UNIT 9 MATRICES AND DETERMINANTS

Structure

9.1 INTRODUCTION

The knowledge of matrices has become necessary for the individuals working in different branches of science, technology, commerce, management and social sciences. In this unit, we introduce the concept of matrices and its elementary properties. The unit also discusses the determinant, which is a number associated with a square matrix and its properties. Trace of a matrix is also defined.

Objectives

After completing this unit, you should be able to:

- define a matrix and give examples of matrices;
- explain the types of matrices;
- know how operations on matrices are done;
- find multiplication of a matrix by a scalar;
- compute transpose of a matrix;
- find the trace of a square matrix;
- evaluate determinants find minors and cofactors of square matrices of different orders; and
- apply properties of determinants.

9.2 DEFINITION OF A MATRIX

Let us consider the following example to arrive at the definition of a matrix: Suppose there are three girls “Kavita, Preksha and Tanu” Kavita has 9 hundred rupees notes, 4 fifty rupees notes and 5 ten rupees notes. Preksha has 17 hundred rupees notes, 6 fifty rupees notes and one ten rupee note. Tanu has 8 hundred rupees notes, 3 fifty rupees notes and 2 ten rupees notes.
This information can be represented as:

$$\begin{array}{ccc} & & \text{Column 1} & \text{Column 2} & \text{Column 3} \\ & & \downarrow & \downarrow & \downarrow \\ & & \text{Rs.100 Notes} & \text{Rs.50 Notes} & \text{Rs.10 Notes} \\ \text{Row 1} \rightarrow & \text{Kavita} & 9 & 4 & 5 \\ \text{Row 2} \rightarrow & \text{Preksha} & 17 & 6 & 1 \\ \text{Row 3} \rightarrow & \text{Tanu} & 8 & 3 & 2 \end{array}$$

This is an arrangement of 9 (3×3) numbers in 3 rows and 3 columns. Such an arrangement is nothing but a matrix. Let us now define a matrix as follows:

Definition of a Matrix

An arrangement of $m \times n$ elements in m rows and n columns enclosed by the brackets ( ) or [ ] only, is called a matrix of order $m \times n$ and is generally denoted by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ . & . & . & & . \\ . & . & . & \dots & . \\ . & . & . & & . \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \text{ or } \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ . & . & . & & . \\ . & . & . & \dots & . \\ . & . & . & & . \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

where a_{ij} denotes the $(i, j)^{th}$ element of the matrix, i.e. element of i^{th} row and j^{th} column is denoted by a_{ij}.

Remark 1:

(i) A matrix is denoted by capital letters A, B, C, etc. of the English alphabets.

(ii) First suffix of an element of the matrix indicates the position of row and second suffix of the element of the matrix indicates position of column. e.g. a_{23} means it is an element in the second row and the third column.

(iii) The order of a matrix is written as "number of rows $\times$ number of columns"

For example,

(i) $A = \begin{bmatrix} 2 & 5 & 7 \\ 3 & 8 & 9 \end{bmatrix}$ is a matrix of order 2×3

(ii) $B = \begin{bmatrix} 9 & 6 \\ 1 & 0 \\ 8 & 4 \end{bmatrix}$ is a matrix of order 3×2

Let us consider some examples:

Example 1: Write the order of the matrix

$$A = \begin{bmatrix} 9 & 7 & 8 & -3 & -8 \\ 4 & 3 & 6 & 1 & -10 \\ 10 & 12 & 15 & 2 & 5 \end{bmatrix}$$

Also write the elements $a_{23}, a_{14}, a_{35}, a_{22}, a_{31}, a_{32}$.

Solution: Order of the matrix A is 3×5 and the desired elements are:

$a_{23} = 6,\ a_{14} = -3,\ a_{35} = 5,\ a_{22} = 3,\ a_{31} = 10,\ a_{32} = 12$

Example 2: Write all the possible orders of the matrix having following elements. (i) 8 (ii) 13

Solution:

(i) All the 8 elements can be arranged in single row, i.e. 1 row and 8 columns.

Or

They can be arranged in two rows with 4 elements in each row, i.e. 2 rows and 4 columns.

Or

in four rows with 2 elements in each row, i.e. 4 rows and 2 columns.

Or

in eight rows with 1 element in each row, i.e. 8 rows and 1 column.

$\therefore$ the possible orders are $1 \times 8,\ 2 \times 4,\ 4 \times 2,\ 8 \times 1$.

(ii) All the 13 element can be arranged in single row, i.e. 1 row and 13 columns.

Or

in 13 rows with 1 element in each row, i.e. 13 rows and 1 column.

$\therefore$ the possible orders are $1 \times 13,\ 13 \times 1$.

Example 3: Construct the matrix A = $[a_{ij}]_{2\times3}$, where $a_{ij} = \dfrac{(i-j)^2}{2}$

Solution: A = $[a_{ij}]_{2\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$,where $a_{ij} = \dfrac{(i-j)^2}{2}$

$$a_{11} = \frac{(1-1)^2}{2} = \frac{0}{2} = 0,\ a_{12} = \frac{(1-2)^2}{2} = \frac{(-1)^2}{2} = \frac{1}{2},$$

$$a_{13} = \frac{(1-3)^2}{2} = \frac{(-2)^2}{2} = \frac{4}{2} = 2,\ a_{21} = \frac{(2-1)^2}{2} = \frac{(1)^2}{2} = \frac{1}{2},$$

$$a_{22} = \frac{(2-2)^2}{2} = \frac{0}{2} = 0,\ a_{23} = \frac{(2-3)^2}{2} = \frac{(-1)^2}{2} = \frac{1}{2}$$

$$\therefore A = \begin{bmatrix} 0 & 1/2 & 2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

Here is an exercise for you.

E 1) Construct A = $[a_{ij}]_{3\times2}$, where $a_{ij} = |i - j|$

9.3 TYPES OF MATRICES

On the basis of number of rows and number of columns and depending on the values of elements, the type of a matrix gets changed. Various types of matrix are explained as below:

Row Matrix

A matrix having only one row is called a row matrix.

For example, $\begin{bmatrix} 2 & 5 & 7 \end{bmatrix}$, $\begin{bmatrix} 8 & 9 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 3 & 2 \end{bmatrix}$ all are row matrices.

Column Matrix

A matrix having only one column is called a column matrix.

For example, $\begin{bmatrix} 9 \\ 6 \\ 7 \end{bmatrix}$, $\begin{bmatrix} 9 \\ -3 \\ 2 \\ 8 \end{bmatrix}$, $\begin{bmatrix} 5 \\ -11 \end{bmatrix}$ all are column matrices.

Remark 2: If a matrix has one element, e.g. $A = \begin{bmatrix} 6 \end{bmatrix}$, then matrix A has only one row and only one column. So, it is both row matrix as well as column matrix.

Rectangular Matrix

A matrix having m rows and n columns is called a rectangular matrix if $m \neq n$.

For example, $\begin{bmatrix} 2 & 5 & 7 \\ 3 & 8 & 9 \end{bmatrix}$ is a rectangular matrix having 2 rows and 3 columns.

Square Matrix

A matrix having equal number of rows and columns is called a square matrix.

For example,

(i) $\begin{bmatrix} 4 & 6 \\ 5 & 3 \end{bmatrix}$ is a square matrix of order 2.

(ii) $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 3 & -4 & 8 \end{bmatrix}$ is a square matrix of order 3.

Remark 3: For a square matrix, there is no need of mentioning the number of columns, e.g. in example (i) the order has been written as 2 and not 2×2.

Diagonal Matrix

Principal Diagonal of a Matrix

If $A = [a_{ij}]_{n \times n}$ be a square matrix of order n then the elements $a_{11}, a_{22}, a_{33}, ..., a_{nn}$ are called diagonal elements of the square matrix A, and the diagonal along which these elements lie is called principal diagonal or **main diagonal** or simply **diagonal** of the matrix A.

For example,

(i) Diagonal elements of the matrix $A = \begin{bmatrix} 8 & 9 \\ 5 & 6 \end{bmatrix}$ are 8, 6.

(ii) Write the diagonal elements (if possible) of the matrix $A = \begin{bmatrix} 8 & 9 & 7 \\ 6 & 5 & 2 \end{bmatrix}$

Here, A is not a square matrix, so writing diagonal elements of a rectangular matrix is impossible.

Diagonal Matrix

A square matrix A = $[a_{ij}]_{n\times n}$ is said to be diagonal matrix if
$a_{ij} = 0, \ \forall\ i \neq j$

For example,

(i) If A = $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ then it is a diagonal matrix because all its non-diagonal elements are zero. Sometimes, we denote it by writing diag. [4, 3, 6].

(ii) A = $\begin{bmatrix} 2 & 9 \\ 0 & 5 \end{bmatrix}$ is not a diagonal matrix because non-diagonal element $a_{12} \neq 0$.

Remark 4:

(i) For a diagonal matrix all non diagonal elements must be zero.
(ii) In a diagonal matrix some or all the diagonal elements may be zero.

Example 4: Write all the diagonal matrices of order 2×2 having its elements only 0 or 1.

Solution: For a diagonal matrix, all the non-diagonal elements are zero. Therefore, we are to write 0 and 1 in the diagonal elements in different ways, i.e. 0, 0; 0, 1; 1, 0; and 1, 1.
$\therefore$ possible diagonal matrices with elements only 0 and 1 are given below:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Scalar Matrix

A diagonal matrix is said to be scalar matrix if all its diagonal elements are same.

For example, $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ all are scalar matrices.

Identity Matrix

A diagonal matrix is said to be Identity or **Unit matrix** if all the diagonal elements are equal to unity.

For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ all are identity (or Unit) matrices of order 2, 3, 4 respectively.

Upper Triangular Matrix

A square matrix A = $[a_{ij}]_{n\times n}$ is said to be upper triangular matrix if all the elements below the principal diagonal are zero.

For example, $\begin{bmatrix} 2 & 5 \\ 0 & 7 \end{bmatrix}$, $\begin{bmatrix} 9 & 0 & 6 \\ 0 & 5 & 4 \\ 0 & 0 & 7 \end{bmatrix}$, $\begin{bmatrix} 8 & 0 & 5 & 0 \\ 0 & 9 & 3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ all are upper triangles matrices.

But $\begin{bmatrix} 2 & 9 & 7 \\ 0 & 5 & 8 \\ 2 & 0 & 9 \end{bmatrix}$ is not an upper triangular matrix because one element below the diagonal line, i.e. a_{31} is non zero, which is 2, in this case.

Lower Triangular Matrix

A square matrix $A = [a_{ij}]_{n\times n}$ is said to be lower triangular matrix if all the elements above the principal diagonal are zero.

For example, $\begin{bmatrix} 5 & 0 \\ 3 & 2 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 6 & 0 \\ 1 & 9 & 7 \end{bmatrix}$, are lower triangular matrices of orders 2 and 3 respectively.

Null Matrix

A matrix $A = [a_{ij}]_{m\times n}$ is said to be null matrix if all its elements are equal to zero.

i.e. $a_{ij} = 0, \quad \forall$ i, j

$\therefore$ a null matrix is generally denoted by **O.**

For example, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, etc. are null matrices.

Comparable Matrices

Two matrices are said to be comparable if they are of the same order.

For example,

if $A = \begin{bmatrix} 2 & 5 & 3 \\ 6 & 8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix}$ then A and B are comparable because both are of the same order, i.e. of order 2×3.

Equal Matrices

Two matrices are said to be equal if

(i) they are of same order, and

(ii) the corresponding elements of the matrices are equal.

For example, if $A = \begin{bmatrix} 2 & 8 \\ 3 & x \end{bmatrix}$, $B = \begin{bmatrix} a & 8 \\ 3 & 5 \end{bmatrix}$, then A = B, if a = 2, x = 5.

Example 5: Write orders and types of the following matrices:

(i) $\begin{bmatrix} 2 & 9 \\ 3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & 5 & 7 \\ 0 & 8 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ (vi) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 7 & 6 \end{bmatrix}$ (vii) $\begin{bmatrix} 2 \\ 9 \\ 6 \end{bmatrix}$ (viii) $\begin{bmatrix} 8 & 9 & 1 & 5 \end{bmatrix}$ (ix) $\begin{bmatrix} 2 & 9 & 3 \\ 6 & 4 & 5 \end{bmatrix}$

Solution:

	Order	**Type**
(i)	2×2	Square matrix [$\because$ rows and columns are equal in number.]
(ii)	2×2	Diagonal matrix [$\because$ all the non-diagonal elements are zero.]
(iii)	2×2	Scalar matrix [$\because$ all the diagonal elements are equal and non diagonal element, are zero.]
(iv)	2×2	Identify matrix [$\because$ all the diagonal elements are unity and non diagonal element are zero.]
(v)	3×3	Upper triangular matrix [$\because$ all the elements below the principal diagonal are zero.]
(vi)	3×3	Lower triangular matrix [$\because$ all the elements above the principal diagonal are zero.]
(vii)	3×1	Column matrix [$\because$ it has only one column.]
(viii)	1×4	Row matrix [$\because$ it has only one row.]
(ix)	2×3	Rectangular matrix [$\because$ number of rows $\neq$ numbers of columns.]

Example 6:

(i) If $\begin{bmatrix} 3 & x+y \\ xy & 7+z \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 8 & 4 \end{bmatrix}$, find x, y, z.

(ii) If $\begin{bmatrix} a+5 & -2a & b+6 \\ 2c & 3b+2 & x \\ y+1 & z+3 & x+2 \end{bmatrix} = \begin{bmatrix} 2 & a+9 & 11 \\ c+4 & -b+22 & 3-x \\ 2 & 2z+3 & 5-x \end{bmatrix}$ find a, b, c, x, y, z.

Solution:

(i) We know that two matrices A and B are equal if

(a) their orders are same, and

(b) the corresponding elements of A and B are equal.

$\therefore$ on comparing corresponding elements of two matrices, we have

$3 = 3$

$x + y = 6$ … (1)

$xy = 8$ … (2)

$7 + z = 4 \Rightarrow z = -3$

From (1), $y = 6 - x$ … (3)

Putting y from (3) in (2), we get

$$x(6-x) = 8$$

$$\Rightarrow 6x - x^2 - 8 = 0 \Rightarrow x^2 - 6x + 8 = 0 \Rightarrow x^2 - 4x - 2x + 8 = 0$$

$$\Rightarrow x(x-4) - 2(x-4) = 0 \Rightarrow (x-4)(x-2) = 0 \Rightarrow x = 4, 2$$

When x = 4, y = 6 – 4 = 2 and when x = 2, y = 6 – 2 = 4

$\therefore x = 4,\ y = 2,\ z = -3$ or $x = 2,\ y = 4,\ z = -3.$

(ii) We know that two matrices A and B are equal if

(a) their orders are same, and

(b) the corresponding elements of A and B are equal.

$\therefore$ on comparing corresponding elements of two matrices, we have

$a + 5 = 2 \Rightarrow a = -3$

$-2a = a + 9 \Rightarrow -3a = 9 \Rightarrow a = -3$

$b + 6 = 11 \Rightarrow b = 5$

$2c = c + 4 \Rightarrow c = 4$

$3b + 2 = -b + 22 \Rightarrow 4b = 20 \Rightarrow b = 5$

$x = 3 - x \Rightarrow 2x = 3 \Rightarrow x = \frac{3}{2}$

$y + 1 = 2 \Rightarrow y = 1$

$z + 3 = 2z + 3 \Rightarrow -z = 0 \Rightarrow z = 0$

$x + 2 = 5 - x \Rightarrow 2x = 3 \Rightarrow x = \frac{3}{2}$

$\therefore a = -3,\ b = 5,\ c = 4,\ x = \frac{3}{2},\ y = 1,\ z = 0.$

Here is an exercise for you.

E 2) Find the values of x, y, z, w if $\begin{bmatrix} 3x - 2y & z + w \\ 3z - w & x + y \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 5 & 3 \end{bmatrix}.$

9.4 OPERATIONS ON MATRICES

In school times, a child first learns the natural numbers and then learns how these numbers are added, subtracted, multiplied and divided. Similarly, here also we now see as to how such operations (except division) are applied on matrices.

These operations are explained by first giving a general formula and then examples followed by some exercises.

Remark 5: Division of a matrix by another matrix is meaning less and hence it is not permitted in case of matrices.

9.4.1 Addition of Matrices

Addition of two matrices A and B make sense only if they are of the same order and obtained by adding their corresponding elements. It is denoted by A + B.

That is, if $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ then $A + B = [a_{ij} + b_{ij}]_{m\times n}$

For example,

(i) If $A = \begin{bmatrix} 2 & 3 & 4 \\ 7 & 5 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 & 6 \\ 2 & 9 & 8 \end{bmatrix}$ then

$$A + B = \begin{bmatrix} 2+1 & 3+5 & 4+6 \\ 7+2 & 5+9 & 1+8 \end{bmatrix} = \begin{bmatrix} 3 & 8 & 10 \\ 9 & 14 & 9 \end{bmatrix}.$$

(ii) If A = $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$, B= $\begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 5 \end{bmatrix}$ then A + B does not make any sense because A and B are of different orders.

Properties of Addition of Matrices

If A, B, C are of the same orders over R, (i.e. elements of A, B, C are real numbers) then

(i) A + B = B + A (commutative law)

(ii) (A + B) + C = A + (B + C) (associative law)

(iii) A + **O** = **O** + A = A, where **O** is a null matrix. (existence of additive identity)

(iv) For a given matrix A, there exists a matrix B of the same order such that A + B **=** **O** = B + A.
Here B is called additive inverse of A. (existence of additive inverse)

9.4.2 Scalar Multiplication

Let A = $[a_{ij}]_{m\times n}$ and k is any scalar then scalar multiplication of A by k is denoted by kA and obtained by multiplying each element of A by k.

i.e. kA= $[ka_{ij}]_{m\times n}$

For example,

If A = $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ and k = 7, then kA = 7A = $\begin{bmatrix} 7\times 3 & 7\times 4 \\ 7\times 5 & 7\times 6 \end{bmatrix} = \begin{bmatrix} 21 & 28 \\ 35 & 42 \end{bmatrix}$.

Properties of Scalar Multiplication

If A and B are two matrices of the same order and α, β are scalars (real numbers), then

(i) $\alpha(A + B) = \alpha A + \alpha B$

(ii) $\alpha(\beta A) = (\alpha\beta)A$

(iii) $(\alpha+\beta)A = \alpha A + \beta A$

(iv) 1A = A

9.4.3 Subtraction of Matrices

Subtraction of two matrices A and B make sense only if they are of the same order, and is given by

A – B = A + (– B) = A+ (–1) B, i.e. A – B means addition of two matrices A and – B. So, if A = $[a_{ij}]_{m\times n}$, B = $[b_{ij}]_{m\times n}$, then

A – B = $[a_{ij} + (-1)b_{ij}]_{m\times n} = [a_{ij} - b_{ij}]_{m\times n}$

For example,

(i) If A = $\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$, B = $\begin{bmatrix} 6 & 2 \\ 1 & 10 \end{bmatrix}$, then A – B = $\begin{bmatrix} 2-6 & 4-2 \\ 6-1 & 8-10 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 5 & -2 \end{bmatrix}$.

(ii) If A = $\begin{bmatrix} 2 & 9 & 3 \\ 8 & 4 & 5 \end{bmatrix}$, B = $\begin{bmatrix} 8 & 7 \\ 6 & 5 \end{bmatrix}$, then A – B does not make any sense because A and B are of different orders.

9.4.4 Matrix Multiplication

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ be two matrices, then product of A and B is denoted by AB and is defined only if number of columns in A = number of rows in B and is given by

$AB = C = [c_{ij}]_{m\times p}$

where c_{ij} = (i, j)th element of C and is equal to (ith row of A) (jth column of B)

$$= [a_{i1} \quad a_{i2} \quad \ldots \quad a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ . \\ . \\ . \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

$= \sum_{k=1}^{n} a_{ik}b_{kj}$, i.e. sum of product of first, second, third, … elements of ith row of A with first, second, third, … , elements of jth column of B respectively.

You may notice that the number of rows in AB = number of rows in A, and number of columns in AB = number of columns in B.

Let us make the above concept more clear by taking the following matrices, in particular let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Here A is a matrix of order 3×2 and B be a matrix of order 2×2. As number of columns of A = 2 = number of rows of B.

$\therefore$ AB is defined and is given by

$$AB = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}_{3\times 2}$$

where c_{11} = Product of first row of A and first column of B

= Sum of product of first, second elements of first row of A with first, second elements of first column of B respectively.

= $a_{11}b_{11} + a_{12}b_{21}$,

c_{12} = Product of first row of A and second column of B

= Sum of product of first, second elements of first row of A with first, second elements of second column of B respectively.

= $a_{11}b_{12} + a_{12}b_{22}$,

c_{21} = …, etc.

Properties of Matrix Multiplication

If A, B, C are three matrices such that corresponding multiplications hold then

(1) A(BC) = (AB)C (associative law)

(2) (i) A(B + C) = AB + AC (left distributive law)

(ii) $(A + B)C = AC + BC$ (right distributive law)

(3) If A is a square matrix of order n, then
$I_n A = AI_n = A$, where I_n is the identity matrix of order n.

Remark 6: Commutative law does not hold, in general,

i.e. $AB \neq BA$, in general. But for some cases AB may be equal to BA. This has been explained below:

(i) AB may be defined but BA may not be defined and hence $AB \neq BA$ in this case.

For example, let A be a matrix of order 3×2 and B be a matrix of order 2×4.
Here AB is defined and is of order 3×4.
But BA is not defined ($\because$ number of columns of $B \neq$ number of rows of A).

(ii) AB and BA both may defined but may not be of same order and hence $AB \neq BA$.
For example, let A be a matrix of order be 3×2 and B be a matrix of order 2×3.

Here as number of columns of A = number of rows of B.
$\therefore$ AB is defined and is of order 3×3.
Also, number of columns of B = number of rows of A.
Hence BA is defined but of order 2×2.
$\therefore AB \neq BA$.

(iii) AB and BA both may be defined and of same order but even then they may not be equal.

Let $A = \begin{bmatrix} 5 & 4 \\ 2 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 8 \\ 2 & 9 \end{bmatrix}$

Here, AB and BA both are defined and are of same order.

But $AB = \begin{bmatrix} 5 & 4 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} 30+8 & 40+36 \\ 12+14 & 16+63 \end{bmatrix} = \begin{bmatrix} 38 & 76 \\ 26 & 79 \end{bmatrix}$ and

$BA = \begin{bmatrix} 6 & 8 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 30+16 & 24+56 \\ 10+18 & 8+63 \end{bmatrix} = \begin{bmatrix} 46 & 80 \\ 28 & 71 \end{bmatrix}$.

So, $AB \neq BA$.

However, sometimes, we may observe that AB = BA.

For example, Let $A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$, and $B = \begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix}$.

Here $AB = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 10+12 & 6-6 \\ 20-20 & 12+10 \end{bmatrix} = \begin{bmatrix} 22 & 0 \\ 0 & 22 \end{bmatrix}$ and

$BA = \begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 10+12 & -15+15 \\ -8+8 & 12+10 \end{bmatrix} = \begin{bmatrix} 22 & 0 \\ 0 & 22 \end{bmatrix}$.

Here, AB = BA.

Example 7: If $A = \begin{bmatrix} 2 & 4 & 5 \\ -3 & 6 & 7 \\ 1 & 8 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 6 & 2 \\ 1 & 4 & 5 \\ 8 & 7 & -1 \end{bmatrix}$, then evaluate the following

(i) 3A + 2B (ii) 2A – 3B (iii) AB (iv) BA

Solution:

(i) $3A + 2B = 3\begin{bmatrix} 2 & 4 & 5 \\ -3 & 6 & 7 \\ 1 & 8 & 9 \end{bmatrix} + 2\begin{bmatrix} 3 & 6 & 2 \\ 1 & 4 & 5 \\ 8 & 7 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 12 & 15 \\ -9 & 18 & 21 \\ 3 & 24 & 27 \end{bmatrix} + \begin{bmatrix} 6 & 12 & 4 \\ 2 & 8 & 10 \\ 16 & 14 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 6+6 & 12+12 & 15+4 \\ -9+2 & 18+8 & 21+10 \\ 3+16 & 24+14 & 27-2 \end{bmatrix} = \begin{bmatrix} 12 & 24 & 19 \\ -7 & 26 & 31 \\ 19 & 38 & 25 \end{bmatrix}$$

(ii) $2A - 3B = 2\begin{bmatrix} 2 & 4 & 5 \\ -3 & 6 & 7 \\ 1 & 8 & 9 \end{bmatrix} - 3\begin{bmatrix} 3 & 6 & 2 \\ 1 & 4 & 5 \\ 8 & 7 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 10 \\ -6 & 12 & 14 \\ 2 & 16 & 18 \end{bmatrix} - \begin{bmatrix} 9 & 18 & 6 \\ 3 & 12 & 15 \\ 24 & 21 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 4-9 & 8-18 & 10-6 \\ -6-3 & 12-12 & 14-15 \\ 2-24 & 16-21 & 18+3 \end{bmatrix} = \begin{bmatrix} -5 & -10 & 4 \\ -9 & 0 & -1 \\ -22 & -5 & 21 \end{bmatrix}$$

(iii) $AB = \begin{bmatrix} 2 & 4 & 5 \\ -3 & 6 & 7 \\ 1 & 8 & 9 \end{bmatrix}\begin{bmatrix} 3 & 6 & 2 \\ 1 & 4 & 5 \\ 8 & 7 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 6+4+40 & 12+16+35 & 4+20-5 \\ -9+6+56 & -18+24+49 & -6+30-7 \\ 3+8+72 & 6+32+63 & 2+40-9 \end{bmatrix}$$

$$= \begin{bmatrix} 50 & 63 & 19 \\ 53 & 55 & 17 \\ 83 & 101 & 33 \end{bmatrix} \quad \ldots (1)$$

(iv) $BA = \begin{bmatrix} 3 & 6 & 2 \\ 1 & 4 & 5 \\ 8 & 7 & -1 \end{bmatrix}\begin{bmatrix} 2 & 4 & 5 \\ -3 & 6 & 7 \\ 1 & 8 & 9 \end{bmatrix}$

$$= \begin{bmatrix} 6-18+2 & 12+36+16 & 15+42+18 \\ 2-12+5 & 4+24+40 & 5+28+45 \\ 16-21-1 & 32+42-8 & 40+49-9 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 64 & 75 \\ -5 & 68 & 78 \\ -6 & 66 & 80 \end{bmatrix} \quad \ldots (2)$$

9.4.5 Integral Powers of a Square Matrix

Here, we will learn how higher powers of A are evaluated. We define

$A^2 = A.A$

$A^3 = A^2A$ or $A^3 = AA^2$

$A^4 = A^3A$ or $A^4 = AA^3$ or $A^4 = A^2A^2$

and so on

in general $A^{p+q} = A^pA^q = A^qA^p$.

Remark 7:

(i) We define $A^0 = I$, where I is the identity matrix of the same order as A.

(ii) $(A+B)^2 = A^2 + AB + BA + B^2$.

(iii) $(A+B)^2 = A^2 + 2AB + B^2$ if and only if AB = BA.

Example 8: If A = $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ then find A^4.

Solution: $A^2 = AA = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1+8 & 2+6 \\ 4+12 & 8+9 \end{bmatrix} = \begin{bmatrix} 9 & 8 \\ 16 & 17 \end{bmatrix}$

$A^4 = A^2A^2 = \begin{bmatrix} 9 & 8 \\ 16 & 17 \end{bmatrix}\begin{bmatrix} 9 & 8 \\ 16 & 17 \end{bmatrix} = \begin{bmatrix} 81+128 & 72+136 \\ 144+272 & 128+289 \end{bmatrix} = \begin{bmatrix} 209 & 208 \\ 416 & 417 \end{bmatrix}$

Now, you can try the following exercises.

E 3) If $3X + 2Y = \begin{bmatrix} 4 & 13 \\ 18 & 13 \end{bmatrix}$ and $2X - 3Y = \begin{bmatrix} 7 & 0 \\ -1 & -13 \end{bmatrix}$, then find matrices X and Y.

E 4) Find AB, if defined, in each of the following cases:

(i) $A = \begin{bmatrix} 5 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$

E 5) Evaluate the product $\begin{bmatrix} 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 0 & 1 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 2 & 4 & 1 \\ 5 & 6 & 8 \end{bmatrix}$.

E 6) If $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$, then find A^8.

9.5 TRANSPOSE OF A MATRIX

Transpose of a matrix A is denoted by A' or A^T and is obtained by interchanging rows and columns of A.

For example, if $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \end{bmatrix}$ then $A' = \begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}$.

Properties of Transpose

(i) $(A')' = A$

(ii) $(kA)' = kA'$, where k is a scalar

(iii) $(A+B)' = A' + B'$

(iv) $(A-B)' = A' - B'$

(v) $(AB)' = B'A'$

Symmetric Matrix

A square matrix A is said to be symmetric matrix if $A' = A$.

For example, let $A = \begin{bmatrix} 2 & 5 & 6 \\ 5 & 4 & 3 \\ 6 & 3 & 8 \end{bmatrix}$, then $A' = \begin{bmatrix} 2 & 5 & 6 \\ 5 & 4 & 3 \\ 6 & 3 & 8 \end{bmatrix} = A$.

$\therefore$ A is symmetric.

Skew-Symmetric Matrix

A square matrix A is said to be skew-symmetric matrix if $A' = -A$.

For example, let $A = \begin{bmatrix} 0 & 5 & -3 \\ -5 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$ then

$$A' = \begin{bmatrix} 0 & -5 & 3 \\ 5 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 5 & -3 \\ -5 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} = -A.$$

$\therefore$ A is skew-symmetric.

Remark 8: A square matrix $A = [a_{ij}]_{m\times n}$ will be symmetric if $a_{ij} = a_{ji}, \forall\ i, j$ and will be skew-symmetric if $a_{ij} = -a_{ji}, \forall i, j$ and hence for a skew-symmetric matrix

$a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$

That is, all the diagonal elements of a skew-symmetric matrix are zero.

Example 9: If $A = \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}$ then show that

(i) $\frac{1}{2}(A + A')$ is symmetric, and (ii) $\frac{1}{2}(A - A')$ is skew-symmetric.

Solution:

(i) Let $P = \frac{1}{2}(A + A') = \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}'\right)$

$$= \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 6 & 3 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 3/2 \\ 3/2 & 4 \end{bmatrix} \quad \ldots (1)$$

$$P' = \begin{bmatrix} 3 & 3/2 \\ 3/2 & 4 \end{bmatrix}' = \begin{bmatrix} 3 & 3/2 \\ 3/2 & 4 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2)

$P' = P \Rightarrow P$ is symmetric, i.e. $\frac{1}{2}(A + A')$ is symmetric.

(ii) Let $Q = \frac{1}{2}(A - A') = \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}'\right) = \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}\right)$

$$= \frac{1}{2}\begin{bmatrix} 3-3 & 5+2 \\ -2-5 & 4-4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 7 \\ -7 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 7/2 \\ -7/2 & 0 \end{bmatrix}$$

$$Q' = \begin{bmatrix} 0 & 7/2 \\ -7/2 & 0 \end{bmatrix}' = \begin{bmatrix} 0 & -7/2 \\ 7/2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 7/2 \\ -7/2 & 0 \end{bmatrix} = -Q$$

$\Rightarrow$ Q is skew symmetric, i.e. $\frac{1}{2}(A - A')$ is skew symmetric.

Remark 9:

$$A = \frac{1}{2}A + \frac{1}{2}A = \frac{1}{2}A + \frac{1}{2}A' + \frac{1}{2}A - \frac{1}{2}A' = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = P + Q$$

i.e. A = P + Q, where P is symmetric and Q is skew symmetric.
i.e. every square matrix can be expressed as a sum of a symmetric and a skew-symmetric matrix.

9.6 TRACE OF A MATRIX

In this section we will define trace of a matrix.

Trace of a square matrix $A = [a_{ij}]_{n \times n}$ is denoted by tr (A) and is defined as
tr (A) = sum of diagonal elements of the matrix.
i.e. $tr(A) = a_{11} + a_{22} + a_{33} + \ldots + a_{nn}$

For example, if $A = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 8 & 4 \\ 9 & 1 & -3 \end{bmatrix}$ then $tr(A) = 2 + 8 + (-3) = 7$.

Properties of Trace of a Matrix

If $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$ then

(i) tr (A + B) = tr (A) + tr (B)

(ii) tr (kA) = k tr (A), where k is a scalar

(iii) tr (AB) = tr (BA)

Remark 10: tr (AB) $\neq$ tr (A) tr (B)

Here is an exercise for you.

E 7) (i) Find trace of the matrix A, where $A = \begin{bmatrix} 8 & 7 \\ 5 & 6 \end{bmatrix}$

(ii) Find trace of the matrices I_2, I_3, I_n.

9.7 DETERMINANT OF SQUARE MATRICES

Determinant is a number associated with each square matrix. In this section, we will deal with determinant of square matrices of order 1, 2, 3 and 4. Determinants of square matrices of order greater than 4 can be evaluated in a similar fashion.

9.7.1 Determinant of a Square Matrix of Order 1

If $A = [a_{11}]$ be a square matrix of order 1 then determinant of A is given by $|A| = |a_{11}| = a_{11}$.

For example,

(i) If $A=[5]$ then $|A|=|5|=5$.

(ii) If $A=[-3]$ then $|A|=|-3|=-3$.

Remark 11:

(i) $|A|$ is read as determinant of A, do not read it modulus of A, i.e.

if $A=[-8]$ then $|A|=|-8|=-8$.

But in case of modulus $|-8|=-(-8)=8$.

(ii) The context in which we are using $|\;|$ will clear whether it represents modulus or determinant.

9.7.2 Determinant of a Square Matrix of Order 2×2

If $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22}-a_{21}a_{12}$

Let us take an example:

Example 10: Evaluate the following determinants:

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & 5 \\ 8 & 9 \end{vmatrix}$ (iii) $\begin{vmatrix} x^2 & x^2+1 \\ x & x+1 \end{vmatrix}$

Solution:

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

(ii) $\begin{vmatrix} 3 & 5 \\ 8 & 9 \end{vmatrix} = 27 - 40 = -13$

(iii) $\begin{vmatrix} x^2 & x^2+1 \\ x & x+1 \end{vmatrix} = x^3+x^2-(x^3+x)=x^2-x$

Now, you can try the following exercise.

E 8) Find x in each of the following cases:

(i) $\begin{vmatrix} x & 7 \\ 9 & x+2 \end{vmatrix}=0$ (ii) $\begin{vmatrix} x & x^2 \\ 15 & 5 \end{vmatrix}=0$

9.7.3 Determinant of a Square Matrix of Order 3×3

Before evaluating, the determinant of order 3×3, let us define the minors and cofactors of a square matrix as follows:

Minors and Cofactors

Minor

If $A=[a_{ij}]_{n\times n}$ be a square matrix of order n then minor of $(i, j)^{th}$ element a_{ij} is denoted by M_{ij} and is defined as

M_{ij} = determinant of sub matrix of order n – 1 obtained after deleting i^{th} row and j^{th} column from A.

Example 11: Find the minor of each element of the following matrices:

(i) $\begin{bmatrix} 2 & 5 \\ 4 & -7 \end{bmatrix}$ (ii) $\begin{bmatrix} -3 & 4 & -2 \\ 6 & 5 & 7 \\ -8 & 9 & 1 \end{bmatrix}$

Solution:

(i) Let A = $\begin{bmatrix} 2 & 5 \\ 4 & -7 \end{bmatrix}$

Let M_{ij} denotes the minor of $(i, j)^{th}$ element of the matrix A, i, j = 1, 2.

$\therefore M_{11} = |-7| = -7$ $\begin{bmatrix} \text{Determinant obtained after deleting first} \\ \text{row and first column of matrix A } = |-7| \end{bmatrix}$

Similarly, $M_{12} = |4| = 4$, $M_{21} = |5| = 5$, $M_{22} = |2| = 2$

(ii) Let A = $\begin{bmatrix} -3 & 4 & -2 \\ 6 & 5 & 7 \\ -8 & 9 & 1 \end{bmatrix}$

Let M_{ij} denotes the minor of $(i, j)^{th}$ element of the matrix A, where i, j = 1, 2, 3.

$\therefore M_{11} = \begin{vmatrix} 5 & 7 \\ 9 & 1 \end{vmatrix} = 5 - 63 = -58$ $\begin{bmatrix} \text{After deleting the first row and} \\ \text{first column from A.} \end{bmatrix}$

$M_{12} = \begin{vmatrix} 6 & 7 \\ -8 & 1 \end{vmatrix} = 6 + 56 = 62$ $\begin{bmatrix} \text{After deleting the first row and} \\ \text{second column from A.} \end{bmatrix}$

Similarly,

$$M_{13} = \begin{vmatrix} 6 & 5 \\ -8 & 9 \end{vmatrix} = 54 + 40 = 94$$

$$M_{21} = \begin{vmatrix} 4 & -2 \\ 9 & 1 \end{vmatrix} = 4 + 18 = 22$$

$$M_{22} = \begin{vmatrix} -3 & -2 \\ -8 & 1 \end{vmatrix} = -3 - 16 = -19$$

$$M_{23} = \begin{vmatrix} -3 & 4 \\ -8 & 9 \end{vmatrix} = -27 + 32 = 5$$

$$M_{31} = \begin{vmatrix} 4 & -2 \\ 5 & 7 \end{vmatrix} = 28 + 10 = 38$$

$$M_{32} = \begin{vmatrix} -3 & -2 \\ 6 & 7 \end{vmatrix} = -21 + 12 = -9$$

$$M_{33} = \begin{vmatrix} -3 & 4 \\ 6 & 5 \end{vmatrix} = -15 - 24 = -39$$

Cofactor

If $A = [a_{ij}]_{n\times n}$ be a square matrix of order n then cofactor of $(i, j)^{th}$ element a_{ij} of matrix A is denoted by C_{ij} and is defined by

$C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} denotes the minor of $(i, j)^{th}$ element of the matrix A.

Example 12: Find the cofactor of each element of the following matrices:

(i) $\begin{bmatrix} 2 & 5 \\ 4 & -7 \end{bmatrix}$ (ii) $\begin{bmatrix} -3 & 4 & -2 \\ 6 & 5 & 7 \\ -8 & 9 & 1 \end{bmatrix}$

Solution:

(i) Let $A = \begin{bmatrix} 2 & 5 \\ 4 & -7 \end{bmatrix}$

Let C_{ij} denotes the cofactor of $(i, j)^{th}$ element of the matrix A, i, j = 1, 2.

$\therefore C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (-7) = -7$ [Using Example 11 (i)]

Similarly,

$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 (4) = -4$

$C_{21} = (-1)^{2+1} M_{21} = (-1)^3 (5) = -5$

$C_{22} = (-1)^{2+2} M_{22} = (-1)^4 (2) = 2$

(ii) Let $A = \begin{bmatrix} -3 & 4 & -2 \\ 6 & 5 & 7 \\ -8 & 9 & 1 \end{bmatrix}$

Let C_{ij} denotes the cofactor of $(i, j)^{th}$ element of the matrix A, then

$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (-58) = -58$ [Using Example 11 (ii)]

Similarly,

$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 (62) = -62$

$C_{13} = (-1)^{1+3} M_{13} = (-1)^4 (94) = 94$

$C_{21} = (-1)^{2+1} M_{21} = (-1)^3 (22) = -22$

$C_{22} = (-1)^{2+2} M_{22} = (-1)^4 (-19) = -19$

$C_{23} = (-1)^{2+3} M_{23} = (-1)^5 (5) = -5$

$C_{31} = (-1)^{3+1} M_{31} = (-1)^4 (38) = 38$

$C_{32} = (-1)^{3+2} M_{32} = (-1)^5 (-9) = 9$

$C_{33} = (-1)^{3+3} M_{33} = (-1)^6 (-39) = -39$

Here is an exercise for you.

E 9) Find minor and cofactor of the elements $a_{12}, a_{23}, a_{31}, a_{13}$ where

$$A = [a_{ij}]_{3\times 3} = \begin{bmatrix} 5 & 6 & -2 \\ -3 & 8 & 9 \\ 7 & 10 & -4 \end{bmatrix}$$

Now, we discuss the determinant of a square matrix of order 3×3.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ = Sum of products of the elements of any line (row or column) with their corresponding co-factors.

Let us expand along first row (R_1), we have

$|A| = a_{11}$ (co-factor of a_{11}) $+ a_{12}$ (co-factor of a_{12}) $+ a_{13}$(co-factor of a_{13})

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Remark 12:

(i) We can expand the determinant along any row or column, we will get the same value.

(ii) When we expand a determinant along any row or column we attach + or – sign with each term containing the product of elements of a row (or column) and its corresponding minor. Pattern of +, – signs is shown as under.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

We put + at (1, 1) position and then alternatively– and + are placed, provided either we can move along row or column (we cannot walk diagonally).

(iii) There is no hard and fast rule, to choose a row or column to expand a determinant. But if we choose that row or column which contains maximum number of zero, it will reduce a lot of our calculation work.

Example 13: Evaluate the following determinants:

(i) $\begin{vmatrix} 3 & 2 & -1 \\ 5 & 4 & 6 \\ -3 & 1 & 7 \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{vmatrix}$ (iii) $\begin{vmatrix} 3 & 1 & 2 \\ 0 & 9 & 6 \\ 0 & 5 & 4 \end{vmatrix}$

Solution:

(i) Let $\Delta = \begin{vmatrix} 3 & 2 & -1 \\ 5 & 4 & 6 \\ -3 & 1 & 7 \end{vmatrix}$

Expanding along R_1 (first row)

$\Delta = 3(28 - 6) - 2(35 + 18) - 1(5 + 12) = 66 - 106 - 17 = -57$

(ii) Let $\Delta = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{vmatrix}$

Expanding along R_1 (first row)

$\Delta = 2(2-4) - 1(1-4) + 2(2-4) = -4 + 3 - 4 = -5$

(iii) Let $\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 0 & 9 & 6 \\ 0 & 5 & 4 \end{vmatrix}$

Expanding along C_1 (first column) $\left[\because \text{ it contains maximum number of zeros.}\right]$

$\Delta = 3(36-30) - 0 + 0 = 18$

Here is an exercise for you.

E 10) If $A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 5 & 1 \\ 2 & 4 & 8 \end{bmatrix}$ then show that $|A| = 0$.

9.7.4 Determinant of Square Matrices of Order 4×4 and of Higher Order

The procedure of expanding the determinant of order 4 or more is the same as we discussed in case of order 3×3.

Example 14: Evaluate $\Delta = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 3 & 5 \\ 6 & 4 & -2 & 7 \\ -4 & 8 & 9 & -5 \end{vmatrix}$

Solution: Expanding along R_1, we get

$$\Delta = 1\begin{vmatrix} -1 & 3 & 5 \\ 4 & -2 & 7 \\ 8 & 9 & -5 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 & 5 \\ 6 & -2 & 7 \\ -4 & 9 & -5 \end{vmatrix} + 3\begin{vmatrix} 2 & -1 & 5 \\ 6 & 4 & 7 \\ -4 & 8 & -5 \end{vmatrix} - 4\begin{vmatrix} 2 & -1 & 3 \\ 6 & 4 & -2 \\ -4 & 8 & 9 \end{vmatrix}$$

Expanding each determinant of order 3×3 along R_1, we get

$$\begin{aligned}\Delta &= 1[-1(10-63) - 3(-20-56) + 5(36+16)] - 2[2(10-63) \\ &\quad - 3(-30+28) + 5(54-8)] + 3[2(-20-56) - (-1)(-30+28) + 5(48+16)] \\ &\quad - 4[2(36+16) - (-1)(54-8) + 3(48+16)] \\ &= (53+228+260) - 2(-106+6+230) + 3(-152-2+320) - 4(104+46+192) \\ &= 541 - 260 + 498 - 1368 \\ &= -589\end{aligned}$$

Remark 13:

(i) If A is square matrix then determinant of A is unique.

(ii) If A is not a square matrix then determent of A does not exist.

9.8 PROPERTIES OF DETERMINANTS

In Sec. 9.7 of this unit you have become familiar about how to expand the determinants of orders 1, 2, 3, or of higher order. But as you have seen that it requires lot of calculations and is a time consuming process. To avoid such calculations and to reduce the time of evaluation, we will use properties of determinants.

In this section, we will discuss some properties of the determinants. We shall give the proofs of these properties only for determinants of order 3×3. But remember that these properties hold good for all orders of the determinants. Let us discuss these one by one. Our way to move further is that, first we list all the properties and then some examples will be solved to get the idea how these properties are used and useful.

P 1 $|A'| = |A|$, i.e. determinants of a matrix and its transpose are equal.

Proof: Let $A = \begin{bmatrix} a & b & c \\ x & y & z \\ l & m & n \end{bmatrix}$ … (1)

$$|A| = \begin{vmatrix} a & b & c \\ x & y & z \\ l & m & n \end{vmatrix}$$

Expanding along R_1

$|A| = a(ny - mz) - b(nx - lz) + c(mx - ly)$ … (2)

From (1), we get

$$A' = \begin{bmatrix} a & x & l \\ b & y & m \\ c & z & n \end{bmatrix}$$

$$\therefore\ |A'| = \begin{vmatrix} a & x & l \\ b & y & m \\ c & z & n \end{vmatrix}$$

Expanding along R_1

$$\begin{aligned} |A'| &= a(ny - mz) - x(bn - cm) + l(bz - cy) \\ &= a(ny - mz) - bnx + cmx + lbz - cly \\ &= a(ny - mz) - b(nx - lz) + c(mx - ly) \quad \dots (3) \end{aligned}$$

From (2) and (3), we get

$|A'| = |A|$

P 2 If any two rows (or columns) of a determinant are interchanged, then sign of determinant is multiplied by (–1).

Proof: Let $\Delta = \begin{vmatrix} a & b & c \\ x & y & z \\ l & m & n \end{vmatrix}$ … (1)

Expanding along R_1

$\Delta = a(ny - mz) - b(nx - lz) + c(mx - ly)$ … (2)

Let us interchange the first and second rows of the given determinant we have a new determinant Δ_1 (say) as

$$\Delta_1 = \begin{vmatrix} x & y & z \\ a & b & c \\ l & m & n \end{vmatrix}$$

Expanding along R_1

$$\begin{aligned}\Delta_1 &= x(bn - cm) - y(an - cl) + z(am - bl) \\ &= bnx - cmx - any + cly + amz - blz \\ &= -a(ny - mz) + b(nx - lz) - c(mx - ly) \\ &= -[a(ny - mz) - b(nx - lz) + c(mx - ly)] \\ &= -\Delta \qquad\qquad [\text{Using } (2)]\end{aligned}$$

Remark 14: Here we interchanged R_1 and R_2. In fact we can interchange any two rows or any two columns, result remains the same in each case.

P 3 If any two rows or columns of a determinant are identical then value of the determinant vanishes.

Proof: Let $\Delta = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix}$, where R_1 and R_2 are identical

Expanding along R_1, we get

$$\Delta = a(bz - cy) - b(az - cx) + c(ay - bx) = abz - acy - abz + bcx + acy - bcx = 0$$

P 4 If each element of a row (or a column) of a determinant is multiplied by a scalar k (say), then value of the new determinant is k times the original given determinant.

Proof: Let $\Delta = \begin{vmatrix} a & b & c \\ x & y & z \\ l & m & n \end{vmatrix}$

Expanding along R_1

$$\Delta = a(ny - mz) - b(nx - lz) + c(mx - ly) \qquad \ldots (1)$$

Let $\Delta_1 = \begin{vmatrix} ka & b & c \\ kx & y & z \\ kl & m & n \end{vmatrix}$ $\left[\begin{array}{l}\text{Here, the elements of first column of} \\ \Delta \text{ have been multiplied with k.}\end{array}\right]$

Expanding along R_1

$$\begin{aligned}\Delta_1 &= ka(ny - mz) - b(knx - klz) + c(kmx - kly) \\ &= k[a(ny - mz) - b(nx - lz) + c(mx - ly)] = k\Delta \qquad \ldots (2)\ [\text{Using } (1)]\end{aligned}$$

From (1) and (2)

$$\Delta_1 = k\Delta$$

Hence proved

Remark 15: This property implies that if there is some factor common in all elements of any line then we can write it as the factor of the whole determinant.

For example, $\begin{vmatrix} 5a & b & c \\ 5x & y & z \\ 5l & m & n \end{vmatrix} = 5\begin{vmatrix} a & b & c \\ x & y & z \\ l & m & n \end{vmatrix}$

P 5 If each element of a row (or column) of a determinant is expressed as a sum of two (or more) terms, then the determinant can be expressed as the sum of two (or more) determinants.

Proof: Let $\Delta = \begin{vmatrix} a+\lambda & b+\mu & c+\nu \\ x & y & z \\ l & m & n \end{vmatrix}$, then expanding along R_1, we get

$$\Delta = (a+\lambda)\begin{vmatrix} y & z \\ m & n \end{vmatrix} - (b+\mu)\begin{vmatrix} x & z \\ l & n \end{vmatrix} + (c+\nu)\begin{vmatrix} x & y \\ l & m \end{vmatrix}$$

$$= \left[a\begin{vmatrix} y & z \\ m & n \end{vmatrix} - b\begin{vmatrix} x & z \\ l & n \end{vmatrix} + c\begin{vmatrix} x & y \\ l & m \end{vmatrix} \right] + \left[\lambda\begin{vmatrix} y & z \\ m & n \end{vmatrix} - \mu\begin{vmatrix} x & z \\ l & n \end{vmatrix} + \nu\begin{vmatrix} x & y \\ l & m \end{vmatrix} \right]$$

$$= \begin{vmatrix} a & b & c \\ x & y & z \\ l & m & n \end{vmatrix} + \begin{vmatrix} \lambda & \mu & \nu \\ x & y & z \\ l & m & n \end{vmatrix}$$

P 6 If to each element of any row (or column), we add some scalar multiple of another row (or column) and some other scalar multiple of some other row (or column), the value of determinant remains unaltered.

Proof: Let $\Delta = \begin{vmatrix} a & b & c \\ x & y & z \\ l & m & n \end{vmatrix}$ … (1)

and $\Delta_1 = \begin{vmatrix} a & b & c \\ x & y & z \\ l+ka & m+kb & n+kc \end{vmatrix}$

where Δ_1 is obtained from Δ by operating $R_3 \to R_3 + kR_1$

i.e. k times R_1 has been added to R_3.

$$\Delta_1 = \begin{vmatrix} a & b & c \\ x & y & z \\ l & m & n \end{vmatrix} + \begin{vmatrix} a & b & c \\ x & y & z \\ ka & kb & kc \end{vmatrix} \quad \text{[Using property 5]}$$

$$= \Delta + k\begin{vmatrix} a & b & c \\ x & y & z \\ a & b & c \end{vmatrix} \quad \begin{bmatrix} \text{Using (1) and taking k} \\ \text{common from third} \\ \text{rows of second determanant} \end{bmatrix}$$

$$= \Delta + k(0) = \Delta + 0 = \Delta \quad \begin{bmatrix} \because R_1 \text{ and } R_2 \text{ are identical and} \\ \text{so using property 3} \end{bmatrix}$$

Hence proved

Remark 16: If operations of the type $R_i \to R_i + kR_j$ are used more than one in a single step then keep it always in mind that row which has been affected in one operation cannot be used in other operation.

For example,

(i) $R_1 \to R_1 + 2R_3,\ R_2 \to R_2 + 5R_1$ is not allowed because R_1 has been affected by first operation, so it cannot be used in second operation in the same step.

(ii) $R_1 \to R_1 + 3R_3,\ \ R_2 \to R_2 + 2R_3,$ etc. are allowed.

P 7 If all the elements of any line (row or column) are zero then value of the determinant vanishes.

Proof: Let $\Delta = \begin{vmatrix} 0 & 0 & 0 \\ x & y & z \\ l & m & n \end{vmatrix}$, then evaluating along R_1, we get

$\Delta = (0)(ny - mz) - (0)(nx - lz) + (0)(mx - ly) = 0 - 0 + 0 = 0$

Example 15: Evaluate the following determinants:

(i) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$ (ii) $\begin{vmatrix} -3 & 5 & -2 \\ 8 & 9 & -17 \\ 3 & -6 & 3 \end{vmatrix}$ (iii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$

(iv) $\begin{vmatrix} 3 & x & xyz \\ 3 & y & xyz \\ 3 & z & xyz \end{vmatrix}$ (v) $\begin{vmatrix} 2 & 3 & 30 \\ 5 & 4 & 54 \\ 6 & 1 & 42 \end{vmatrix}$ (vi) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

(vii) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ (viii) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$ (ix) $\begin{vmatrix} 2x+3 & x & x \\ x & 2x+3 & x \\ x & x & 2x+3 \end{vmatrix}$

Solution:

(i) Let $\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

Operating $C_1 \to C_1 + C_2 + C_3$

$\Delta = \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0$ [$\because$ all elements of C_1 are zero, so using P7.]

(ii) Let $\Delta = \begin{vmatrix} -3 & 5 & -2 \\ 8 & 9 & -17 \\ 3 & -6 & 3 \end{vmatrix}$

Operating $C_1 \to C_1 + C_2 + C_3$

$\Delta = \begin{vmatrix} -3+5-2 & 5 & -2 \\ 8+9-17 & 9 & -17 \\ 3-6+3 & -6 & 3 \end{vmatrix}$

$= \begin{vmatrix} 0 & 5 & -2 \\ 0 & 9 & -17 \\ 0 & -6 & 3 \end{vmatrix} = 0$ [$\because$ all the element of C_1 are zero, so using P7.]

(iii) Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$

Operating $R_3 \to R_3 + R_2$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a+b+c & a+b+c & a+b+c \end{vmatrix}$$

Taking (a + b + c) common from R_3

$$\Delta = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = (a+b+c)(0) = 0 [\because R_1 \text{ and } R_3 \text{ are identical.}]$$

(iv) Let $\Delta = \begin{vmatrix} 3 & x & xyz \\ 3 & y & xyz \\ 3 & z & xyz \end{vmatrix}$

Taking 3, xyz common from C_1 and C_3 respectively

$$\Delta = 3xyz\begin{vmatrix} 1 & x & 1 \\ 1 & y & 1 \\ 1 & z & 1 \end{vmatrix} = 3xyz\,(0) = 0 \qquad [\because C_1 \text{ and } C_3 \text{ are identical.}]$$

(v) Let $\Delta \begin{vmatrix} 2 & 3 & 30 \\ 5 & 4 & 54 \\ 6 & 1 & 42 \end{vmatrix}$

Taking 6 common from C_3

$$\Delta = 6\begin{vmatrix} 2 & 3 & 5 \\ 5 & 4 & 9 \\ 6 & 1 & 7 \end{vmatrix}$$

Operating $C_3 \to C_3 - C_1 - C_2$

$$\Delta = 6\begin{vmatrix} 2 & 3 & 0 \\ 5 & 4 & 0 \\ 6 & 1 & 0 \end{vmatrix} = 6\,(0) = 0 [\because \text{all the elements of } C_3 \text{ are zero, so using P7.}]$$

(vi) Let $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

Operating $R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1$

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0.(y-x) & (y-x).1 & (y-x)(y+x) \\ 0.(z-x) & (z-x).1 & (z-x)(z+x) \end{vmatrix}$$

Taking y – x, z – x common from R_2, R_3 respectively

$$\Delta = (y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$$

Operating $R_3 \to R_3 - R_2$

$$\Delta = (y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 0 & z-y \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0.(z-y) & 0.(z-y) & 1.(z-y) \end{vmatrix}$$

Taking (z – y) common from R_3

$$\Delta = (y-x)(z-x)(z-y)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1, we get

$$\Delta = (y-x)(z-x)(z-y)[1(1-0)-0+0]$$
$$= (x-y)(y-z)(z-x)$$

(vii) Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Operating $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix}$$

Taking (a + b+ c) common from C_1

$$\Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Operating $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$\Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

Expanding along C_1

$$\Delta = (a+b+c)[1\{(c-b)(b-c)-(a-b)(a-c)\}-0+0]$$
$$= (a+b+c)[bc-c^2-b^2+bc-(a^2-ac-ab+bc)]$$
$$= (a+b+c)(ab+bc+ca-a^2-b^2-c^2)$$

(viii) Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$

Operating $C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^3 & y^3-x^3 & z^2-x^3 \end{vmatrix}$$

Taking y – x, z – x common from C_2, C_3 respectively

$$\Delta = (y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^3 & y^2+x^2+xy & z^2+x^2+zx \end{vmatrix}$$

Operating $C_3 \to C_3 - C_2$

$$\Delta = (y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x^3 & x^2+y^2+xy & z^2-y^2+x(z-y) \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x^3 & x^2+y^2+xy & (z-y)(z+y+x) \end{vmatrix}$$

Taking (z – y) (x + y + z) common from C_3

$$\Delta = (y-x)(z-x)(z-y)(x+y+z)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x^3 & x^2+y^2+xy & 1 \end{vmatrix}$$

Expanding along R_1

$$\Delta = (y-x)(z-x)(z-y)(x+y+z)[1(1-0)-0+0]$$
$$= (x-y)(y-z)(z-x)(x+y+z)$$

(ix) Let $\Delta = \begin{vmatrix} 2x+3 & x & x \\ x & 2x+3 & x \\ x & x & 2x+3 \end{vmatrix}$

Operating $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} 4x+3 & x & x \\ 4x+3 & 2x+3 & x \\ 4x+3 & x & 2x+3 \end{vmatrix}$$

Taking 4x + 3 common from C_1

$$\Delta = (4x+3)\begin{vmatrix} 1 & x & x \\ 1 & 2x+3 & x \\ 1 & x & 2x+3 \end{vmatrix}$$

Operating $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$\Delta = (4x+3)\begin{vmatrix} 1 & x & x \\ 0 & x+3 & 0 \\ 0 & 0 & x+3 \end{vmatrix}$$

Expanding along C_1

$$\Delta = (4x+3)[1\{(x+3)^2 - 0\} - 0 + 0] = (4x+3)(x+3)^2$$

Now, you can try the following exercise.

E 11) Prove the following

(i) $\begin{vmatrix} ab & 1 & c(a+b) \\ bc & 1 & a(b+c) \\ ca & 1 & b(c+a) \end{vmatrix} = 0$ [Without expanding]

(ii) $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1+x^3 & 1+y^3 & 1+z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(1+xyz)$

[Using properties]

(iii) $\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} = 4abc$ [Using properties]

9.9 SUMMARY

In this unit we have covered following topics:

1) Definition with examples of a matrix.
2) Types of matrices with examples.
3) Operations on matrices.
4) Integral powers of a square matrix.
5) Trace of a matrix.
6) Determinant and its properties.

9.10 SOLUTIONS/ANSWERS

E 1) $A = [a_{ij}]_{3\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$**,** where $a_{ij} = |i-j|$

$a_{11} = |1-1| = |0| = 0$, $a_{12} = |1-2| = |-1| = -(-1) = 1$, $a_{21} = |2-1| = |1| = 1$,

$a_{22} = |2-2| = |0| = 0$, $a_{31} = |3-1| = |2| = 2$, $a_{32} = |3-2| = |1| = 1$

$\therefore A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$

E 2) We know that two matrices A and B are equal if

(a) their orders are same, and

(b) the corresponding elements of A and B are equal.

$\therefore$ on comparing corresponding elements of two matrices, we have

$3x - 2y = -1$ … (1)

$z + w = 7$ … (2)

$3z - w = 5$ … (3)

$x + y = 3$ … (4)

Equation (1) + 2× equation (4) gives

$$\begin{array}{l} 3x - 2y = -1 \\ \underline{2x + 2y = 6} \\ 5x \quad\quad = 5 \end{array}$$

$\Rightarrow x = 1$

Putting x = 1 in (4), we get

$1 + y = 3 \Rightarrow y = 2$

(2) + (3) gives.

$4z = 12 \Rightarrow z = 3$

Putting z = 3 in (2), we get

$3 + w = 7 \Rightarrow w = 4$

$\therefore x = 1, y = 2, z = 3, w = 4.$

E 3) $3X + 2Y = \begin{bmatrix} 4 & 13 \\ 18 & 13 \end{bmatrix}$ … (1)

$2X - 3Y = \begin{bmatrix} 7 & 0 \\ -1 & -13 \end{bmatrix}$ … (2)

Equation (1) $\times 3 + 2 \times$ equation (2) gives

$$9X + 6Y + 4X - 6Y = 3\begin{bmatrix} 4 & 13 \\ 18 & 13 \end{bmatrix} + 2\begin{bmatrix} 7 & 0 \\ -1 & -13 \end{bmatrix}$$

$$13X = \begin{bmatrix} 12 & 39 \\ 54 & 39 \end{bmatrix} + \begin{bmatrix} 14 & 0 \\ -2 & -26 \end{bmatrix} = \begin{bmatrix} 26 & 39 \\ 52 & 13 \end{bmatrix}$$

$$\Rightarrow X = \frac{1}{13}\begin{bmatrix} 26 & 39 \\ 52 & 13 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$ [By scalar multiplication property]

Putting this value of X in (1), we get

$$3\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} + 2Y = \begin{bmatrix} 4 & 13 \\ 18 & 13 \end{bmatrix}$$

$$\Rightarrow 2Y = \begin{bmatrix} 4 & 13 \\ 18 & 13 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 13 \\ 18 & 13 \end{bmatrix} + \begin{bmatrix} -6 & -9 \\ -12 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 6 & 10 \end{bmatrix}$$

$$\Rightarrow Y = \frac{1}{2}\begin{bmatrix} -2 & 4 \\ 6 & 10 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & 5 \end{bmatrix}$$ [By scalar multiplication property]

$$\therefore X = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \text{ and } Y = \begin{bmatrix} -1 & 2 \\ 3 & 5 \end{bmatrix}.$$

E 4) (i) Order of A is 1×2 and order of B is 3×1.

$\therefore$ number of columns in A $\neq$ number of rows in B.

$\Rightarrow$ AB is not defined.

(ii) Number of columns in A = number of rows in B = 1.

$\Rightarrow$ AB is defined and is given by

$$AB = \begin{bmatrix} 3 \\ 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \end{bmatrix} = \begin{bmatrix} 3\times 5 & 3\times 6 \\ 4\times 5 & 4\times 6 \end{bmatrix} = \begin{bmatrix} 15 & 18 \\ 20 & 24 \end{bmatrix}$$

(iii) AB is defined and is given by

$$AB = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 3+12+5 & 6+16+6 \\ 2+15+30 & 4+20+36 \end{bmatrix} = \begin{bmatrix} 20 & 28 \\ 47 & 60 \end{bmatrix}$$

(iv) AB is defined and is given by

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 10+9 & 8+6 \\ 5+0 & 4+0 \end{bmatrix} = \begin{bmatrix} 19 & 14 \\ 5 & 4 \end{bmatrix}$$

E 5) $$\begin{bmatrix} 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 0 & 1 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 2 & 4 & 1 \\ 5 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 8+0+10 & 10+3+30 \end{bmatrix}\begin{bmatrix} 2 & 4 & 1 \\ 5 & 6 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 43 \end{bmatrix}\begin{bmatrix} 2 & 4 & 1 \\ 5 & 6 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 36+215 & 72+258 & 18+344 \end{bmatrix}$$

$$= \begin{bmatrix} 251 & 330 & 362 \end{bmatrix}$$

E 6) $$A^2 = AA = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 2+6 & 0+9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix}$$

$$A^4 = A^2A^2 = \begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 8+72 & 0+81 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 80 & 81 \end{bmatrix}$$

$$A^8 = A^4A^4 = \begin{bmatrix} 1 & 0 \\ 80 & 81 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 80 & 81 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 0+0 \\ 80+6480 & 0+6561 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 6560 & 6561 \end{bmatrix}$$

E 7) (i) tr(A) = sum of diagonal elements = 8 + 6 = 14

(ii) We know that in an identity matrix, all the diagonal elements are unity.

$\therefore$ tr (I_2) = 1+ 1 = 2 [$\because I_2$ is identity matrix of order 2×2].

Similarly, $tr(I_3) = 1+1+1 = 3$, $tr(I_n) = \underbrace{1+1+1+\ldots+1}_{\text{n times}} = n$.

E 8) (i) $$\begin{vmatrix} x & 7 \\ 9 & x+2 \end{vmatrix} = 0 \Rightarrow x(x+2) - 63 = 0$$

$$\Rightarrow x^2 + 2x - 63 = 0 \Rightarrow x^2 + 9x - 7x - 63 = 0$$

$$\Rightarrow x(x+9) - 7(x+9) = 0$$

$$\Rightarrow (x + 9)(x - 7) = 0 \Rightarrow x = 7, -9$$

(ii) $$\begin{vmatrix} x & x^2 \\ 15 & 5 \end{vmatrix} = 0 \Rightarrow 5x - 15x^2 = 0$$

$$\Rightarrow 15x^2 - 5x = 0 \Rightarrow 5x(3x-1) = 0 \Rightarrow x = 0, 1/3$$

E 9) Let M_{ij} and C_{ij} denote the minor and cofactor of $(i, j)^{th}$ element in the matrix A respectively then

$$M_{12} = \begin{vmatrix} -3 & 9 \\ 7 & -4 \end{vmatrix} = 12 - 63 = -51, \; M_{23} = \begin{vmatrix} 5 & 6 \\ 7 & 10 \end{vmatrix} = 50 - 42 = 8$$

$$M_{31} = \begin{vmatrix} 6 & -2 \\ 8 & 9 \end{vmatrix} = 54 - (-16) = 54 + 16 = 70$$

$$M_{13} = \begin{vmatrix} -3 & 8 \\ 7 & 10 \end{vmatrix} = -30 - 56 = -86$$

$$C_{12} = (-1)^{1+2} M_{12} = (-1)^3(-51) = 51,\ C_{23} = (-1)^{2+3} M_{23} = (-1)^5(8) = -8$$

$$C_{31} = (-1)^{3+1} M_{31} = (-1)^4(70) = 70$$

$$C_{13} = (-1)^{1+3} M_{13} = (-1)^4(-86) = -86$$

E 10) $|A| = \begin{vmatrix} 1 & 2 & 4 \\ -3 & 5 & 1 \\ 2 & 4 & 8 \end{vmatrix}$

Expanding along R_1

$$|A| = 1(40-4) - 2(-24-2) + 4(-12-10) = 36 + 52 - 88 = 0$$

E 11) (i) L.H.S. $= \begin{vmatrix} ab & 1 & c(a+b) \\ bc & 1 & a(b+c) \\ ca & 1 & b(c+a) \end{vmatrix} = \begin{vmatrix} ab & 1 & ac+bc \\ bc & 1 & ab+ac \\ ca & 1 & bc+ab \end{vmatrix}$

Operating $C_3 \to C_3 + C_1$

$$\text{L.H.S.} = \begin{vmatrix} ab & 1 & ab+bc+ca \\ bc & 1 & ab+bc+ca \\ ca & 1 & ab+bc+ca \end{vmatrix}$$

Taking ab + bc + ca common from C_3

$$\text{L.H.S.} = (ab+bc+ca)\begin{vmatrix} ab & 1 & 1 \\ bc & 1 & 1 \\ ca & 1 & 1 \end{vmatrix}$$

$= (ab+bc+ca)(0) = 0 = \text{R.H.S.}$ [$\because C_2$ and C_3 are identical.]

(ii) Let $\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1+x^3 & 1+y^3 & 1+z^3 \end{vmatrix}$

$= \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$ [Using property 5]

Taking x, y, z common from C_1, C_2, C_3 of the second determinant respectively.

$$\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} + xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Operating $R_1 \leftrightarrow R_3$ on first determinant

$$\Delta = (-1)\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix} + xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Operating $R_2 \leftrightarrow R_3$ on first determinant

$$\Delta = (-1)(-1)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Taking determinant common from both terms

$$\Delta = (1+xyz)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Operating $C_2 \to C_2 - C_1;\ C_3 \to C_3 - C_1$

$$\Delta = (1+xyz)\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix}$$

Taking $y-x$, $z-x$ common from C_2, C_3 respectively

$$\Delta = (1+xyz)(y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & x+y & z+x \end{vmatrix}$$

Operating $C_3 \to C_3 - C_2$

$$\Delta = (1+xyz)(y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x^2 & x+y & z-y \end{vmatrix}$$

Expanding along R_1

$$\Delta = (1+xyz)(y-x)(z-x)[1\{(z-y)-0\}-0+0]$$

$$= (1+xyz)(y-x)(z-x)(z-y) = (x-y)(y-z)(z-x)(1+xyz) = \text{R.H.S.}$$

(iii) $\text{L.H.S} = \begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix}$

Operating $R_1 \to R_1 - R_2 - R_3$

$$\text{L.H.S.} = \begin{vmatrix} 0 & -2b & -2a \\ a & b+c & a \\ b & b & c+a \end{vmatrix}$$

Operating $C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$

$$\text{L.H.S.} = \begin{vmatrix} 0 & -2b & -2a \\ a & b+c-a & 0 \\ b & 0 & c+a-b \end{vmatrix}$$

Expanding along R_1

$$\text{L.H.S.} = 0-(-2b)[a(c+a-b)-0]+(-2a)[0-b(b+c-a)]$$

$$= 2ab(c+a-b)+2ab(b+c-a) = 2ab(c+a-b+b+c-a)$$

$$= 2ab(2c) = 4abc = \text{R.H.S.}$$