UNIT 10 APPLICATIONS OF MATRICES AND DETERMINANTS

Structure

10.1 INTRODUCTION

In the previous unit, we have discussed matrices, types of matrices and determinants of square matrices of orders 1, 2, 3 and higher orders. We have also discussed minors, cofactors of square matrices and properties of determinants.

In this unit, we will learn some applications of matrices and determinants such as solutions of simultaneous linear equations by using matrix method and Cramer's rule.

Objectives

After completing unit, you should be able to:

- find the adjoint of a square matrix;
- find the inverse of a square matrix;
- solve the simultaneous linear equations with the help of matrix methods; and
- solve the simultaneous linear equations with the help of Cramer's rule.

10.2 ADJOINT OF A MATRIX

In next section, i.e. Sec.10.3, we will discuss inverse of a square matrix. But in order define inverse of a square matrix we use the concept of adjoint of the square matrix, so in this section we are going to discuss adjoint of the square matrix.

Adjoint of a Matrix: Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order $n \times n$, then adjoint of A is denoted by adjA and is defined as

$$\text{adj}A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \dots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \dots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \dots & A_{3n} \\ . & . & . & & . \\ . & . & . & \dots & . \\ . & . & . & & . \\ A_{n1} & A_{n2} & A_{n3} & \dots & A_{nn} \end{bmatrix}'$$

where A_{ij} denotes the cofactor of $(i, j)^{th}$ element of the matrix A.

It may be verified that $A(adjA) = (adjA)A = |A|\, I$,
where I is an identity matrix of order n.

Example 1: Find the adjA, where $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 0 \\ -1 & -2 & 1 \end{bmatrix}$.

Also verify that $A(adjA) = (adjA)A = |A| I$.

Solution: $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 0 \\ -1 & -2 & 1 \end{bmatrix}$

Let A_{ij} denotes the cofactor of $(i, j)^{th}$ element of the matrix A.

$$\therefore A_{11} = (-1)^{1+1}\begin{vmatrix} 5 & 0 \\ -2 & 1 \end{vmatrix} = (-1)^2(5-0) = 5$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} -4 & 0 \\ -1 & 1 \end{vmatrix} = (-1)^3(-4-0) = 4$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} -4 & 5 \\ -1 & -2 \end{vmatrix} = (-1)^4(8+5) = 13$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} = (-1)^3(2+6) = -8$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = (-1)^4(1+3) = 4$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} = (-1)^5(-2+2) = 0$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix} = (-1)^4(0-15) = -15$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ -4 & 0 \end{vmatrix} = (-1)^5(0+12) = -12$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ -4 & 5 \end{vmatrix} = (-1)^6(5+8) = 13$$

$$\therefore adjA = \begin{bmatrix} 5 & 4 & 13 \\ -8 & 4 & 0 \\ -15 & -12 & 13 \end{bmatrix}' = \begin{bmatrix} 5 & -8 & -15 \\ 4 & 4 & -12 \\ 13 & 0 & 13 \end{bmatrix}$$

To verify $A(adjA) = (adjA)A = |A| I$, we proceed as follows

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 0 \\ -1 & -2 & 1 \end{vmatrix}$$

Expanding along C_3

$$|A| = 3(8+5) - 0 + 1(5+8) = 39 + 13 = 52 \quad \dots (1)$$

$$A(\text{adj}A) = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 0 \\ -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 5 & -8 & -15 \\ 4 & 4 & -12 \\ 13 & 0 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 5+8+39 & -8+8+0 & -15-24+39 \\ -20+20+0 & 32+20+0 & 60-60+0 \\ -5-8+13 & 8-8+0 & 15+24+13 \end{bmatrix} = \begin{bmatrix} 52 & 0 & 0 \\ 0 & 52 & 0 \\ 0 & 0 & 52 \end{bmatrix}$$

$$= 52\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 52I = |A|I \qquad \ldots (2) \quad [\text{Using } (1)]$$

$$(\text{adj}A)A = \begin{bmatrix} 5 & -8 & -15 \\ 4 & 4 & -12 \\ 13 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 0 \\ -1 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5+32+15 & 10-40+30 & 15-0-15 \\ 4-16+12 & 8+20+24 & 12+0-12 \\ 13+0-13 & 26+0-26 & 39+0+13 \end{bmatrix} = \begin{bmatrix} 52 & 0 & 0 \\ 0 & 52 & 0 \\ 0 & 0 & 52 \end{bmatrix}$$

$$= 52\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 52I = |A|I \qquad \ldots (3)$$

From (2) and (3), we have A(adjA) = (adjA)A= $|A|$I

Now, you can try the following exercise.

E 1) If A = $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$ then verify that A (adjA) = (adjA) A = $|A|I_2$.

10.3 INVERSE OF A MATRIX

You are familiar with the concept of the multiplicative inverse of a real number.

e.g. you know the multiplicative inverse of 5, $\frac{2}{3}$, etc.

Multiplicative inverse of 5 is $\frac{1}{5}$ and of $\frac{2}{3}$ is $\frac{3}{2}$, etc. $\left[\because 5\times\frac{1}{5}=1, \frac{2}{3}\times\frac{3}{2}=1\right]$

i.e. when a number is multiplied with its multiplicative inverse, it gives 1. Similarly, in case of matrices, when a square matrix is multiplied with its multiplicative inverse, it gives identity matrix (I). That is, in case of matrices, identity, matrix plays the role as 1 plays in case of real numbers.

Inverse of a Matrix

Let A be a square matrix of order n. If there exists a square matrix B of order n such that

AB = BA = I_n, then B is known as inverse of A and is denoted by A^{-1}.

As verified above that A(adjA) = (adjA)A = $|A|$I,

$\therefore A(\text{adj}A) = |A|I$

$$\Rightarrow A\left[\frac{1}{|A|}(\text{adj}A)\right] = I, \text{ provided } |A| \neq 0$$

Now, by definition of inverse of a matrix, $\frac{1}{|A|}(\text{adj}A)$ is inverse of A.

i.e. $A^{-1} = \frac{1}{|A|}(\text{adj}A)$

Properties of Inverse of a Matrix

(i) The inverse of a square matrix, if exists is unique.

(ii) A^{-1} exists if and only if $|A| \neq 0$.

(iii) $(A^{-1})' = (A')^{-1}$

(iv) $(AB)^{-1} = B^{-1}A^{-1}$

To find inverse of a square matrix use following steps

I Find $|A|$

II If $|A| = 0$ then A^{-1} does not exists.

III If $|A| \neq 0$ then find adjA and $A^{-1} = \frac{1}{|A|}(\text{adj}A)$.

Example 2: Find A^{-1}, where $A = \begin{bmatrix} 4 & 2 & 3 \\ 0 & -1 & -2 \\ 5 & -3 & 6 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 4 & 2 & 3 \\ 0 & -1 & -2 \\ 5 & -3 & 6 \end{bmatrix}$

$|A| = 4(-6-6) - 2(0+10) + 3(0+5) = -48 - 20 + 15 = -53 \neq 0$

$\therefore A^{-1}$ exists.

Let A_{ij} denotes the cofactor of (i, j) th element of the matrix A.

$\therefore A_{11} = (-1)^{1+1}(-6-6) = -12$, $A_{12} = (-1)^{1+2}(0+10) = -10$

$A_{13} = (-1)^{1+3}(0+5) = 5$, $A_{21} = (-1)^{2+1}(12+9) = -21$

$A_{22} = (-1)^{2+2}(24-15) = 9$, $A_{23} = (-1)^{2+3}(-12-10) = 22$

$A_{31} = (-1)^{3+1}(-4+3) = -1$, $A_{32} = (-1)^{3+2}(-8-0) = 8$

$A_{33} = (-1)^{3+3}(-4-0) = -4$

$$\therefore \text{adj}A = \begin{bmatrix} -12 & -10 & 5 \\ -21 & 9 & 22 \\ -1 & 8 & -4 \end{bmatrix}' = \begin{bmatrix} -12 & -21 & -1 \\ -10 & 9 & 8 \\ 5 & 22 & -4 \end{bmatrix}$$

and $A^{-1} = \frac{1}{|A|}(\text{adj}A) = \frac{1}{-53}\begin{bmatrix} -12 & -21 & -1 \\ -10 & 9 & 8 \\ 5 & 22 & -4 \end{bmatrix}$

Here is an exercise for you.

E 2) For $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 5 & -2 \end{bmatrix}$, verify the result $(AB)^{-1} = B^{-1}A^{-1}$.

10.4 APPLICATION OF MATRICES

Matrix Method

Inverse of a matrix can be used to solve a system of linear simultaneous equations. Consider a system of n equations in n unknowns $x_1, x_2, x_3, ..., x_n$ given below.

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
.
.
.
$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$$

This system of equations can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & & & \\ . & & & \\ . & & & \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix} \qquad \ldots (1)$$

Or $AX = B$

$$\text{,where } A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & & & \\ . & & & \\ . & & & \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

If n = 3, $x_1 = x$, $x_2 = y$, $x_3 = z$, then (1) reduces to

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Or $AX = B$ … (2)

Or $X = A^{-1}B$ … (3)

Or $X = \frac{1}{|A|}(\text{adj}A)B$ $\quad \left[\because A^{-1} = \frac{1}{|A|}(\text{adj}A)\right]$

Or $|A|X = (\text{adj}A)B$ … (4)

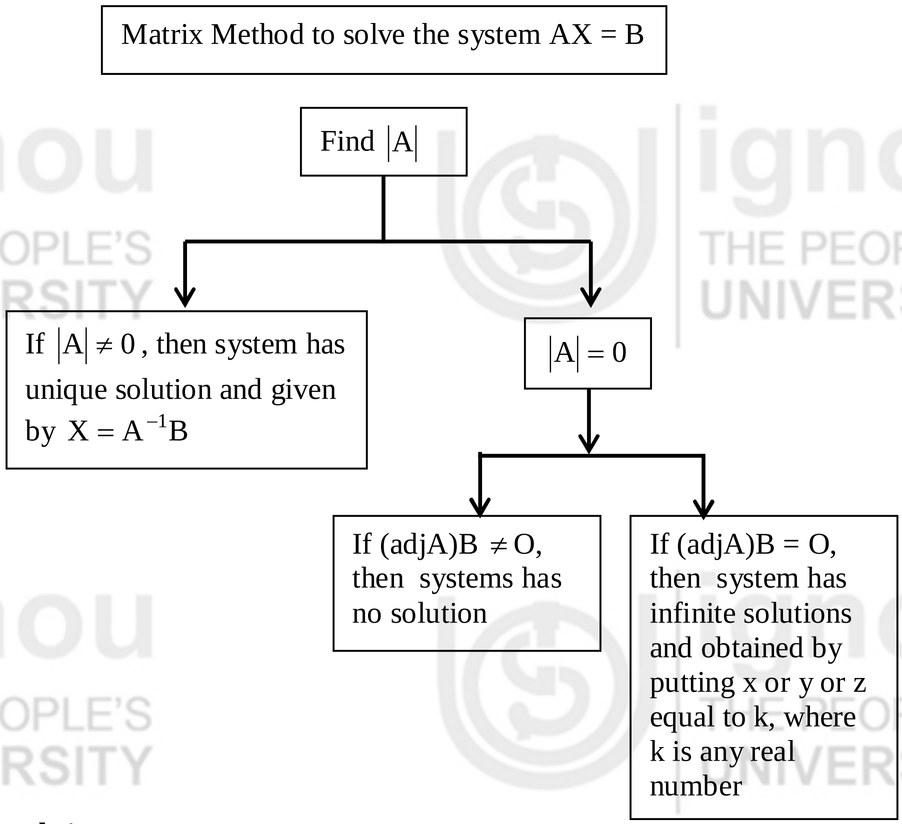
If $|A| = 0$, then A^{-1} does not exist. In this case, the given system of equations either has no solution or infinitely many solutions. In this case, we find (adjA)B. If (adjA)B = O, then system has infinitely many solutions

$$\left[\begin{array}{l}\because \text{in this case, } (4) \Rightarrow 0X = (\text{adj}A)B \\ \text{which holds for all matrices } X, \text{ i.e.} \\ \text{for all real values of } x, y, z.\end{array}\right]$$

and if $(\text{adj}A)B \neq O$, then system has no solution.

$$\left[\begin{array}{l}\because \text{in this case, L.H.S. of } (4) = O = \text{null matrix} \\ \text{but R.H.S. of } (4) = \text{non zero matrix.}\end{array}\right]$$

Above discussion can be summarised in the following diagram.



Remark 1:

(i) If B = O, i.e. if B is null matrix, then system given by (2) reduces to AX = O and is known as linear homogeneous system of equations.

(ii) Homogeneous system is always consistent.

> **Consistent system:** A system of equations is said to be consistent if there is either unique solution or infinite number of solutions.
> **Inconsistent system:** If system has no solution then it is known as inconsistent system.

Let us explain this method with the help of following example.

Example 3: The cost of 2 pens, 3 note-books, and 1 book is Rs 90. The cost of 1 pen, 4 note-books and 2 books is Rs 120. The cost of 2 pens, 4 note-books and 5 books is Rs 205. Find the cost of 1 pen, 1 note-book and 1 book by matrix method.

Solution: Let Rs x, y, z be the cost of 1 pen, 1 note-book and 1 book respectively, then according to given

$2x + 3y + z = 90$

$x + 4y + 2z = 120$

$2x + 4y + 5z = 205$

In matrix form this system can be written as

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 2 \\ 2 & 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 90 \\ 120 \\ 205 \end{bmatrix}$$

Or $AX = B$ … (1)

$$|A| = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 2 \\ 2 & 4 & 5 \end{bmatrix}$$

Expanding along R_1

$|A| = 2(20 - 8) - 3(5 - 4) + 1(4 - 8)$

$= 24 - 3 - 4 = 17 \neq 0$

$\therefore A^{-1}$ exists.

Let A_{ij} denotes the cofactor of $(i, j)^{th}$ element of the matrix A.

$\therefore A_{11} = (-1)^{1+1}(20-8) = 12$

$A_{12} = (-1)^{1+2}(5-4) = -1$

$A_{13} = (-1)^{1+3}(4-8) = -4$

$A_{21} = (-1)^{2+1}(15-4) = -11$

$A_{22} = (-1)^{2+2}(10-2) = 8$

$A_{23} = (-1)^{2+3}(8-6) = -2$

$A_{31} = (-1)^{3+1}(6-4) = 2$

$A_{32} = (-1)^{3+2}(4-1) = -3$

$A_{33} = (-1)^{3+3}(8-3) = 5$

$$\text{adj}A = \begin{bmatrix} 12 & -1 & -4 \\ -11 & 8 & -2 \\ 2 & -3 & 5 \end{bmatrix}' = \begin{bmatrix} 12 & -11 & 2 \\ -1 & 8 & -3 \\ -4 & -2 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj}A) = \frac{1}{17}\begin{bmatrix} 12 & -11 & 2 \\ -1 & 8 & -3 \\ -4 & -2 & 5 \end{bmatrix}$$

$$\text{Equation (1)} \Rightarrow X = A^{-1}B = \frac{1}{17}\begin{bmatrix} 12 & -11 & 2 \\ -1 & 8 & -3 \\ -4 & -2 & 5 \end{bmatrix}\begin{bmatrix} 90 \\ 120 \\ 205 \end{bmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} 1080 - 1320 + 410 \\ -90 + 960 - 615 \\ -360 - 240 + 1025 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 170 \\ 255 \\ 425 \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ 25 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ 25 \end{bmatrix}$$

By definition of equality of two matrices, we have

x = 10, y = 15, z = 25

$\therefore$ costs of 1 pen, 1 note-book and one book are Rs 10, Rs 15, Rs 25, respectively.

Example 4: Solve the following system of equations:

(i) $4x + 2y = 6$
$6x + 3y = 8$

(ii) $3x + 6y - 4z = 3,\ 3x - z = 0,$
$12x - 6y - z = -3$

Solution:

(i) Given system of equations is

$4x + 2y = 6$ … (1)

$6x + 3y = 8$ … (2)

This system of equations can be written is matrix form as

$$\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

Or AX = B … (3), where $A = \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

$$|A| = \begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 12 - 12 = 0$$

$\Rightarrow$ system has either no solution or infinite many solutions.

Let A_{ij} denotes the cofactor of $(i, j)^{th}$ element of the matrix A.

$$\therefore A_{11} = (-1)^{1+1}(3) = 3, \quad A_{12} = (-1)^{1+2}(6) = -6$$

$$A_{21} = (-1)^{2+1}(2) = -2, \quad A_{22} = (-1)^{2+2}(4) = 4$$

$$\text{adj}A = \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}^1 = \begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix}$$

$$(\text{adj}A)B = \begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 18-16 \\ -36+32 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} \neq O = \text{null matrix}$$

Hence system has no solution.

i.e. given system of equations is inconsistent.

(ii) Given system of equations is

$3x + 6y - 4z = 3$ … (1)

$3x - z = 0$ … (2)

$12x - 6y - z = -3$ … (3)

This system of equations can be written in matrix forms as

$$\begin{bmatrix} 3 & 6 & -4 \\ 3 & 0 & -1 \\ 12 & -6 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$$

Or AX = B … (4)

where $A = \begin{bmatrix} 3 & 6 & -4 \\ 3 & 0 & -1 \\ 12 & -6 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 6 & -4 \\ 3 & 0 & -1 \\ 12 & -6 & -1 \end{vmatrix}$$

Expanding along R_1

$$|A| = 3(0-6) - 6(-3+12) - 4(-18-0) = -18 - 54 + 72 = 0$$

$\therefore A^{-1}$ does not exist.

$\Rightarrow$ system has either no solution or infinite many solutions.

Let A_{ij} denotes the cofactor of $(i, j)^{th}$ element of the matrix A.

$\therefore A_{11} = (-1)^{1+1}(0-6) = -6$

$A_{12} = (-1)^{1+2}(-3+12) = -9$

$A_{13} = (-1)^{1+3}(-18-0) = -18$

$A_{21} = (-1)^{2+1}(-6-24) = 30$

$A_{22} = (-1)^{2+2}(-3+48) = 45$

$A_{23} = (-1)^{2+3}(-18-72) = 90$

$A_{31} = (-1)^{3+1}(-6-0) = -6$

$A_{32} = (-1)^{3+2}(-3+12) = -9$

$A_{33} = (-1)^{3+3}(0-18) = -18$

$$\text{adj}A = \begin{bmatrix} -6 & -9 & -18 \\ 30 & 45 & 90 \\ -6 & -9 & -18 \end{bmatrix}' = \begin{bmatrix} -6 & 30 & -6 \\ -9 & 45 & -9 \\ -18 & 90 & -18 \end{bmatrix}$$

$$(\text{adj}A)B = \begin{bmatrix} -6 & 30 & -6 \\ -9 & 45 & -9 \\ -18 & 90 & -18 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -18+0+18 \\ -27+0+27 \\ -54+0+54 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = O = \text{null matrix}$$

Hence system has infinite many solutions and given by putting either x = k or y = k or z = k in any two equations given by (1), (2) and (3). Let us put z = k in (1) and (2), we get, where k is any real number

$3x + 6y = 3 + 4k$ …. (5)

$3x = k$ … (6)

$(6) \Rightarrow x = \dfrac{k}{3}$

Putting $x = k/3$ in (5), we get

$3(k/3) + 6y = 3 + 4k \Rightarrow 6y = 3 + 4k - k \Rightarrow 6y = 3 + 3k \Rightarrow y = (k+1)/2$

$\therefore x = \dfrac{k}{3}, y = \dfrac{k+1}{3}, z = k,$ where k is any real number.

Here is an exercise for you.

E 3) Solve the following system of equations by matrix method:

(i)	(ii)	(iii)
$x + y = 3$	$2x - 3y = 3$	$2x + 3y = 5$
$4x - 3y = 5$	$6x - 9y = 5$	$4x + 6y = 10$

6.5 APPLICATION OF DETERMINANTS

There are many applications of determinants but here we shall discuss as to how the concept of determinant provides the solution of a given system of linear equations. The procedure which we will use is known as Cramer's rule.

Cramer's Rule: Cramer's rule can be used in any system of n linear equations in n variables. But here we shall discuss Cramer's rule only for system of 2 (or 3) equations in 2 (or 3) variables respectively. Suppose we are given following system of equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Cramer's rule suggests the following:

(i) Write the determinant for the coefficients. For the above given system of equations, it is $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \Delta(\text{say})$.

(ii) Write determinant by interchanging first column of Δ with the right side constants of the given equations. Let this determinant denoted by Δ_1.

$$\therefore \Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

Similarly, writing the determinants by interchanging the second and third columns of Δ with the right side constants and denoting them by Δ_2, Δ_3 respectively, we have

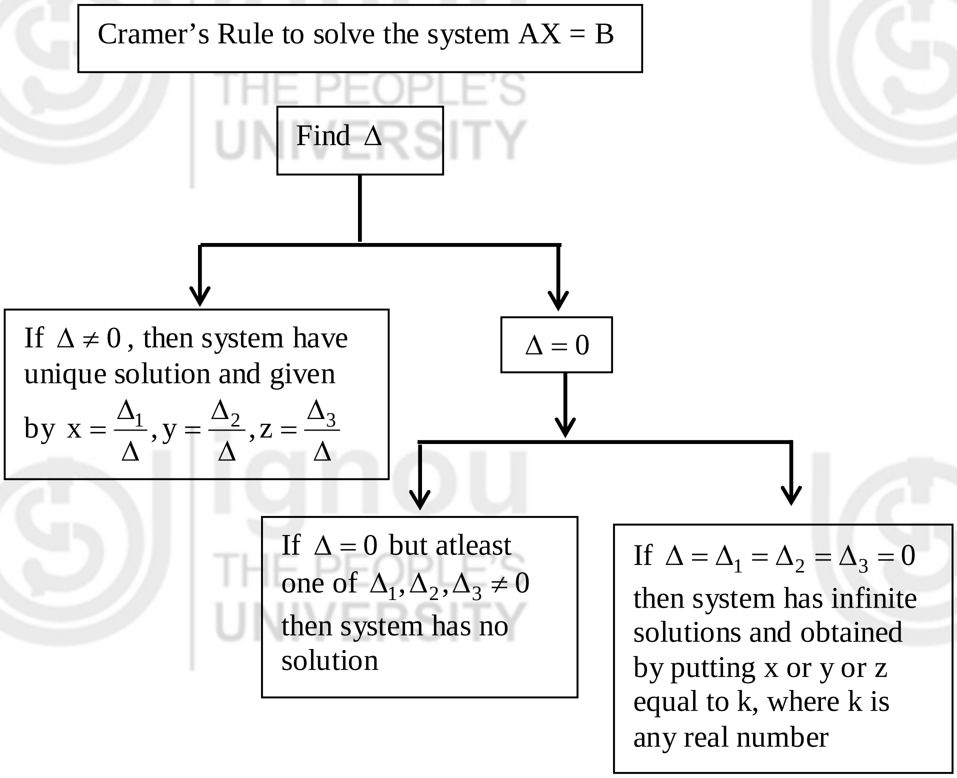
$$\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}.$$

(iii) If $\Delta \neq 0$, system has unique solution and is given by

$$x = \frac{\Delta_1}{\Delta},\ y = \frac{\Delta_2}{\Delta},\ z = \frac{\Delta_3}{\Delta}.$$

If $\Delta = 0$ but at least one of the other determinants is non-zero then system has no solution. If $\Delta = 0$ and other determinants are also zero then system has infinitely many solutions given by putting x or y or z equal to k, where k is any real number.

Above discussion can be summarised in the following diagram.



Let us now consider some examples wherein Cramer's rule is applied.

Example 5: Solve the following system of equations:

2x + 3y + z = 4
x – y +2z = 9
3x + 2y –z =1
using Cramer's rule

Solution: Given system of equations is

2x + 3y + z = 4 … (1)
x – y + 2z = 9 … (2)
3x + 2y – z = 1 … (3)

Here, $\Delta = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 2 \\ 3 & 2 & -1 \end{vmatrix}$

Expanding along R_1, we get

$\Delta = 2(1-4) - 3(-1-6) + 1(2+3) = -6 + 21 + 5 = 20 \neq 0$

$\therefore$ system has unique solution.

$\Delta_1 = \begin{vmatrix} 4 & 3 & 1 \\ 9 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix}$

Expanding along R_1

$\Delta_1 = 4(1-4) - 3(-9-2) + 1(18+1) = -12 + 33 + 19 = 40$

$\Delta_2 = \begin{vmatrix} 2 & 4 & 1 \\ 1 & 9 & 2 \\ 3 & 1 & -1 \end{vmatrix}$

Expanding along R_1

$\Delta_2 = 2(-9-2) - 4(-1-6) + 1(1-27) = -22 + 28 - 26 = -20$

$$\Delta_3 = \begin{vmatrix} 2 & 3 & 4 \\ 1 & -1 & 9 \\ 3 & 2 & 1 \end{vmatrix}$$

Expanding along R_1

$\Delta_3 = 2(-1-18) - 3(1-27) + 4(2+3) = -38 + 78 + 20 = 60$

$\therefore$ by Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{40}{20} = 2,\ y = \frac{\Delta_2}{\Delta} = \frac{-20}{20} = -1,\ z = \frac{\Delta_3}{\Delta} = \frac{60}{20} = 3$$

$\therefore x = 2,\ y = -1,\ z = 3$

Example 6: Solve the following system of equations:

x+ y + 2z = 4
x – y + 3z = 3
2x + 2y + 4z = 7
using Cramer's rule.

Solution: Given system of equations is

x+ y + 2z = 4 …(1)
x – y + 3z = 3 …(2)
2x + 2y + 4z = 7 … (3)

$$\text{Here, } \Delta = \begin{vmatrix} 1 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 2 & 4 \end{vmatrix}$$

Taking 2 common from R_3

$$\Delta = 2\begin{vmatrix} 1 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 1 & 2 \end{vmatrix} \qquad \left[\begin{array}{l} \because \text{if a row is multiplied with some} \\ \text{number, the whole determinant} \\ \text{is multiplied with that number.} \end{array}\right]$$

$= 2(0) = 0 \qquad [\because R_1 \text{ and } R_3 \text{ are identical}]$

$\therefore$ system has either no solution or infinite many solutions.

$$\Delta_1 = \begin{vmatrix} 4 & 1 & 2 \\ 3 & -1 & 3 \\ 7 & 2 & 4 \end{vmatrix}$$

Expanding along R_1

$\Delta_1 = 4(-4-6) - 1(12-21) + 2(6+7) = -40 + 9 + 26 = -5 \neq 0$

$\therefore$ systems have no solution.

Example 7: Solve the following system of equations:

x + 3y + 2z = 6
–x + 4y + 5z = 8
2x + 5y + 3z = 10

Solution: Given system of equations is

x + 3y + 2z = 6 … (1)
–x + 4y + 5z = 8 … (2)
2x + 5y + 3z = 10 … (3)

Here, $\Delta = \begin{vmatrix} 1 & 3 & 2 \\ -1 & 4 & 5 \\ 2 & 5 & 3 \end{vmatrix}$

Expanding along R_1

$\Delta = 1(12-25) - 3(-3-10) + 2(-5-8) = -13 + 39 - 26 = -39 + 39 = 0$

$\therefore$ system has either no solution or infinite many solutions.

Now, $\Delta_1 = \begin{vmatrix} 6 & 3 & 2 \\ 8 & 4 & 5 \\ 10 & 5 & 3 \end{vmatrix}$

Expanding along R_1

$\Delta_1 = 6(12-25) - 3(24-50) + 2(40-40) = -78 + 78 + 0 = 0$

$\Delta_2 = \begin{vmatrix} 1 & 6 & 2 \\ -1 & 8 & 5 \\ 2 & 10 & 3 \end{vmatrix}$

Expanding along R_1

$\Delta_2 = 1(24-50) - 6(-3-10) + 2(-10-16) = -26 + 78 - 52 = -78 + 78 = 0$

$\Delta_3 = \begin{vmatrix} 1 & 3 & 6 \\ -1 & 4 & 8 \\ 2 & 5 & 10 \end{vmatrix}$

Expanding along R_1

$\Delta_3 = 1(40-40) - 3(-10-16) + 6(-5-8) = 0 + 78 - 78 = 0$

As $\Delta = 0$ and also $\Delta_1 = \Delta_2 = \Delta_3 = 0$

$\Rightarrow$ system has infinite many solutions which are given by putting z = k (where k is any real number) in any of the two equations given by (1), (2) and (3).

Let us put z = k in (1) and (2), we get

$x + 3y = 6 - 2k$ … (4)

$-x + 4y = 8 - 5k$ … (5)

Again for this system of equations

$\Delta = \begin{vmatrix} 1 & 3 \\ -1 & 4 \end{vmatrix} = 4 + 3 = 7$

$\Delta_1 = \begin{vmatrix} 6-2k & 3 \\ 8-5k & 4 \end{vmatrix} = 24 - 8k - 24 + 15k = 7k$

$\Delta_2 = \begin{vmatrix} 1 & 6-2k \\ -1 & 8-5k \end{vmatrix} = 8 - 5k + 6 - 2k = 14 - 7k$

$\therefore$ by Cramer's rule

$x = \dfrac{\Delta_1}{\Delta} = \dfrac{7k}{7} = k$, $y = \dfrac{\Delta_2}{\Delta} = \dfrac{14-7k}{7} = 2 - k$

$\therefore x = k,\ \ y = 2 - k,\ \ z = k,$ where k is any real number

Remark 2: Here, in the above example, we have taken z = k. You may also take y = k or x = k and then can solve by taking any two equations out of (1), (2), (3) in remaining two unknowns using Cramer's rule.

Now, you can try the following exercise.

E 4) Solve the following system of equations using Cramer's rule:

(i) $3x + 5y = -11$, $2x - 3y = 18$ (ii) $x - 2y = 5$, $-2x + 4y = 8$ (iii) $2x - y = 6$, $-6x + 3y = -18$

6.6 SUMMARY

Let us summarise the topics that we have covered in this unit:

1) Adjoint of a square matrix.
2) Inverse of a square matrix.
3) Application of matrices for solving a given system of linear equations, i.e. matrix method.
4) Application of determinants for solving a given system of linear equations, i.e. Cramer's rule.

6.7 SOLUTIONS/ANSWERS

E 1) $A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$

Let A_{ij} denotes the cofactor of $(i, j)^{th}$ element of the matrix A.

$\therefore A_{11} = (-1)^{1+1}(5) = 5, \quad A_{12} = (-1)^{1+2}(4) = -4$

$A_{21} = (-1)^{2+1}(-3) = 3, \quad A_{22} = (-1)^{2+2}(2) = 2$

$$\therefore \text{adj}A = \begin{bmatrix} 5 & -4 \\ 3 & 2 \end{bmatrix}' = \begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix} = 10 - (-12) = 10 + 12 = 22$$

$$A(\text{adj}A) = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 10+12 & 6-6 \\ 20-20 & 12+10 \end{bmatrix} = \begin{bmatrix} 22 & 0 \\ 0 & 22 \end{bmatrix}$$

$$= 22\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 22I_2 = |A|I_2 \qquad \ldots (1)$$

$$(\text{adj}A)A = \begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 10+12 & -15+15 \\ -8+8 & 12+10 \end{bmatrix} = \begin{bmatrix} 22 & 0 \\ 0 & 22 \end{bmatrix}$$

$$= 22\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 22I_2 = |A|I_2 \qquad \ldots (2)$$

From (1) and (2), we get

$A(\text{adj}A) = (\text{adj}A)A = |A|I_2$

E 2) $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 \\ 5 & -2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} = 8 + 3 = 11 \neq 0$$

$\therefore A^{-1}$ exists.

$$|B| = \begin{vmatrix} 1 & 4 \\ 5 & -2 \end{vmatrix} = -2 - 20 = -22 \neq 0$$

$\therefore B^{-1}$ exists.

$$AB = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 2+15 & 8-6 \\ -1+20 & -4-8 \end{bmatrix} = \begin{bmatrix} 17 & 2 \\ 19 & -12 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 17 & 2 \\ 19 & -12 \end{vmatrix} = -204 - 38 = -242 \neq 0$$

$\therefore (AB)^{-1}$ exists.

Let A_{ij}, B_{ij}, C_{ij} denotes the cofactors of $(i, j)^{th}$ element of matrices A, B, AB respectively.

$\therefore A_{11} = (-1)^{1+1}(4) = 4$, $\quad A_{12} = (-1)^{1+2}(-1) = 1$

$A_{21} = (-1)^{2+1}(3) = -3$, $\quad A_{22} = (-1)^{2+2}(2) = 2$

$B_{11} = (-1)^{1+1}(-2) = -2$, $\quad B_{12} = (-1)^{1+2}(5) = -5$

$B_{21} = (-1)^{2+1}(4) = -4$, $\quad B_{22} = (-1)^{2+2}(1) = 1$

$C_{11} = (-1)^{1+1}(-12) = -12$, $C_{12} = (-1)^{1+2}(19) = -19$

$C_{21} = (-1)^{2+1}(2) = -2$, $\quad C_{22} = (-1)^{2+2}(17) = 17$

$$\therefore \text{adj}A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}' = \begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}, \text{adj}B = \begin{bmatrix} -2 & -5 \\ -4 & 1 \end{bmatrix}' = \begin{bmatrix} -2 & -4 \\ -5 & 1 \end{bmatrix}$$

$$\text{adj}(AB) = \begin{bmatrix} -12 & -19 \\ -2 & 17 \end{bmatrix}' = \begin{bmatrix} -12 & -2 \\ -19 & 17 \end{bmatrix},\ A^{-1} = \frac{1}{|A|}(\text{adj}A) = \frac{1}{11}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|}(\text{adj}B) = \frac{1}{-22}\begin{bmatrix} -2 & -4 \\ -5 & 1 \end{bmatrix} = \frac{-1}{22}\begin{bmatrix} -2 & -5 \\ -4 & 1 \end{bmatrix}'$$

$$(AB)^{-1} = \frac{1}{|AB|}(\text{adj}AB) = \frac{-1}{242}\begin{bmatrix} -12 & -2 \\ -19 & 17 \end{bmatrix} \qquad \ldots (1)$$

$$B^{-1}A^{-1} = \frac{-1}{22}\begin{bmatrix} -2 & -4 \\ -5 & 1 \end{bmatrix}\frac{1}{11}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{-1}{242}\begin{bmatrix} -2 & -4 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix} = \frac{-1}{242}\begin{bmatrix} -8-4 & 6-8 \\ -20+1 & 15+2 \end{bmatrix}$$

$$= \frac{-1}{242}\begin{bmatrix} -12 & -2 \\ -19 & 17 \end{bmatrix} \qquad \ldots (2)$$

From (1) and (2), we have

$(AB)^{-1} = B^{-1}A^{-1}$

E3) (i) Given system of equations is

$x + y = 3$

$4x - 3y = 5$

This system of equations can be written in matrix form as

$$\begin{bmatrix} 1 & 1 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Or AX = B …. (1)

$$\text{where } A = \begin{bmatrix} 1 & 1 \\ 4 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 \\ 4 & -3 \end{vmatrix} = -3 - 4 = -7 \neq 0$$

$\therefore A^{-1}$ exists.

Let A_{ij} denotes the cofactor of $(i.j)^{th}$ element of the matrix A.

$$\therefore A_{11} = (-1)^{1+1}(-3) = -3, \quad A_{12} = (-1)^{1+2}(4) = -4$$

$$A_{21} = (-1)^{2+1}(1) = -1, \quad A_{22} = (-1)^{2+2}(1) = 1$$

$$\text{adj}A = \begin{bmatrix} -3 & -4 \\ -1 & 1 \end{bmatrix}' = \begin{bmatrix} -3 & -1 \\ -4 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj}A) = -\frac{1}{7}\begin{bmatrix} -3 & -1 \\ -4 & 1 \end{bmatrix}$$

$$\text{Equation (1)} \Rightarrow X = A^{-1}B = -\frac{1}{7}\begin{bmatrix} -3 & -1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= -\frac{1}{7}\begin{bmatrix} -9-5 \\ -12+5 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -14 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

By definition of equality of two matrices, we have

x = 2, y = 1

(ii) Given system of equations is

2x – 3y = 3

6x – 9y = 5

This system of equations in matrix form can be written as

$$\begin{bmatrix} 2 & -3 \\ 6 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Or AX = B … (1), where $A = \begin{bmatrix} 2 & -3 \\ 6 & -9 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -3 \\ 6 & -9 \end{vmatrix} = -18 + 18 = 0$$

$\therefore A^{-1}$ does not exists.

$\Rightarrow$ system has either no solution or infinite many solutions.

Let A_{ij} denotes the cofactor of $(i, j)^{th}$ element of A.

$$\therefore A_{11} = (-1)^{1+1}(-9) = -9, \quad A_{12} = (-1)^{1+2}(6) = -6$$

$$A_{21} = (-1)^{2+1}(-3) = 3, \quad A_{22} = (-1)^{2+2}(2) = 2$$

$$\text{adj}A = \begin{bmatrix} -9 & -6 \\ 3 & 2 \end{bmatrix}' = \begin{bmatrix} -9 & 3 \\ -6 & 2 \end{bmatrix}$$

$$(\text{adj}A)B = \begin{bmatrix} -9 & 3 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -27+15 \\ -18+10 \end{bmatrix} = \begin{bmatrix} -12 \\ -8 \end{bmatrix} \neq O = \text{null matrix}$$

Hence system has no solution.

(iii) Given system of equations is

$2x + 3y = 5$ … (1)

$4x + 6y = 10$ … (2)

This system of equations can be written in matrix form as

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

Or $AX = B$, where $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 2 \times 6 - 4 \times 3 = 12 - 12 = 0$$

$\therefore A^{-1}$ does not exist.

$\Rightarrow$ system has either no solution or infinite many solutions.

Let A_{ij} denotes the cofactor of $(i, j)^{th}$ element of the matrix A.

$\therefore A_{11} = (-1)^{1+1}(6) = 6$, $A_{12} = (-1)^{1+2}(4) = -4$

$A_{21} = (-1)^{2+1}(3) = -3$, $A_{22} = (-1)^{2+2}(2) = 2$

$$\text{adj}A = \begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}' = \begin{bmatrix} 6 & -3 \\ -4 & 2 \end{bmatrix}$$

$$(\text{adj}A)B = \begin{bmatrix} 6 & -3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 30-30 \\ -20+20 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = O = \text{ null matrix}$$

$\Rightarrow$ system has infinitely many solutions and given by putting y = k in (1), we get

$$2x + 3k = 5 \Rightarrow \ x = \frac{5-3k}{2}$$

$\therefore x = \dfrac{5-3k}{2}, y = k,$ where k is any real number.

E 4) (i) Given system of equations is

$3x + 5y = -11$

$2x - 3y = 18$

Here, $\Delta = \begin{vmatrix} 3 & 5 \\ 2 & -3 \end{vmatrix} = -9 - 10 = -19 \neq 0$

$\Rightarrow$ system has unique solution.

$$\Delta_1 = \begin{vmatrix} -11 & 5 \\ 18 & -3 \end{vmatrix} = 33 - 90 = -57$$

$$\Delta_2 = \begin{vmatrix} 3 & -11 \\ 2 & 18 \end{vmatrix} = 54 + 22 = 76$$

$\therefore$ by Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-57}{-19} = 3,\ y = \frac{\Delta_2}{\Delta} = \frac{76}{-19} = -4$$

$\therefore x = 3, \quad y = -4$

(ii) Given system of equations is

$x - 2y = 5$
$-2x + 4y = 8$

Here, $\Delta = \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} = 4 - 4 = 0$

$\therefore$ system has either no solution or infinite many solutions.

$$\Delta_1 = \begin{vmatrix} 5 & -2 \\ 8 & 4 \end{vmatrix} = 20 + 16 = 36 \neq 0$$

$\therefore \Delta = 0$ but $\Delta_1 = 36 \neq 0$

$\Rightarrow$ system has no solution.

(iii) Given system of equations is

$2x - y = 6$ … (1)
$-6x + 3y = -18$ … (2)

Here, $\Delta = \begin{vmatrix} 2 & -1 \\ -6 & 3 \end{vmatrix} = 6 - 6 = 0$

$$\Delta_1 = \begin{vmatrix} 6 & -1 \\ -18 & 3 \end{vmatrix} = 18 - 18 = 0,\ \Delta_2 = \begin{vmatrix} 2 & 6 \\ -6 & -18 \end{vmatrix} = -36 + 36 = 0$$

$\therefore \Delta = 0, \quad \Delta_1 = \Delta_2 = 0$

$\Rightarrow$ system has infinite many solutions and given by putting either x or y equal to some arbitrary constant.

Let y = k, where k is any real number.

Equation (1) $\Rightarrow 2x = 6 + k \Rightarrow x = \dfrac{6+k}{2}$

$\therefore x = \dfrac{6+k}{2}, \quad y = k,$ where k is any real number