.

Methodology used : The discussion-cum-lecture method is used. The properties of natural numbers are elicited from students using inductive reasoning and by giving illustrations.

5.3.3 Zero and Integers

Main Teaching Point : Properties of integers.

Teaching-Learning Process

The number **zero** "0" is introduced as (i) the number associated with the empty set ϕ or { }. This can be done through an activity. For example, a child could be given say 5 chocolates and asked to eat all of them one by one; so that finally none is left. Since the empty set is a proper subset of every non-empty set, zero is less than any natural number K, that is $0 < K$, (ii) the starting point in arranging the objects in a sequence (on a line), and (iii) an answer to questions like $6 - 6 = ?$; $15 - 15 = ?$ which can be demonstrated through counting backwards or taking away. Zero completes counting both ways – ahead and backwards. The set of numbers zero and natural numbers is called **whole numbers (W).**

Whole Numbers = {0, 1, 2, 3, 4, 5, ... }

Zero is the smallest whole number. There is no greatest whole number.

A more inclusive set than W is the set of **Integers (I)**. The set of integers can be introduced through activities involving (a) opposites such as profit and loss; rise in level and fall in level; earning and spending etc. or (b) direction – high and low, east and west, north and south, etc. The integers consist of · negative integers, zero and positive integers.

{. . . . . – 4, – 3, – 2, – 1, 0, + 1, + 2, + 3, + 4 }

Zero is not included in the set of positive integers or in the set of negative integers. The set of integers has no least or greatest number. It extends infinitely in both directions. The natural numbers are assigned the positive (+) sign and called positive integers. For every positive integer (+ a) we have a negative number (– a) such that

$(+a) + (-a) = 0$

The extension to negative numbers can be demonstrated by presenting a pattern in subtraction

5	5	5	5	5	5	5	5
– 1	– 2	– 3	– 4	– 5	– 6	– 7	– 8
4	3	2	1	0	A	B	C

Clearly A is one less than 0, B is one less than A and so on. Further A is such that when 1 is added to it we get zero, B is such that when 2 is added to it we get zero.

Thus A is the opposite of 1 and B is the opposite of 2 etc These can be represented by – 1 and – 2.

Thus $(1) + (-1) = 0$, $(+2) + (-2) = 0$. But we know that $1 - 1 = 0$ and $2 - 2 = 0$, so we can replace $+ (-1)$ by -1 and $+ (-2)$ by -2. By convention we do not put a sign with positive numbers. $+ (+1) = 1$ and $- (+1) = -1$ etc.

The set of integers is closed under addition, subtraction and multiplication. But try division; we find $\frac{3}{4}$; $\frac{15}{27}$, $\frac{200}{327}$ etc., are not possible, that is the system is not closed under division. We, therefore, extend further and define "rational numbers" (Q).

Methodology used : The discussion-cum-lecture method is used. The properties of integers are enlisted by giving illustrations and using inductive reasoning.

Check Your Progress

Notes : a) Write your answers in the space given below

b) Compare your answers with those given at the end of the unit.

1. Is the sum, difference, product, quotient of two given integers always an integer?

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2. Illustrate with examples the rules for multiplication of integers.

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3 Show that the square of an even integer is even and that the square of an odd integer is odd.

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4 Show how you would establish a one-to-one correspondence between

a) Positive integers and the set of integers.

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b) Negative integers and the set of natural numbers.

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5.3.4 Rational and Irrational Numbers

Main Teaching Points : a) Need for rational numbers

b) Need for irrational numbers

Numbers which can be expressed in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$ are called **rational numbers** (Q). Fractions introduced in earlier classes are a subset of rational numbers. Rational numbers (or fractions) can be easily introduced through activities involving equal sharing or partitioning of sets. These are interpreted as a part of a whole; a part of a group; indicated division and as a ratio. In fact the word rational comes from the word, ratio The same number can be written in many ways as a ratio of integers. For example:

$$3 = \frac{+3}{+1} = \frac{+6}{+2} = \frac{-6}{-2} = \frac{-9}{-3} = \frac{+15}{+5} \qquad \frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12}$$

Since any integer "n" can be written as n/1, integers form a subset of rationals. It can be easily demonstrated that the set of rationals is closed under addition, subtraction. multiplication and division (provided we do not divide by zero) It is also closed under the operation of finding the power of a number (i.e., squaring. cubing, etc). But it is not closed when we try to find roots — square root. cube root etc Sometimes the root of a rational number is itself a rational number. Sometimes it has a rational approximation but cannot be written exactly in the form of a ratio.

i.e. $\frac{p}{q}$, $q \neq 0$

Such numbers are called **irrational numbers.**

2, 3, $\sqrt[3]{6}$, $\sqrt[4]{12}$ etc.

are irrational numbers.

Another way of describing irrational numbers as distinct from rational numbers is to consider the decimal form of numbers. It is possible to prove that for each rational number the pattern of digits in the decimal representation either terminates, e.g., $\frac{1}{20} = 0.05$ or recurs,e.g., $\frac{1}{7} = 0.142857142857...$

In case of irrational numbers the decimal representation neither terminates nor recur, e.g., 2.131133133331 ....

For some numbers such as $\sqrt{2}, \sqrt{3}$ etc., it is easy to prove that they are irrational, while for other numbers such as π it may be quite difficult. The proof for "$\sqrt{2}$ is irrational" is of special importance as we use the method of contradiction; we assume that it is rational and show that this leads to a contradiction

Assume $\sqrt{2} = \frac{m}{n}$, where m and n are coprime integers, that is, they do not have a common integral factor (other than 1 or – 1)

Now $\sqrt{2} = \frac{m}{n} \Rightarrow 2 = \frac{m^2}{n^2} \Rightarrow m^2 = 2n^2$

$\therefore m^2$ is even $\Rightarrow$ m is even

(Since the square of an odd integer is odd)

$$\begin{aligned} \text{Now, } m &= 2p,\ p \in I \\ &\Rightarrow 4p^2 = 2n^2 \\ &\Rightarrow n^2 = 2p^2 \end{aligned}$$

$\therefore n^2$ is even $\Rightarrow$ n is even $\Rightarrow n = 2q,\ q \in I$

Hence m and n are both even i.e., have a common factor 2, which contradicts the assumption.

Hence $\sqrt{2}$ is irrational.

Methodology used : Illustrations are given and mainly the lecture method is used for showing the need for rational numbers. While the deductive method is used to prove that certain numbers are not rational.

Check Your Progress

Notes : a) Write your answers in the space given below.

b) Compare your answers with those given at the end of the unit.

5. Is the sum, difference, product, or quotient of two given rational numbers always a rational number?

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6 Prove that (a) $\sqrt{3}$ and (b) $1 - \sqrt{2}$ are irrational numbers.

7. Is the sum of two irrational numbers always irrational ?

8. Is the product of two irrational numbers always irrational

9. Express each of the following as recurring or terminating decimal : $\frac{1}{15}$, $\frac{1}{25}$, $\frac{1}{9}$, $\frac{1}{7}$, $\frac{1}{8}$. Try to suggest a rule which determines whether a decimal terminates or not.

5.3.5 Real Numbers — Operations and their Properties

Main Teaching Points : a) Properties of real numbers
b) Concept of absolute value of a number

Teaching-Learning Process

The union of the sets of rational numbers and irrational numbers is called the set of real numbers.

{Real Numbers) = {Rationals) ∪ {Irrationals}

Most of the numbers used in school mathematics are real numbers. But there are still some roots that are not even irrational numbers. e g. $\sqrt{-9}$,$\sqrt{-1}$. These are different kinds of numbers sometimes called non-real numbers. These are studied in senior secondary classes. The union of real and non-real numbers is called complex numbers. The diagram below gives the development of the number system from natural numbers to real numbers.

Number system	Closure under operations
1. Natural Numbers (N) 1, 2, 3, 4, ...	addition, multiplication
2. Whole Numbers (W) 0 1 2, 3, 4, ...	addition, multiplication
3. Integers (I) ... – 3, – 2, – 1, 0, 1, 2, 3	addition, multiplication, subtraction
4. Rational Numbers (Q) p/q where p, q ∈ I, q ≠ 0	addition, multiplication, subtraction, division
5. Real Numbers (R)	addition subtraction, multiplication, division

The study of the properties of real numbers is one of the major concepts at the secondary level. It is, therefore, necessary that the meaning of the operations of addition subtraction, multiplication and division be explained through examples and then properties be inductively generalised.

Adding, subtracting, multiplying and dividing operations are binary since we carry them out on two numbers. Addition is counting forward. Multiplication is repeated addition subtraction is the inverse of addition. Division is repeated subtraction. The properties are listed below.

The properties for Real Numbers

1	Closure Laws For all a, b	$a + b$ is a real number	ab is a real number
2.	Commutative Laws	$a + b = b + a$	$ab = ba$
3.	Associative Laws	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
4.	Identity Laws	There is a unique real number 0, such that for all a, $a + 0 = 0 + a = a$	There is a unique real number 1 such that for all a. $a \times 1 = 1 \times a = a$
5.	Inverse Laws	For all a, there is unique real number $-a$ such that $a + (-a) = 0 = (-a) + a$	For all a (except 0) there is a unique real number $\frac{1}{a}$ such that $a(\frac{1}{a}) = 1 = (\frac{1}{a})a$
6.	Distributive Law	For all a, b, c $a(b + c) = ab + ac$	

The Order Laws for Real Numbers

7. The Trichotomy Law

If 'a' is a real number, then out of the three statements below exactly one is true:

(i) $a = 0$, (ii) a is positive, (iii) $-a$ is positive

8. For any real numbers a and b exactly one of the three statements below is true .

(i) $a < b$, (ii) $b < a$, (iii) $a = b$

These laws are used to prove many results which have been taught as rules. Two simple rules are proved here.

1. For any real number a, $a. 0 = 0 = 0. a$

Proof : $1 + 0 = 1$ (identity law)

$a (1 + 0) = a.1$

By distributive law

$a.1 + a.0 = a.1$

But, identity law is

$a.1 + 0 = a.1$

Hence by uniqueness of 0, $a. 0 = 0$

2. For any two numbers $(-a) + (+a) = 0$. (Inverse law), multiplying by $(-b) \times$ we get $(-b)(-a) + (-b)(+a) = 0$

Now $(-b) \times (+a) = (-ba)$. (To be proved earlier)

$(-b)(-a) + (-ba) = 0$

By using Inverse law we argue that

$(-b) \times (-a)$ should be opposite of $(-ba)$

Hence $(-b) \times (-a) = +ba$

An important concept which should be made clear through examples is that of the absolute value of a real number. Recall the idea of opposites and ask pupils to write additive inverses (using signed numbers).

The absolute value of numbers that are additive inverses of each other is the same. We use the symbol for absolute value. Thus

$$\left|\frac{-2}{3}\right| = \left|\frac{+2}{3}\right| = \frac{+2}{3};\ |-5| = |+5| = +5 \text{ etc.}$$

In general, for all x, $|x| = \begin{cases} x, \text{ if } x \geq o \\ -x, \text{ if } x < o \end{cases}$

Notice that the last line means that the absolute value of a negative number is a positive number $|-3| = -(-3) = +3$.

Methodology used : Properties are enlisted on the blackboard by putting questions to the students regarding them. The concept of absolute value is illustrated through examples.

Check Your Progress

Notes : a) Write your answers in the space given below.

b) Compare your answers with those given at the end of the unit.

10. Prove that

a) $a\,(b - c) = ab - ac$

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b) $-[a + (-b)] = -a + b$

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c) $(a + b)\,(a + b) = a^2 + 2ab + b^2$

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11. Illustrate the meaning of

a) additive inverse

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b) multiplicative inverse

12. If a, b and c are real numbers and $a > b$, which is larger (a) $a + c$ or $b + c$ (b) ac or bc?

13. If $a > b > 0$ which is larger (a) $\sqrt{a}$ or $\sqrt{b}$ (b) $\frac{1}{a}$ or $\frac{1}{b}$

14. How are these two sets related

a) $\{X : X = 4, X \in R\}, \{X : |X| = 4, X \in R\}$?

b) $\{X : X^2 = 5, X \in R\}, \{X : |X| = \sqrt{5}\, X \in R\}$?

15. Is each of the following true or false

a) $R = R^- \cup \{0\} \cup R^+$ b) $R^+ \cap \{0\} = \phi$

c) $R^- \cap \{0\} = \phi$ d) $R^+ \cap R^- = \phi$

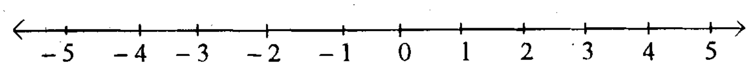
5.3.6 The Numberline

Main Teaching Point : Representation of real numbers on the Numberline.

Teaching-Learning Process

In recent years the idea of Numberline has been used as a very effective aid to familiarize pupils with the properties of the number system.

Draw a horizontal line and mark on it a set of evenly spaced points to represent the integers: this line is called the Numberline.

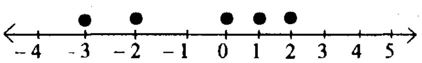


Given a rational number we can mark a point representing it by using a scale or using a geometrical construction. The approximate rational value of an irrational number can be marked as a point by measurement, but the exact position of a point representing an irrational number can be marked by using a geometrical construction. Thus, for every real number we can mark a point on the numberline. The order relation is also indicated by movement on the Numberline.

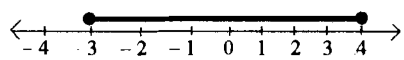
$$\ldots\ldots -5 < -4 < -3 < -2 < -1 < 0 < 1 < 2 < 3 < \ldots\ldots$$

A number on the right is greater than the number on the left. Thus 8 is greater than 6. We write 8 > 6. 0 is greater than all the negative numbers. Sets of numbers can be shown by marking corresponding points.

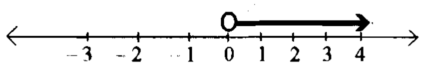
Examples



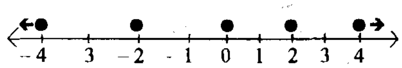
(i) {-3, 2, 0, 1, 2}



(ii) {Real numbers from − 3 to + 4}



(iii) {Positive reals}

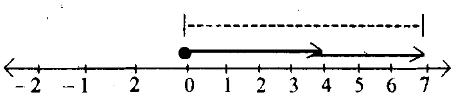


(iv) {Multiples of 2}

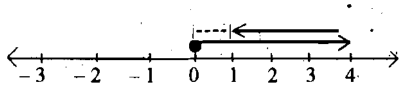
• means point is included. 0 means point is not included. The thick part of line shows continuity between finite boundaries. Infinite sets are shown by an arrow at one end or both ends.

Again we associate numbers with displacements, +5 is a number which is 5 units away from 0 and on the positive (i.e. right) side of it. We show displacement with a segment and direction by putting an arrow head to the segment. Thus any number is represented by a **directed line segment**. To represent the sum of two numbers on a Numberline we draw the first arrow with its tail at the origin, the second with its tail at the head of the first.

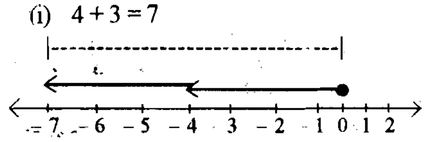
Examples



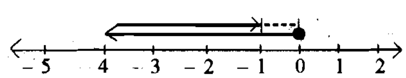
(i) $4 + 3 = 7$



(ii) $4 + (-3) = 1$



(iii) $-4 + (-3) = -7$



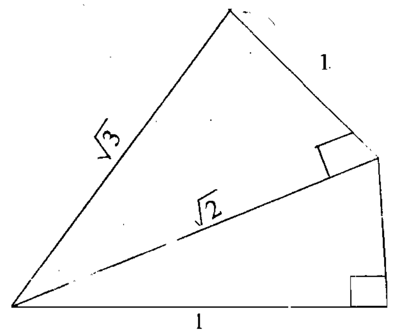
(iv) $-4 + 3 = -1$

The arrow representing the answer has been shown by a dotted line. There is no need to do subtraction on the Numberline because the subtraction sum can be changed to an addition sum.

$$4 - 3 = 4 + (-3) = 1, 4 - (-3) = 4 + 3 = 7$$

Multiplication can be represented as repeated addition Division can be represented as repeated subtraction. Thus, the Numberline is a useful aid to illustrate the properties of rational numbers. The Numberline can also furnish an excellent model for portraying the meaning of directed numbers. The negative number is designated as a direction-changing operator. Different directional situations can be described by different rules.

The irrational numbers such as $\sqrt{2}$, $\sqrt{3}$ can be represented with the help of geometrical constructions using the pythagorous theorem. Draw a line segment of unit length. With this as the base, erect a right-angle triangle of altitude 1 unit. The hypotenuse of this triangle will be of length $\sqrt{2}$. Now use this hypotenuse as base and on it erect another right-angle triangle of altitude 1. The hypotenuse of this triangle is $\sqrt{3}$ and so on.



Methodology used : Recapitulating the method used for dividing a line segment into number of parts, the skill of plotting the numbers on a Numberline is developed by the practice method using proper geometrical instruments

Check Your Progress

Notes : a) Write your answers in the space given below

b) Compare your answers with those given at the end of the unit.

16. Pick out pairs of numbers from {– 3, – 2, – 1, 1, 2, 3}, construct addition sums and draw Numberline graphs. Verify the commutative and associative laws.

17. Draw a Numberline graph to represent :

i) {Real numbers greater than – 5 and less than 5}

ii) $x < 30$

iii) $x > 30$

18. Write the statements below in two ways, one involving $|x - 2|$ and the other involving $(x - 2)^2$

a) $-5 < x - 2 < 5$

b) $x - 2 > 5$ or $x - 2 < -5$

5.5 POWER, ROOTS AND LOGARITHMS

Computation is an important aspect of mathematics. In many problems from real life we find large numbers and that too connected by complicated terms formed from monomials. Suppose we have a problem involving an expression like :

$$\frac{53296 \times 32847}{243941} \times \frac{5 \times 10^2}{27}$$

The usual process of computation may take a long time and cause frustration. Why should numerical complexities be imposed upon the students? Many devices to facilitate solving of such sums have been developed and logarithms is one of these.

5.4.1 Exponents, Power and Root

Main Teaching Points : a) Properties of exponents
b) Power and root as inverse processes

Teaching-Learning Process

Power Notation
Number = $(\text{Base})^{\text{Exponent}}$

$16 = 2^4$
is the base
4 is the exponent
16 is the power of 2

The exponent tells how many times the base is taken as a factor. Successive multiplication leads to the idea of taking power.

2^1	=	2	=	2	first power of 2
2^2	=	2×2	=	4	squared or second power of 2
2^3	=	$2 \times 2 \times 2$	=	8	cubed or third power of 2
2^4	=	$2 \times 2 \times 2 \times 2$	=	16	fourth power of 2
2^n	=	$\frac{2 \times 2 \times 2 \times \ldots\ldots \times 2}{n \text{ factors}}$	=	2^n	n^{th} power of 2

x^n means a variable/literal number for which a number can be substituted has been raised to n^{th} power.

In teaching powers, n should be first of all taken as a positive integer, then a negative integer and lastly a rational number.

For any real number x the following laws hold :

1. $x^m \times x^n = x^{m+n}$
2. $x^m/x^n = x^{m-n}$
3. $(x/y)^n = x^n/y^n$
4. $x^0 = 1$
5. $x^{-n} = 1/x^n$

We can also consider $X^{m/n}$ where m/n is rational.

What does $X^{m/n}$, $n > 0$ mean.

Now, $Y = X^{m/n}$ and so $Y^n = X^m$ or $Y = \sqrt[n]{X^m}$

It is always possible to find n^{th} root, $\sqrt[n]{a^m}$ is the **positive n^{th} root of a^m.**

The inverse operation of taking a power of a number is that of taking the corresponding root. A square root is indicated by the radical symbol $\sqrt{\ }$. Thus $\sqrt{16} = 4$. With higher roots an index is placed on the radical to show the order of the root:

$$\begin{tikzcd} 4 \arrow[r, bend left, "\textbf{Square}"] & 16 \arrow[l, bend left, "\textbf{Square Root}"] \end{tikzcd}$$

$\sqrt[3]{27}$; $\sqrt[4]{16}$; $\sqrt[5]{243}$ etc.,

The student should be exposed to a large number of examples so that they grasp the methods of finding roots by observing the pattern. The easiest method is that of finding roots by prime factorisation. Another method for finding the square root is called "division method" or "square root algorithm" by which the successive digits of the root are found one by one. Sometimes it is difficult to explain why the rule works. A proper demonstration with proper explanation at each step is, therefore, necessary.

In short, field and laboratory roots are read from tables or on slide rule, or occasionally are computed by the rule or division method. Secondary level students should be acquainted with the root table and the methods of interpolation.

Methodology used : Inductive method is used to arrive at the different properties of exponents and a number of illustrations are given so that students are able to grasp the pattern.

5.4.2 Indices and Logarithms

Main Teaching Points : a) Indices and logarithm as inverse processes.

b) Properties of logarithms.

Teaching-Learning Process

We have considered sets of powers of various numbers. To give meaning to logarithm, closer attention is given to the indices (or exponents) themselves. The logarithm of a number is the index to which the base is raised to get the number. For example, the logarithm of the number 243 to the base 3 is 5 because $3^5 = 243$. This is abbreviated as $\log_3 243 = 5$.

Each statement in the logarithm form has its equivalent statement in the index form.

$\log_3 243 = 5 \qquad 3^5 = 243$

$\log_2 64 = 6 \qquad 2^6 = 64$

Early in the seventeenth century Henry Briggs expressed each of the natural numbers from 1 to 20,000 and from 90,000 to 100,000 in the form $\log^n_{10}$ where n was calculated correct to 14 decimal places. Adriam Vlacq completed the table for natural numbers between 20,000 and 90,000. The calculations were the basis of tables of logarithms. The first practical system of logarithms was published by Napier in 1614. The word logarithm is derived from the Greek word "logos" meaning ratio.

In our calculations we use logarithms to the base 10 or common logarithms only. In practice, therefore, we do not write the base, we simply write log 10 = 1; log 100 = 2 etc. when calculating.

Since logarithm is really the index corresponding to a number, we can write index laws using logarithm notation. Thus :

i) $\log xy = \log x + \log y$

ii) $\log x/y = \log x - \log y$

iii) $\log x^m = m \log x$

iv) if $x = a^n$, $n = \log_a x$ then $x = a^{\log_a^x}$

Since we use base 10, it is necessary to introduce the idea of expressing a number in standard form. Thus

$$5636 = \frac{5636}{1000} \times 1000 = 5.636 \times 10^3;\ 349 = 3.49 \times 10$$

$$56.36 = 5.636 \times 10^1;\ .002 = 2.0 \times 10^{-3}$$

Any number, "n", can be expressed as $n = m \times 10^P$, where p is an integer (positive, zero or negative) and $1 < m < 10$. This is called the standard form of any number,. "n".

The index "p" in the standard form gives the integral part of the logarithm. This is called the characteristic. The decimal part of the logarithm is taken from a table. This is called the mantissa.

Methodology used : The discussion cum lecture method is used to make the concept understood by students. Many examples are given to illustrate $b^n = m \Leftrightarrow \log_b m = n$. A suitable drill exercise may be provided.

5.4.3 Use of Logarithm Tables

Main Teaching Points : a) Reading mantissa from logarithm tables.

b) Use of logarithm tables for calculations.

Teaching-Learning Process

Sufficient practice is required in writing the characteristics and reading the mantissa from the table Examples such as the following may be given:

	0	1	2	3	4	5	6	7	8	9		1	2	3	4	5	6	7	8	9	
61	.7853	.7860	.7868	.7875	.7882	.7889	.7896	.7903	.7910	.7917		1	1	2	3	4	4	5	6	6	

To find log 6.176, we note that log 61 has the decimal column headed by 7, where the decimal is .7903. Log 6.176 is obtained by adding to .7903 the number 4, actually .0004, found by noting that 4 is the number headed by 6 in the difference column. This gives log 6.176 as .7907. Since 6 lies between 1 and 10, log 6.176 = 0.7907.

Log 61.76 = 1.7907 (61 lies between 10 and 100), log 617.6 = 2.7907 (617 lies between 100 and 1000). It should be emphasized that (1) the mantissa always lies between 0 and 1, and hence the mantissa is always positive, (2) to find the characteristic we have only to count the number of digits in the integral part of n and subtract 1 from it, provided $n > 1$. For $n < 1$ we count the number of zeros after the decimal place to the first non-zero digit and add 1 to it. The number so obtained with a negative sign gives the characteristic.

The inverse process of finding the number whose logarithm is given is called finding the anti-logarithm. We use the symbol antilog to denote the phrase "the antilogarithm of ....". We have a separate table for finding the antilogarithm. In using the table of antilogarithms we disregard the characteristic and make sure that the fractional part of the given number is indeed positive. We then find the number by the use of the mantissa and finally insert a decimal point by rule of the characteristic.

In the teaching of logarithm, sufficient practice should be given in the reading of tables. The fact that the mantissa is always positive should be carefully emphasized. Separate examples for multiplication, division and finding roots should be worked out. Speed and accuracy should be the final goal in calculating with logarithm.

Methodology used : Every student must possess a logarithm table when this is discussed in the class. To develop skill in using logarithm tables, sufficient practice should be provided. Basically, the drill method is required.

Check Your Progress

Notes : a) Write your answers in the space given below.

b) Compare your answers with those given at the end of the unit.

19. Is $2^5 = 5^2$? Is $3^2 = 2^3$? Is the operation "to the power of" commutative ? If not, can you find the conditions for which the statement $a^m = m^a$ is true.

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20. Is the statement $a^2 + a^2 = a^4$ true for all values of "a"? Can you find any value of "a" (real numbers) that makes the statement true ?

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21. Find the values of the following logarithm by first writing down a statement in index form: e.g., for $\log_2 32$, $32 = 2^5$ $\log_2 32 = 5$

(i) $\log_8 512$ (ii) $\log_{11} 1331$ (iii) $\log_7 343$ (iv) $\log_5 625$

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5.5 HISTORICAL NOTE

Man's present interaction with numbers has developed from extremely limited beginnings. The teacher should tell the students how our ancestors first acquired the sense of numbers, then gradually evolved methods for recording numerical data and making simple calculations. Man saw and experienced the discrete in the physical universe. There were very many objects, separate and distinct, but how many? The simple one-to-one matching adopted by man was not the answer to the question. But certainly it was first the thinking process which opened a new era in man's life. The teacher should show how strokes in clay, knots on strings, wooden tally sticks and fingers were used to keep a record of numbers. In India the earliest known examples of written numerals are found in the inscriptions of King Asoka who ruled in the third century B C. These inscriptions are:

I	II	+	ɕ
1	2	4	6

In inscriptions of about a century later found on the walls of a cave of the Nana Ghat hill there are number symbols whose probable form was:

1	2	3	4	6	7	9	10	20	60	80	100	1000
–	=		≠	Ψ	>	?	α	0	γ	ω	≻≺	T

Later, about the eighth century A D., a set of numerals known as Devanagari numerals were used. They contained the idea of zero (sunya).

१	२	३	४	५	S	)	(	ε	0
1	2	3	4	5	6	7	8	9	0

At the same time a very important development was the use of the principle of place value in the Hindu methods of writing numbers.

The early devices — counting boards, abacus etc. — also form an interesting subject to discuss with students. Students may be provided with models of these devices.

In the early stages the use of numbers was restricted to counting and measuring. But when numbers began to occur in physical situations that had nothing to do with linear measure, a new theory of numbers became an absolute necessity. The problem of incommensurables, the lore of the infinite and the foundations of the calculus needed a more precise formulation of real numbers. The totality of real numbers should be illustrated with examples so that the pupils understand the significance of applied and pure mathematics.

5.6 LET US SUM UP

The teaching of numbers starts with the arithmetic of natural numbers. This in turn helps in defining rational numbers and their arithmetic. By assuming that the rational numbers are ordered pairs (p q) where p and q are natural numbers, one can proceed to define real numbers in several ways For example as a decimal, in which case both rational and irrational can be expressed as a sequence of decimal digits. One may, however, prefer to conceptualise real numbers as measuring numbers (magnitudes) based on the $|1-1|$ correspondence with points of the Euclidean line. It may be noted that defining real numbers in terms of rationals and irrationals is of great importance. It enables us free analysis from geometric intution. In actual teaching the concept of real numbers may be taught as relative to (1) the structure of the real line and (2) its arithmetic and algebraic properties. In the former case (1) the real number is essentially treated as a geometric entity: in the latter case, (2) as the arithmetized entity based on the natural (via the rational) numbers. Through the device of labeling points on a line with arithmetized real numbers the two concepts are usefully combined.

5.7 UNIT-END ACTIVITIES

Discuss the following questions

1. ⊗ means "increase the first number by 3 and multiply by the second number" Evaluate .

(i) $4 \otimes 7$ (ii) $7 \otimes 4$ (iii) $(3 \otimes 5) \otimes 6$ (iv) $3 \otimes (5 \otimes 6)$

Is the operation $\otimes$ commutative? Is it associative?

2. If $n \in N$, prove that n^2+n is always even.
3. If $n/13 < 5/17$ find the greatest integral value for n.
4. Find all fractions p/q between 0 and 1 where p and q are relatively prime natural numbers and $2 \leq q \leq 5$. Find the mean value of all these fractions. Which is the largest and which is the smallest of these fractions?
5. State, whether each of the following statements is true or false :

i) If $a \in I$, and $b \in I$ then

$a + b \in I$, where I is the set of integers.

ii) If $a \in W$, and $b \in W$ then

$a - b \in W$, where W is the set of whole numbers.

iii) If $x \in I$, then x^4 cannot be negative

iv) If $a < b$ then $-a < -b$ where $a, b \in R$.

v) $(-3)(-5) < 6 \times (-3)$.

vi) $\sqrt[3]{216}$ is an irrational number.

5.8 ANSWERS TO CHECK YOUR PROGRESS

1. The sum, the difference and the product of two integers is always an integer but it is not necessary for the quotient of two integers to be an integer. For example, $(-3) \div 2$ is not an integer.
2. The product of two positive integer is positive

$3 \times 5 = 15$; $7 \times 2 = 14$

The product of two negative integers is positive

$(-3) \times (-5) = 15$; $(-2) \times (-7) = 14$

The product of a positive and a negative integer is negative

$(-2) \times 7 = -14$; $3 \times (-4) = -12$.

3. Let 2n be an even number.

$(2n)^2 = 4n^2$ which is divisible by two, and hence it is even. Let $2n + 1$ be an odd number.

$$(2n+1)^2 = 4n^2 + 4n + 1$$

$4n^2$ and $4n$ are both divisible by 2.

$\therefore 4n^2 + 4n+1$ is not divisible by 2, hence it is odd.

4. a) $f(x) = \begin{cases} 2x, & X>0 \\ 1, & X=0 \\ -2x+1, & X<0 \end{cases}$

Where x is an integer.

There is a one-to-one correspondence between integers and positive integers.

b) $f(x) = -x$, for all $x \in I^- =$ Set of all negative integers.

This is a one-to-one correspondence between negative integers and set of natural numbers.

5. Yes, the sum, the difference, the product and the quotient of two rational numbers is always a rational number
6. a) Let if possible $\sqrt{3}$ be a rational number. Therefore, there exists integers p and q such that p and q are coprime and

$$\frac{p}{q} = \sqrt{3}$$

Squaring, we get $\frac{p^2}{q^2} = 3$

$\Rightarrow \quad p^2 = 3q^2$

$\Rightarrow \quad$ p is divisible by 3

Let, $p = 3r$

Squaring, we get $p^2 = (3r)^2 = 9r^2$

$\Rightarrow \quad 9r^2/q^2 = 3$

$\Rightarrow \quad q^2 = \quad = 3r^2 \quad \Rightarrow q^2$ is divisible by 3

$\Rightarrow$ q is divisible by 3.

$\therefore$ 3 is a common factor of both p and q, a contradiction.

$\Rightarrow \quad \sqrt{3}$ is irrational.

b) Let $1 - \sqrt{2} = r$ where r is rational.

$\therefore \sqrt{2} = 1 - r$

$1 - r$ is rational $\Rightarrow \sqrt{2}$ is rational,

This is a contradiction. Hence, our supposition is wrong and $1 - \sqrt{2}$ is irrational.

7. The sum of two irrational numbers is not always irrational. Ex. $\sqrt{3} + (-\sqrt{3}) = 0$ and "0" is a rational number.

8. The product of two irrational numbers is not always irrational.

For example. $\sqrt[3]{2} \times \sqrt[3]{4} = \sqrt[3]{2 \times 4} = \sqrt[3]{8} = 2$, which is rational

9. $\frac{1}{15} = 0.0\overline{6}$; terminating decimal, $(15 = 5 \times 3)$

$\frac{1}{25} = 0.04$, terminating decimal. $(25 = 5 \times 5)$

$\frac{1}{9} = 0.\overline{1}$; recurring decimal, $(9 = 3 \times 3)$

$\frac{1}{7} = 0.\overline{142857}$, recurring decimal, $(7 = 1 \times 7)$

$\frac{1}{8} = 0.125$; terminating decimal, $(8 = 2 \times 2 \times 2)$

Rule : If 2 and/or 5 are the only prime factors of the denominator, then the rational number is a terminating decimal, otherwise it is a recurring decimal.

10. a) $a(b - c) = ab - ac$

Proof : LHS $= a(b - c) = a\{b + (-c)\} = ab + a(-c)$

$= ab + (-ac)$

$= ab - ac =$ RHS

b) $-[a + (-b)] = -a + b$

$\{a + (-b)\} + \{-a + b\} = a + (-b) + (-a) + b = a + (-a) + (-b) + b$

(by commutative and associative properties)

$= 0 + 0 = 0$

$\therefore -\{a + (-b)\} = -a + b.$

c) $(a + b)(a + b) = a^2 + 2ab + b^2$

Proof : LHS $= (a + b)(a + b) \quad = (a + b)a + (a + b)b.$

$= a.a + b.a + a.b + b.b$

$= a^2 + ab + ab + b^2$

(by commutative property)

$= a^2 + 2ab + b^2 =$ RHS

11. a) Additive inverse of 'a' is the number 'b' such that

$a + b = 0 = b + a,$

Thus, the additive inverse of a is $-a$.

For example, the additive inverse of $-5 = -(-5) = 5$

b) The multiplication inverse of $a \neq 0$ is the number b such that

$a \times b = 1 = b \times a$

Thus, the multiplicative inverse of b is $\frac{1}{b}$

For example, the multiplicative inverse of $\frac{-2}{3}$ is $\frac{-2}{3}$

12. a) $a > b \Rightarrow a + c > b + c$

b) $a > b,\ c > 0 \Rightarrow ac > bc$

$a > b,\ c < 0 \Rightarrow ac < bc.$

13. a) $a > b > 0 \Rightarrow \sqrt{a} > \sqrt{b}$

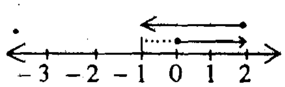
b) $a > b > 0 = \frac{1}{a} < \frac{1}{b}$

14. a) $\{X{:}X = 4, X \in R\} \subset \{X{:} \left|X\right| = 4, X \in R\}$

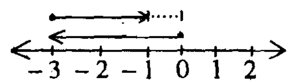
b) Both sets are equal.

15. All are true.

16. Find $(-3) + 2$ and $2 + (-3)$ using the number line



$2 + (-3) = -1$



$(-3) + 2 = -1$

The dotted arrow represents the sum.

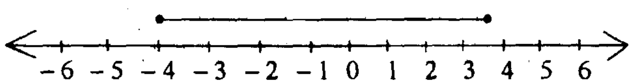
For a positive number we move to the right and for a negative number we move to the left.

$\therefore 2 + (-3) = (-3) + 2$

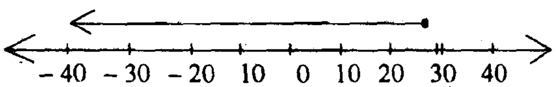
The commutative property is verified.

Similarly, the associative property can also be verified.

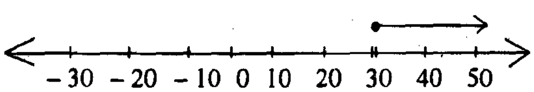
17. i) Real numbers greater than −5 and less than 5.



ii) $X < 30$

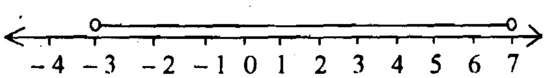


iii) $X > 30$



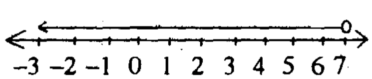
18. a) i) $\left|X - 2\right| < 5$ $\left|X - 2\right| > -5$

$X < 7$ $X > -3$



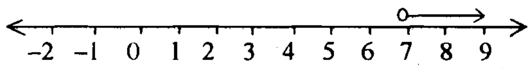
ii) $(X - 2)^2 < 25$

$x(x - 4) < 21 \therefore x < 7$



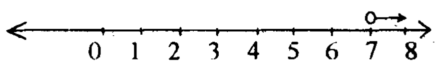
b) i) $|X-2| > 5$

$X > 7$



ii) $(X-2)^2 > 25$

$x(x-4) > 21$. $x < 7$



19 No, the operation 'to the power of' is not a commutative operation.

$a^m = m^a$ if $a = m$

20. No. It is true if $a = 0$

21. i) $8^3 = 512$ $\therefore \log_8 512 = 3$

ii) $11^3 = 1331$ $\therefore \log_{11} 1331 = 3$

iii) $7^3 = 343$ $\therefore \log_7 343 = 3$

iv) $5^4 = 625$ $\therefore \log_5 625 = 4$

5.9 SUGGESTED READINGS

Sawyer *Prelude to Mathematics*, Pelican.

Richardson, *Fundamentals of Mathematics*, Collier, Macmillan.

N.C T.M., USA; *Insights in Modern Mathematics*, 28th Year Book, New York.

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