UNIT 10 INTRODUCING 3D-GEOMETRY

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10.1 INTRODUCTION

It is generally felt by almost all teachers that students find it difficult **to visualise** abstract three-dimensional (3D) figures when they are discussed in the classroom. The main reason is that while teaching them concepts in three-dimensional geometry, we have to draw the figures on the blackboard or on paper, i.e., in 2D. Often, these representations do not reflect the entire picture, and a lot is left to the imagination.

In Sec. 10.2 of this unit, we begin with suggesting ways of helping your students to visualise 3D from 2D pictures. We have also suggested strategies for helping them to represent 3D in 2D.

In Sec. 10.3 and Sec. 10.4 we continue this discussion in the context of shifting the coordinate system. We look at ways in which the experiences of the students in 2D can be used for helping them to understand rigid body transformations in 3D. We also consider ways of helping them to visualise pairs of skew lines, and ways of generating infinitely many such pairs.

Finally, in Sec. 10.5, we take up a major area of difficulty that students face while studying three-dimensional geometry. This is related to solving problems about loci. Here, we have suggested some strategies that would clarify this concept in their minds.

Throughout this unit, you will find that we have tried to use the Cartesian and the vectorial approaches. This is important because students must realise their equivalence, and when it is convenient to use which approach.

A word about the abbreviations we use! In this unit we shall interchangeably use '2D' or '2-space' for 'two-dimensional space'. Similarly, '3D' or '3-space' will mean 'three-dimensional space'.

Objectives

After studying this unit, you should be able to improve the ability of your learners to

- visualise a 3D coordinate system;
- draw and read 2D representations of 3D objects and situations;
- visualise the translation and rotation of axes in 3D, and solve related problems;
- visualise skew lines and differentiate between skew and parallel lines;
- explain what 'locus' is, and identify it algebraically as well as geometrically in some simple situations.

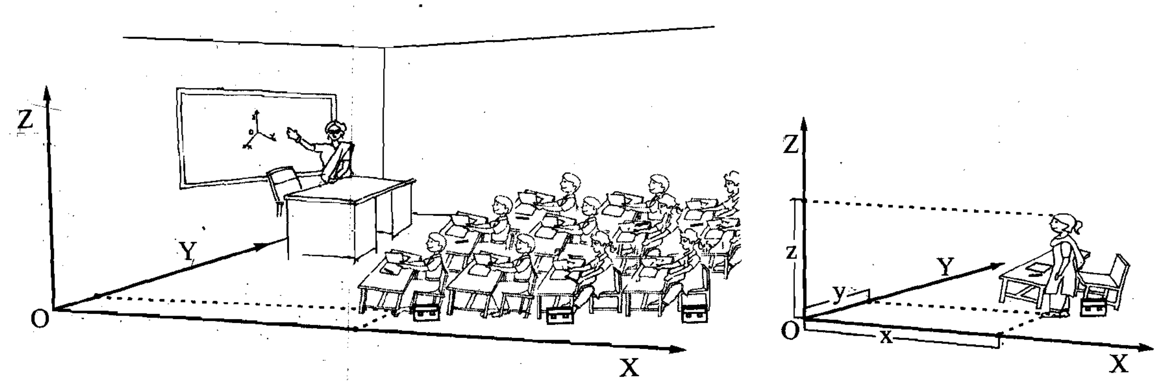
10.2 VISUALISING THE COORDINATE SYSTEM

While talking to students and teachers of Classes 11 and 12, we realised that a major problem faced by many students is their inability to read or draw 2D representations of objects in 3D space. This is probably because we teachers assume that the students will understand the representation of 3D space that we draw on the board for them. As long as the students are studying/reading/drawing in two-dimensional space, there is no problem. This is because there is not much left to the imagination, for instance, a circle or disc on paper is the same as it is in reality. But, when they start studying positions and movement in three-dimensional space, which is represented on paper (i.e., in 2D), there is a lot of imagining that they need to do. They need to 'see' how the third dimension (i.e., depth) is added to the diagram, merely by drawing an axis (a line) on the paper at an acute angle to one of the other axes. This ability of imagining the third dimension is not acquired easily. For visualising the complete image, a person needs several opportunities to correlate a variety of concrete models with their 2D representations. It is this kind of exposure that will help her to become familiar with the conventions of drawing 3D in 2D.

Sometimes, the students may not realise all the conventions that are being used. You could ensure that this doesn't happen by asking them questions that require them to focus on these conventions. You could also draw some concrete situations in which you deliberately ignore a particular convention, and ask them if it matters, and in what way. You should also ask students to draw 3D situations or objects on paper. Then ask each student to clearly explain which conventions she has used for drawing the 2D representations of the 3D objects and of 3-space. Of course, some teachers say that all this is not necessary. According to them, the students are already familiar with these conventions because they see photographs, diagrams, etc., around them all the time — all of which are 2D representations of 3D objects. However, remember that the student is familiar with these 3D objects, and therefore, would usually not look at them carefully regarding the way details related to the third dimension are presented.

Let us now look at an example of how the classroom can be used for familiarising the learner with visualisation of 3D in 2D. Of course, the assumption (which needs to be checked!) is that the students are familiar with finding coordinates in 2D. With this assumption, let us consider the following activity.

To start with, how is your classroom usually arranged? Are you, the teacher, facing about 30 students with the board on the wall behind you? Let us assume the room is as in Fig. 1(a).



(a) (b)

Fig. 1

You could ask your students to assume that the corner of the floor to your right, where two walls and the floor meet, is the origin. You could mark the x and y axes along the edges of the floor, and the z-axis upwards, showing the height from the floor. Here, it is important that the students realise the order in which the 3 axes are taken — according to the right-hand rule. Give them an example of an order that is not acceptable, according to convention.

Each student could be asked to find the coordinates of, say, her left foot, w.r.t. these x and y axes. Then you could ask her to stand at her place, and take her height as the third dimension, measured along the z-axis (see Fig. 1(b)).

'w.r.t' is often used in maths, and stands for 'with respect to'.

Next, you could draw a schematic view of the classroom on the blackboard, showing the 3 axes and explaining how and why this is an acceptable 2D representation. You could pinpoint the positions of the heads of one or two students in this diagram by writing the coordinates next to the point, and showing how those points are placed in the diagram. The rest of the students could, then, add their relative positions in the same diagram. This would help them in understanding how the diagram represents the real 3D situation. However, we must remember that some of the students may need several hints regarding how to find the 3 coordinates of any point in space.

You need to do several activities of this kind with them. This activity could be extended by, for instance, asking them to form teams. These teams could ask each other the coordinates of various objects w.r.t. the axes chosen — the tip of a bulb, the centre of the fan, the foot of a chair leg, and so on. Then, one of them could mark a point on your diagram, giving its coordinates, and ask what object in the room corresponds to it.

With such activities and exercises, the students would be able to relate any given point P in the room, with the three perpendicular distances of this point from the mutually perpendicular planes, namely, the planes of the floor and two adjacent walls of the room. They would be able to do the converse too — given any three positive numbers, they could take these three as the distances measured from the three planes and locate a corresponding point P in the room.

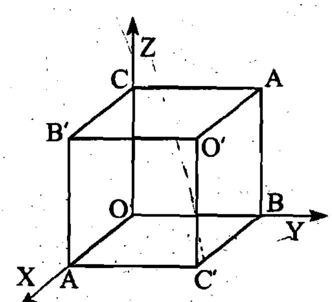


Fig. 2

These 'room-related' activities require to be followed by more abstract activities to ensure that the students can improve their visualisation. As we've mentioned earlier, let us look at the problem students can have, for instance in 'reading' Fig. 2. In this figure of a cube, $\angle$ BOC is clearly a right angle. However, $\angle$ AOB and $\angle$AOC are not right angles. You have to help your students look at Fig.2 in such a manner that they visualise the edge OA as sticking out of the sheet of paper/blackboard, perpendicular to the plane of the paper/blackboard. They have to realise that $\angle$ AOB and $\angle$ AOC, when coming out of the plane, are at right angles to OBA′C. It is then that we have three mutually perpendicular lines OA, OB and OC, and three mutually perpendicular planes OAC′B, OBA′C and OCB′A. So, if they see this cube as representing their classroom, then they should see that it shows the opposite edges of the floor and the ceiling as being parallel. Similarly, can they see that it shows the floor and the ceiling as parallel planes, and the opposite walls as parallel planes?

With the different kinds of experiences suggested above, the students would see that if the axes were chosen as in Fig. 1, it is adequate to have 3 positive numbers to locate a point inside the room. Now, ask the students what happens if any of the positive numbers is greater than the dimension of the room (or the cube in Fig. 2)? Can they reply that the corresponding point would be outside the room in the direction that this number corresponds to? And, what happens if you had taken P to be a point in the verandah behind the wall (i.e., the plane XOZ in Fig. 1), or to the left of the plane YOZ in the corridor, or below the plane (floor) XOY? Do they realise that the corresponding distances measured would be in the opposite directions, and so the corresponding coordinates would become negative? It is not easy for the students to understand which points correspond to negative coordinates along any axis. You would need to be patient about this aspect, and get them to look at it again and again through a variety of appropriate activities.

Having discussed these points with your students, you could give them some simple exercises which allow them to practise identifying co-ordinates and locating points. Some are given below. Try them with your students.

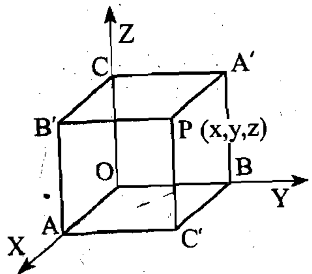


Fig. 3

E1) i) In Fig. 3, the point P has coordinates (x,y,z). Write down the coordinates of the points A,B,C, A′,B′,C′ and O.

ii) Write down the coordinates of the feet of the perpendiculars from the point $(-1,2,-4)$ to the coordinate planes.

iii) Find the perpendicular distance of the point $(-3,-1,4)$ from the y-axis.

iv) Find the image of the point $(-3,2,1)$ in the ZX-plane.

E2) Frame some questions to assess if your students are able to distinguish between the eight octants. Try them out with your students. What is the outcome? If the assessment is negative, how would you help your students to become familiar with the octants?

E3) What kind of detailed activity would you do with your students to familiarise them with negative numbers as coordinates?

In fact, it is with several activities/exercises/focussed questions of this kind that your students would realise that given **any** three real numbers r_1, r_2 and r_3, **there is one and only one point** located in space corresponding to them in that order, i.e., the point (r_1, r_2, r_3). Here, you would need to do some exercises with them to focus their attention on the fact that **(r_1, r_2, r_3) is not the same point as (r_1, r_2, r_3)**, in general. Ask them to think about the reason for this, and to give you examples in support of their reason.

At the next level of abstraction, you would need to help your students realise the arbitrariness of the choice of the origin and the axes. This arbitrariness is in several ways. For one, through the same origin, there are infinitely many sets of 3 mutually perpendicular lines. Secondly, any point in space can be taken as the origin. However, we will look at the formal aspect of this arbitrariness in detail in the next section.

There is yet another source of confusion that students have. This is related to the unit lengths to be chosen along the 3 axes. Students think that it should be the same for all the axes. This is because in our examples we usually choose the unit lengths to be the same. This is regardless of axes in 2D or in 3D. We need to consciously give them examples that force a choice of different unit lengths along the axes. These examples would first need to be related to 2D, which the students are more familiar with.

E4) Write down two problems you would do with students to help them realise that the unit distances chosen along the axes need not be the same.

Before ending this section, we should mention a difficulty students have about an equivalent approach to locate a point in space, **the vectorial approach**. The students need to understand the equivalence under which any point P(x,y,z) corresponds to the vector **OP**. Helping students to understand and visualise this equivalence would require the use of several activities. Some of them would be built around concrete 3D models. Think about other activities while doing the following exercises.

E5) How would you help your students correlate the two approaches for pinpointing any point in space? Does your method help them understand why the vector **OP** is called the **position vector** of the point P w.r.t. the origin of reference O?

E6) What is the direction of a vector whose magnitude is zero?

E7) What is the difference between 0 and **0**?

Students may also wonder about why they need to bother about two approaches, one is bad enough! Here is where you could tell them instances of where the Cartesian approach can be more useful, and those in which the vectorial approach is more handy. Some examples are alternative representations for a sphere with centre $\mathbf{r}_1$, and radius a, or a plane perpendicular to a given vector and at a distance d from the origin, etc. You should think of many other examples.

Let us now focus on an aspect that students usually don't realise — the arbitrariness of the choice of the origin and axes.

10.3 CHANGE OF AXES

It is important for students to know that there is nothing sacred about the particular point in a particular plane that is chosen as the origin. After all, the axes do not actually exist. It is we who create them — picking any point as an origin and picking any 3 mutually perpendicular lines as the axes. To help students internalise this, you could first discuss the impact of changing the origin and the axes in two dimensions only. You could draw pairs of axes so that neither is horizontal, and ask the students their opinion on this. Let them understand why this is acceptable, as long as the convention is followed that the direction of the y-axis is obtained by rotating the x-axis through 90° in the anti-clockwise direction.

The students could then be asked to think of ways of generalising this to three dimensions. We could also ask them : what happens if we just shift the axes? How are the coordinates of the objects changed by such changes? Do the objects themselves change?

Let us focus on familiarising our students with two particular kinds of transformations of the axes here. We take them up in the following sub-sections, one by one.

10.3.1 Translation

Let us start by asking your students to **pick any point** on the board as an origin O (0,0), and draw the axes OX and OY. Next, they should pick any other point O′ on the board as the new origin. Suppose its coordinates are (a,b) w.r.t the system XOY. Draw O′ X′ and O′ Y′ parallel to OX and OY, respectively, and choose them as the new axes (see Fig. 4). Now you could ask your students what the coordinates of O and O′ are w.r.t. X′ O′ Y′. Are they able to answer (– a, –b) and (0,0), respectively? Next, suppose the coordinates of any point P in the two systems are (x,y) and (x′, y′), respectively. Ask your students if these coordinates are related to each other, and how? You may need to give them a hint (as in Fig. 4.)

From the figure, do they see the following relationship?

$x = x' + a,\ y = y' + b$, or, equivalently, (1)

$x' = x - a,\ y' = y - b$ (2)

This transformation of the axes by shifting the origin is called a **translation**. In the

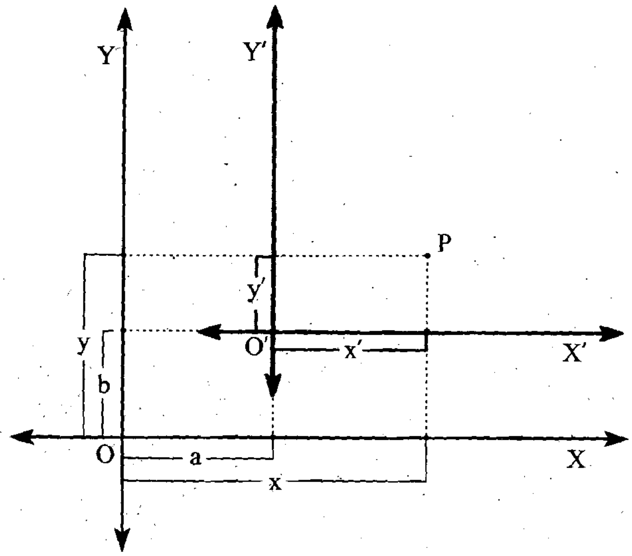


Fig. 4

example above, we have translated the origin from O to O′, keeping the axes parallel to their original position.

Now, students may wonder about **what the point is of translating the axes**. A good way of showing them the utility is to give them several examples of equations which are in non-standard form, and which reduce to 'known' simple forms by shifting the origin to an appropriate point. For instance, if we shift the origin to the centre of a circle, its equation reduces to standard form. Ana this form is much easier to deal with geometrically or algebraically. You could give them examples like the equation

$x^2 + y^2 - 2x + 4y + 1 = 0.$

This represents a circle with centre (1, −2), a fact that is not obvious to students. But, by shifting the origin to (1, −2), and using the equations (2), this equation changes to

$$(x' + 1)^2 + (y' - 2)^2 - 2(x' + 1) + 4(y' - 2) + 1 = 0 \Leftrightarrow x'^2 + y'^2 = 4,$$

a form that students can immediately recognise as a circle with radius 2 units.

As important as the algebraic translation is the geometric understanding of what is going on. This is what many students don't understand. And, this is what you need to explain to them. How are the two objects, before and after translation, related? A good way is to draw the circle $x'^2 + y'^2 = 4$ w.r.t. the X′O′Y′ system. Then, keeping this diagram, delete the X′O′Y′ system and bring back the original XOY system. This way the students see the placing of the circle w.r.t. the original axes.

You should ask them to do this kind of exercise with several objects — lines, conics, etc. Some such exercises are given below.

E8) Ask your students to simplify the following equations:

i) $2x - 3y - 5 = 0$, by shifting the origin to (1, −1).

ii) $x^2 + y^2 - 6x + 4y + 4 = 0$, by shifting the origin to (3, −2).

Also ask them to explain what this means geometrically.

E9) Shift the origin in such a manner that the equation

$x^2 + y^2 + 2x - 8y + 16 = 0$

is in standard form. What is happening geometrically?

Once you think the concept of shifting of the origin in two dimensions is clear to your students, you could ask them in what way this could be extended to three dimensions. You could give them a hint by doing activities with them of the kind suggested in the previous section to make them comfortable with the 2D representation of 3D space. Once you are assured that this aspect is taken care of, you could help the students to extend this to the translation of axes in 3D. Here, you could again begin by asking a student to come up on the board and pick any point as the origin, and draw the 3 coordinate axes through it. Then ask another student to pick another point O′ as the new origin on the board (see Fig. 5).

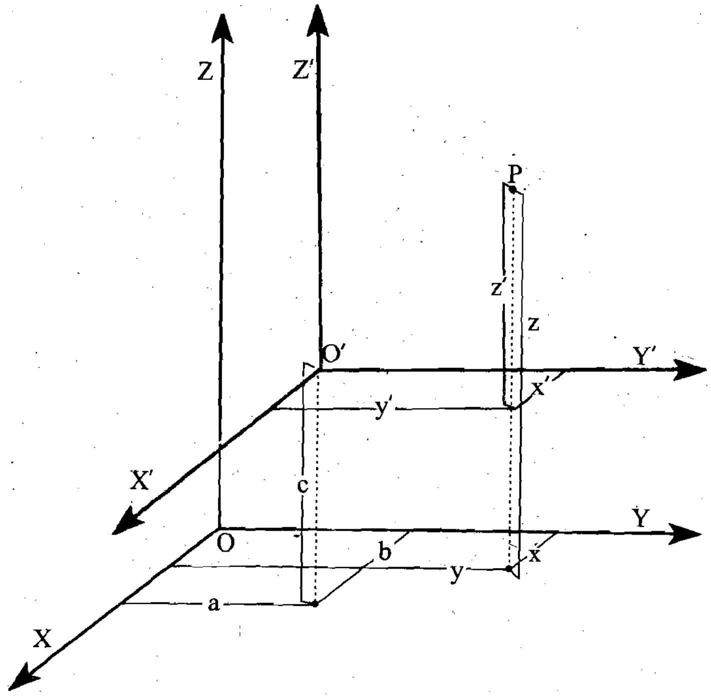


Fig. 5

Suppose O′ has coordinates (a, b, c) in the coordinate system OXYZ and O′X′, O′Y′, O′Z′ are the axes through O′ parallel to OX, OY and OZ, respectively. As in the 2D case, ask your students if they can prove the following relationship geometrically:

$$\left.\begin{aligned} x &= x' + a \\ y &= y' + b \\ z &= z' + c \end{aligned}\right\} \qquad (3)$$

Why don't you try some exercises now?

E10) How would you help your students to see the utility of a translation of the axes in 3-space?

E11) What kinds of activities would you give your learners to **assess** their understanding of the geometric aspect of translation?

Translation is not the only kind of shift in axes that the students encounter. Let us consider another rigid body transformation of the coordinate system.

10.3.2 Rotation

As in the case of translation, rotation would be easier for your students to understand if you first do it in the 2D case. To start with, you could ask your students if they can think of any other way of transforming the axes. Maybe, someone would come out with the idea of rotating the axes. If so, you could ask her/them to explain how this works to the rest of the students. Otherwise, you could prod them by asking each of them to do the following task.

Activity: Take a rectangular sheet of paper and pick any point on it as the origin O. Draw the coordinate axes OX and OY through it. Now, draw OX′ and OY′ through O so that $\angle XOX' = \angle YOY' = \theta$, say. (Here each child could pick the angle she wants.) Now, she should check that X′OY′ is a new coordinate system with coordinate axes OX′ and OY′, and origin O.

Next, pick any point P on the sheet of paper (see Fig. 6). From P, draw perpendiculars PL on OX and PA on OX′. Also, draw from A a perpendicular on OX, namely, AM. Then draw AB perpendicular to PL. Ask them which of these segments PL, AM and AB are parallel to OX or OY. Can they find $\angle$ BAO and $\angle$PAB? Do they realise that $\angle BAO = \theta$, $\angle PAB = \pi/2 - \theta$?

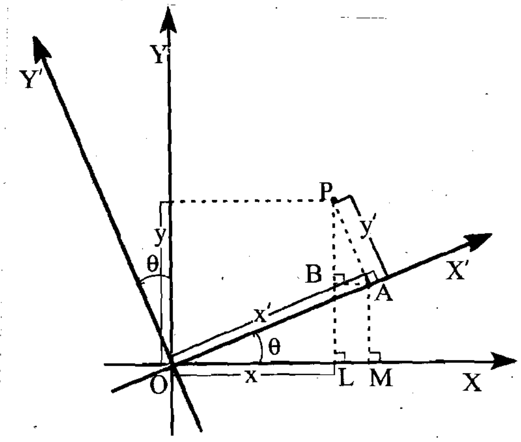


Fig. 6

Continue with the activity, asking them to find x and y in terms of x′, y′ and θ. They could do this task in groups if they feel the need to. Once you have given them enough time to discuss this, you could ask them to come up and present their solutions to the others on the board.

How many of them get the following solution?

$$x = OL = OM - AB = OA\cos\theta - PA\sin\theta = x'\cos\theta - y'\sin\theta$$

$$y = PL = BP + AM \text{ (here } BL = AM)$$

$$= PA\cos\theta + OA\sin\theta = y'\cos\theta + x'\sin\theta$$

What other solutions emerged?

Having obtained

$$\left.\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}\right\} \qquad (4)$$

you could ask your students how they would find (x′, y′) in terms of (x,y). How many of them solved equations (4) for x′ and y′ to obtain the values of these in terms of x,y and θ? How many said, "We can get x and y by rotating the axes in the reverse direction through the same angle. This means that we put −θ in (4), to get the required equation."?

How many other routes did the students use to get the following equations?

$$\left.\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}\right\} \quad (5)$$

As in the case of translation, your students need to get a geometric feel for what is going on when these transformations are applied. How would you help them get this feel? How would they realise that rotation is also a rigid body motion. Through examples, you could let them see that a straight line will remain a straight line, a circle will remain a circle and a conic section (parabola, ellipse, hyperbola) will remain the same conic section under a rotation. Only its equation will change. And, again, as in translation, you could choose examples to help them see that with a proper choice of the coordinate system the equations could be put in standard form. Such forms can then help in studying the objects represented by any given equation. For instance, ask your students to tell you the object represented by the equation

We have discussed rigid body motions in Sec. 12.2 in detail.

$$11x^2 + 2\sqrt{3}xy + 9y^2 = 12\left(x\sqrt{3} + y + 1\right).$$

How many of them can do it? But, if you now tell them to rotate the axes through an angle of 30°, and then shift the origin to (1,0), they will get $\frac{X^2}{2} + \frac{Y^2}{3} = 1$. This would be recognisable to them as an ellipse.

Here are some related exercises for you.

E12) How would you help students to appreciate the underlying geometry taking place when an equation is transformed under a rotation of the axes?

E13) If T is a translation and R a rotation, is $T \circ R = R \circ T$? Give reasons for your answer.

Once students are familiar with the rotation of the axes about the origin in two dimensions, we can encourage them to extend this to three dimensions. Some of them may do this in many different ways. Some generalisations that students make may really surprise you! However, if the students aren't able to extend the rotation to 3D you could give them some 'concrete' help as follows.

You could ask your students to take any cuboidal box, as shown in Fig.2, and rotate it about the origin. Most students rotate the box in such a manner that OA and OB turn, but not OC. This results in a rotation in the XOY plane, i.e., a two-dimensional rotation. A few students may exhibit a general rotation in which OC also turns. They could share this with the rest of the class. To generate the transformation equations in this case, the students would require familiarity with direction cosines. It is important that your students realise that this rotation of the axes is also a rigid body motion. For instance, the three coterminous edges of the box, which are mutually perpendicular, will remain mutually perpendicular after a rotation.

We have discussed 'direction cosines' in the next section.

Now, let us assume that the students can 'see' what is going on geometrically, and are comfortable with the diagrammatic representation. The next step is to ask them to see if they can generalise (4) and (5). They need to see how the equations of the coordinates change when the axes are rotated. You may need to hint to them that they should consider the direction cosines of the new axes w.r.t. the old axes. You could ask each of them to draw the OXYZ and OX′Y′Z′ systems on a sheet of paper (see Fig. 7).

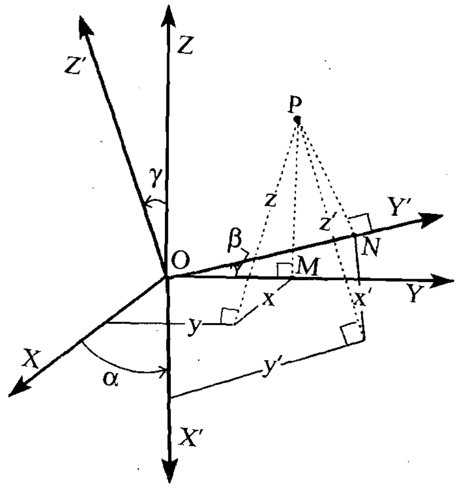


Fig. 7 : Rotating the axes in three-dimensional space

Ask them to call the direction cosines of OX′, OY′ and OZ′, respectively, l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 with reference to the old axes. If they call the coordinates of a point P, (x,y,z) w.r.t. the original axes and (x′, y′, z′) w.r.t the new axes, ask them to draw perpendiculars from P to OY and OY′, respectively, say PM and PN. Then ON = y′ is also the projection of the line segment OP on OY′ with direction cosines l_2, m_2, n_2. Your students would need to know why this implies

$$y' = ON = (x-0)l_2 + (y-0)m_2 + (z-0)n_2 \text{, i.e.,}$$

$$y' = l_2 x + m_2 y + n_2 z .$$

You could ask your students to get similar equations for x′ and z′, viz.,

$$x' = l_1 x + m_1 y + n_1 z \text{, and}$$
$$z' = l_3 x + m_3 y + n_3 z .$$

If they manage this, you could ask them to obtain (x, y, z) in terms of (x′, y′, z′). Do they need help to see that with reference to OX′, OY′ and OZ′, the direction cosines of OY are m_1, m_2 and m_3? In fact, using the fact that $m_1^2 + m_2^2 + m_3^2 = 1$, and the equations above, they should be able to obtain

$$y = m_1 x' + m_2 y' + m_3 z' .$$

Similarly, ask your students to obtain

$$x = l_1 x' + l_2 y' + l_3 z' \text{, and}$$

$$z = n_1 x' + n_2 y' + n_3 z' .$$

Maybe, your learners find it difficult to remember all these equations. Writing them in the form of a table, as below, helps.

	x	y	z
x′	l_1	m_1	n_1
y′	l_2	m_2	n_2
z′	l_3	m_3	n_3

This could be taken as a table for transformation of coordinates through a rotation of the axes, without shifting the origin. To find z′, for instance, in terms of x, y, z, we make use of the elements in the third row, multiplying these with x,y, z, respectively, and add them to get the required equation. Similarly, if we have to find x, say, in terms of x', y', z', we use the elements in the first column, multiply them with x', y', z', respectively, and add them all to get the required equation.

It is important that the students understand the implications of what we have discussed. One is suggested in the exercise below.

E14) In 2D, we say 'rotate the OXY system through an angle θ to get OX′Y′'. In 3D, why can we not say that we rotate OXYZ through an angle θ to get OX′Y′Z′?

We will discuss direction cosines in some detail later. Let us now consider how to help students visualise various aspects of lines in 3-space.

10.4 LINES — SKEW AND OTHERS

I asked my niece, who is in Class 8, if she could give me an example of two parallel lines, two intersecting lines and two lines which are neither parallel nor intersecting. The first two examples were no problem — she pulled out a piece of paper and drew them. But, for the third example, she tried drawing them in various ways without succeeding. Finally, I suggested that she look around the room and see if she finds any such pair of lines. She thought for a bit, looked around the room, thought some more and finally came out with an answer. What do you think it was? One line was the edge where two walls met, and the other was where a third wall and the floor met (see AB and CD in Fig. 8).

Lines in 3-space which are neither parallel nor intersecting, are called skew lines.

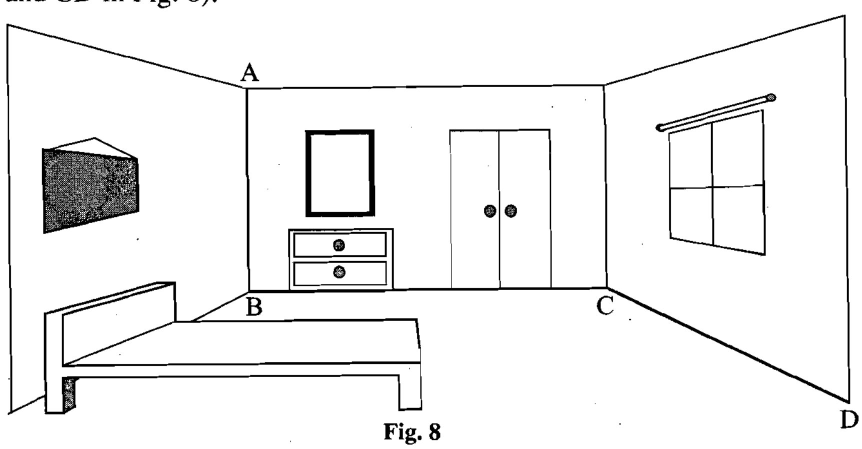


Fig. 8

She was right. This is the kind of example we should use to introduce students to skew lines. You could also ask them to give you several other examples of skew and non-skew lines. With such exercises they are likely to be able to visualise skew lines.

Try some exercises now.

E15) Ask your learners to identify at least 3 planes which pass through a point P and are parallel to a line L. How many such planes are there?

E16) Ask your students how many lines can be drawn through a point P, which will
i) intersect a line L?
ii) be parallel to L?
iii) which will not intersect L?

Also ask them the reasons for their responses.

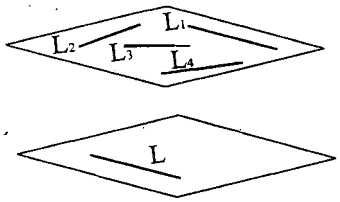


Fig. 9 : L is parallel to L_1, and skew w.r.t L_2, L_3,...

Another useful activity for students in this regard is the following.

Activity : Ask the student to pick any point in space, say P, and draw a line through P, say, L. Ask her how many lines can we have which pass through P and intersect L at some point? She will probably say as many as we like. Now, ask her to take a sheet of paper and draw a line L on it. Then hold another paper parallel to this paper at a certain distance from it (see Fig. 9). Next, ask her: How many lines on the second paper will be parallel to L? How many lines on this sheet will be skew to L?
Do your students realise that only one line on the second sheet can be parallel to L and infinitely many are skew? What is their reasoning behind this? What proof can they give?

Some students may reason that the lines can't intersect because they are on parallel planes. But why are they not parallel then? Here is where they would need to find the angle between any two lines in space. This requires them to use the concept of direction cosines. To help them understand this concept, the following activity may be useful.

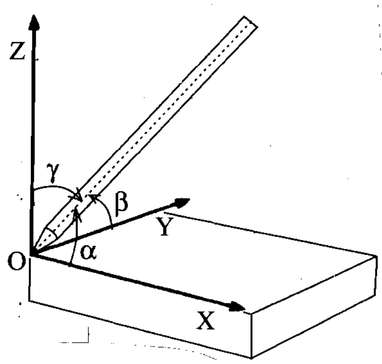


Fig:10

Activity : Divide your students into groups of two. Ask each group to hold a pencil up at an angle to the surface of their desks and at a corner of their desks. You could ask them to take the 2 adjacent corners of the desk as the x and y axes and a perpendicular to these corners as the z-axis (see Fig. 10). Then ask them to find the angles that the pencil makes with each of the three axes, say, α, β, γ.

Is it easy for them to find the angles? Do they try to hold a protractor up to do so? What other strategies do they use? Once they have been given ample time to explore the situation on their own, you could tell them how to find the angle between two lines.

For this, it would be better to take them back to 2D. Ask them how they find the angle between 2 lines in 2-space. It is in terms of the slopes of the two lines. Why is the slope of the line used as the identifier? Let's see. If we assume that the origin is one point on the line, and the other is $P(x_1,y_1)$, then the slope is $\tan^{-1}(y_1/x_1)$. So, knowing x_1 and y_1 allows us to pinpoint the line.

Now ask the students how this can be extended to 3D. Given any line L in 3-space, how do we identify it uniquely? Do they need a hint? If yes, ask them to assume that L passes through the origin O, and $P(x_1,y_1,z_1)$ is another point on it. Then if α, β, γ are the angles L makes with the positive direction of the x, y and z axes, respectively, ask them to find these angles in terms of x_1, y_1 and z_1.
Do they get the following equations?

$$\cos\alpha = \frac{x_1}{|OP|}, \cos\beta = \frac{y_1}{|OP|}, \cos\gamma = \frac{z_1}{|OP|}$$

Do they now realise, how the direction cosines $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ uniquely determine L? Can they also use the equations above to prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$?

Now, ask your students if they can suggest a method for finding the angle between any two lines in space. Do they think in terms of the direction cosines (dcs, in short)? If so, in what way? You may need to tell them to pick any convenient point in space as the origin, O, Through O, they should draw lines parallel to the two given lines. Then **the angle between the two given lines is defined as the angle between the lines parallel to them drawn through O.**

You could ask your students questions like :

- What happens to this angle if the given lines are parallel?
(Do they realise that in this case the lines drawn parallel to them through a point will coincide, so that the angle between them will be zero?)

- If a line L makes an angle α with the x-axis (as shown in Fig.11), then why is $-\cos\alpha$ also a direction cosine of L?
(As L extends indefinitely in both directions, it makes an angle $\pi-\alpha$ with the x-axis. Therefore, $\cos(\pi-\alpha)$ is also a direction cosine of L. Similarly, if the angles made with the positive directions of the y-axis and z-axis are β and γ, respectively, then $\pi-\beta$ and $\pi-\gamma$ are also angles made with the positive directions of these axes. Therefore, if the direction cosines of the line L are $\cos\alpha$, $\cos\beta$, $\cos\gamma$, then so are $-\cos\alpha$, $-\cos\beta$, $-\cos\gamma$.)

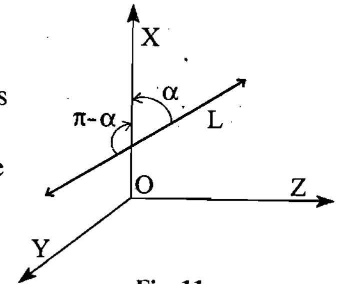


Fig. 11

dc is short for 'direction cosine'.

- If a,b,c are any 3 numbers such that $a^2+b^2+c^2=1$, can you find a line L with dcs a,b,c?
(Let P have coordinates (a,b,c). Then, can they show that $|OP|=1$, and that **OP** is the unit vector along the required line?)

The students could also be told that denoting the direction cosines of a line by *l, m, n* **is a convention.**

- If a,b,c are 3 numbers such that $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$, where $l^2+m^2+n^2=1$, then

prove that $l=\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m=\pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n=\pm\frac{c}{\sqrt{a^2+b^2+c^2}}$.

Here the same sign, + ve or – ve, is to be taken with all of them. (Here is where you could tell them what **direction ratios** (drs, in short) are.)

Some more exercises that you and your students could do are given below.

E17) Are the direction cosines of a line also direction ratios of this line? Give reasons for your answer.

E18) Find the direction cosines of a line which is equally inclined to the three coordinate axes.

E19) If a line makes angles $\alpha,\beta,\gamma,\delta$ with the four diagonals of a cube, prove that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}.$$

E20) Suppose l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 are the direction cosines of three mutually perpendicular lines. Show that l_1, l_2, l_3; m_1, m_2, m_3; n_1, n_2, n_3 are also the direction cosines of three mutually perpendicular lines, and

$$l_1^2+l_2^2+l_3^2=1 \qquad l_1m_1+l_2m_2+l_3m_3=0$$
$$m_1^2+m_2^2+m_3^2=1 \qquad m_1n_1+m_2n_2+m_3n_3=0$$
$$n_1^2+n_2^2+n_3^2=1 \qquad n_1l_1+n_2l_2+n_3l_3=0$$

E21) Find the direction cosines of the vector $\mathbf{i}-2\mathbf{j}+3\mathbf{k}$.

Now, how would your students find the distance between any two lines in 3-space? Given any two lines L_1 and L_2, the distance between them is the shortest distance among all the distances PQ, where P ranges over all points on L_1 and Q ranges over all points of L_2. (Why?) So, if L_1 and L_2 intersect, the distance between them is zero, this being the distance at the point of intersection. If L_1 and L_2 are parallel, the perpendicular distance between them always remains the same. In fact, this distance is the shortest distance between any two points that lie on the two different lines, as shown in Fig. 12.

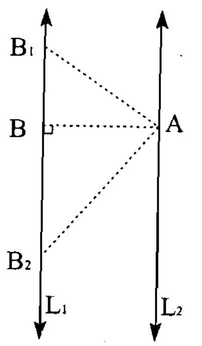


Fig. 12 : AB is the line of shortest distance.

Now, what is the distance if L_1 and L_2 are skew? Ask your students to think of ways of finding this. Here you could hint to them that given any two skew lines, we can find two planes which pass through the two lines respectively, and are parallel to each other. How this can be done by transforming the axes suitably is given in the solution below.

Problem 1: Given a pair of skew lines, reduce their equations to the simplest possible form.

Solution : Let AB and CD be two skew lines and let LM be perpendicular to both the lines, where L is on AB and M is on CD. Then LM is their line of shortest distance. We suppose that this shortest distance is 2c. Choose O as the middle point of the line segment LM. Through O we draw two lines OE and OF parallel to CD and AB, as shown in Fig.13. If 2θ is the angle between AB and CD, then $\angle$ EOF = 2θ.

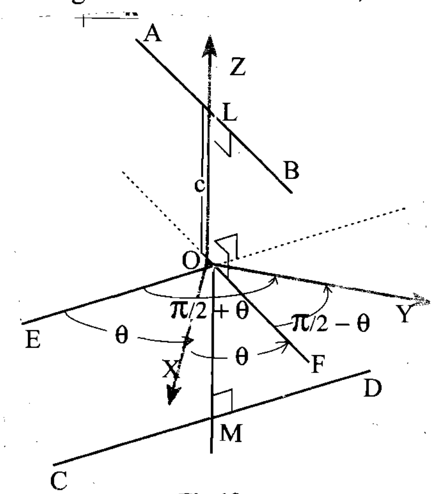


Fig.13

We choose the bisectors of the angles between OE and OF as the x-axis and y-axis, as shown in Fig.13, and ML as the z-axis. Note that the three lines OX, OY and OZ are mutually perpendicular. The line OF makes angles $\theta, \frac{\pi}{2} - \theta, \frac{\pi}{2}$ with the axes OX, OY and OZ, respectively. Therefore, the direction cosines of AB, which is parallel to OF, are $\cos\theta, \sin\theta, 0$.

Similarly, the line OE makes angles $\theta, \frac{\pi}{2} + \theta, \frac{\pi}{2}$ with the x-axis, y-axis and z-axis, respectively. Therefore, the direction cosines of the line CD are $\cos\theta, -\sin\theta, 0$. Also the coordinates of the points L and M are, respectively, (0, 0, c) and (0, 0, –c). Therefore, the equations of the lines AB and CD are

$$\frac{x-0}{\cos\theta} = \frac{y-0}{\sin\theta}, z = c \text{, i.e., } y = x\tan\theta, z = c \text{ and}$$

$$\frac{x-0}{\cos\theta} = \frac{y-0}{\sin\theta}, z = -c, \text{i.e., } y = -x\tan\theta, z = -c, \text{respectively.}$$

These are of the form y = mx, in the parallel planes z = c and z = –c.

———×———

Once students assume that the skew lines can lie in parallel planes, ask each student to draw two skew lines on two sheets of paper (as in Fig. 9). Then she should use a pencil or a long thin stick to find the different perpendiculars between the planes, and the one giving the shortest distance between the two lines. In fact, from different points of either one of these lines, we can draw perpendiculars to the parallel plane containing the other line. One and only one of these perpendiculars will have its foot at a point lying on the other line. But the lengths of all these perpendiculars will be the same. The perpendicular which intersects the two lines is along the line of shortest distance between the two skew lines. This will give the distance between the two skew lines.

Try some related exercises now.

E22) Find the shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$.

Let us now discuss another difficulty many students face while studying geometry.

10.5 LOCUS

Through our interaction with teachers and students of Classes 9 to 12, we realised that the moment students have to solve a problem on 'locus', they become afraid. In this section we discuss ways of removing this problem.

As always, it is better to introduce students to this concept in 2D first. To start with, you could ask each student to mark a point on a blank sheet of paper. Then, she should move her pencil in such a way that its distance from this point remains the same. Does each student draw a circle? You could then bring the students' attention to the fact that they have just drawn the path traced by a point subject to a given condition. Here you could introduce the term 'locus', telling them that it is just another word for 'the path traced by a point subject to some given conditions'. You could also relate this path to **the set of all points that satisfy the given conditions.** So, for example, the set of all points that are at a given distance from the given point is the circle they have drawn.

'Locus' is a Latin word, meaning 'place'. The plural of 'locus' is 'loci'.

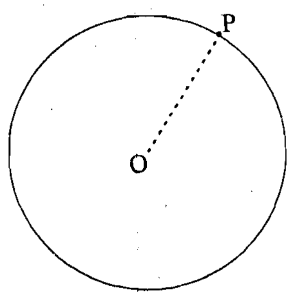
As in the case of any concept, you could introduce your students to loci by giving them several related exercises in two-dimensional space. You could ask them to find, for example, the locus of a point that is always equidistant from the x and y axes. You also need to do 'reverse' exercises like drawing a two-dimensional curve in standard form, say, a line, or an ellipse, and then asking them to formulate a 'locus problem' to which this would be the answer. Such examples would help them become familiar with the idea in 2D.

Loci are not only of points, but also of lines, or segments. Students could be asked to find the locus in 2D of a line parallel to a given line. Do they see that this set of lines is actually the whole plane? Give them several exercises related to finding the loci of lines/segments. One is given below, which you should also try.

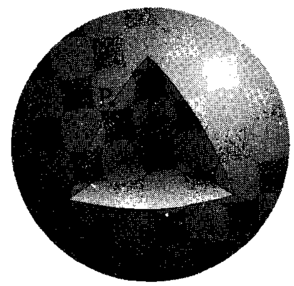
E23) a) Draw the locus of a line segment PQ passing through a fixed point O.

b) Which misconceptions of your students became clear to you after looking at their solutions of (a) above?

Once students are comfortable with the algebraic and geometric views of loci in 2-space, you could extend the concept to 3D. The important point here is to bring home the understanding that the locus of a point, subject to one or more constraints, would be different in 2D and in 3D. For instance, ask them to find the locus of a point which moves such that it is at a constant distance from a fixed point, in 2D first, and then in 3D. In two dimensions, they would find that the locus is a circle. However, in three dimensions the locus is a hollow sphere (see Fig. 14).



(a)

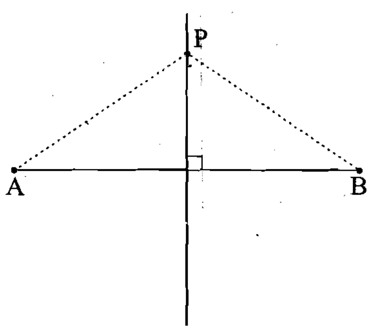


(b)

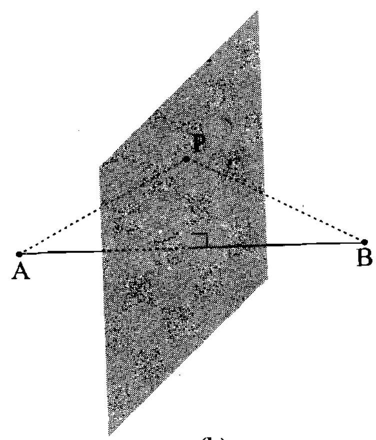
Fig. 14 : Locus in 2D and in 3D , under the same constraint

Your students need to understand this difference. How would you help them in this matter? One way is to show them many points in 3-space which are not on the circle, but are equidistant from the centre. For instance, they can see this by taking a ball that is cut into two semi-circles.

You could, similarly, ask the students to find the 'locus' in 2D, as well as in 3D, of a point which is equidistant from two points (see Fig. 15).



(a)



(b)

Fig.15 : The locus of a point which moves such that it is equidistant from two fixed points A and B is
a) the right bisector of AB in 2-space;
b) a plane which is perpendicular to, and bisects, the line segment AB in 3-space.

The immediate answer you may get is the middle-point O of the line segment AB. You may encourage your learners to think of some more points that are equidistant from A and B. The learners would certainly arrive at the line bisecting the line segment AB at right angles, i.e., the right bisector of AB.

So far as plane geometry is concerned, this is fine. But if they are studying three-dimensional geometry, then they need to go beyond this. You could give them a hint by taking some other points, say $P_1,\ldots,P_n$, lying on the plane passing through the middle point O of the line segment AB, such that AB is along the normal to this plane. Ask them to see if $P_1,\ldots,P_n$, are equidistant from A and B. You could do this by standing up a cardboard piece as the plane and taking some points on it. By actual measurement, they would see that any point P lying on the plane will be such that PA = PB. Ask your learners to think about why this is so.

You could also change your problem of the locus a little, so that your students consider loci in vectorial form. For example, you can have your students do problems like the following ones.

Problem 2 : A point P moves such that the direction of **OP** is always the same, say, along a unit vector **a**. Find the locus of P in vector form and in Cartesian form.

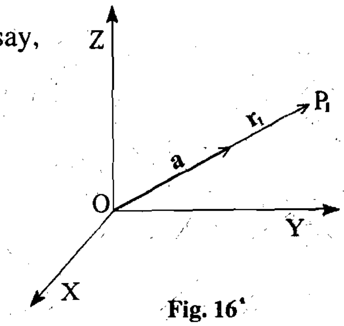


Fig. 16

Solution : We take a point P_1 with position vector $\mathbf{r}_1$. Then, the condition of the problem is shown in Fig. 16. From this figure, we see that

$\mathbf{r}_1 = s\mathbf{a}$, where s is a scalar.

As the point P_1 moves subject to this condition, s will keep changing. So, the equation of the locus is given by

$\mathbf{r} = s\mathbf{a}$, $s \in \mathbf{R}$.

Your students need to realise two things while solving the problem above. Firstly, they need to understand that the second equation is the generalised form of the first one. This is obtained by changing $\mathbf{r}_1$ (the position vector of a particular point P_1) to **r** (the position vector of a generic point P) on the locus.

Secondly, the students must understand which geometric object is represented by this equation. Do they realise that this is the equation of a straight line passing through the origin O with the direction of the line along the unit vector **a**? Do they also notice that the points on one side of O will correspond to positive values of s, and on the other side to negative values of s?

You could tell them why the equation is called the **parametric equation** of the straight line. The scalar s is called a **parameter** (it changes as the point moves along the line).

In the Cartesian form, **r** will be (x,y,z), and **a** can be taken as (l, m, n). (Why?) Then the vector equation will become

$(x, y, z) = s(l, m, n)$.

This is equivalent to the three equations

$x = sl$, $y = sm$, $z = sn$.

These equations could be rewritten as

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = s,$$

which is the usual **symmetric form of the equation** of a straight line, passing through the origin, with direction ratios l, m, n.

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Problem 3 : Find the locus of a line which is at a distance p from the origin and is perpendicular to the unit vector **n**, in vectorial and Cartesian forms.

Solution : Let us assume that p > 0, i.e., O does not lie on the line L. Take any point P_1, with position vector $\mathbf{r}_1$, on L. The shortest distance from O to L is the perpendicular distance. If we assume that O is the initial point of **n**, then the shortest distance lies along **n**. This is also the projection of OP_1 along **n**, and is given by $\mathbf{OP_1.n}$. So, $\mathbf{r}_1.\mathbf{n} = p$.

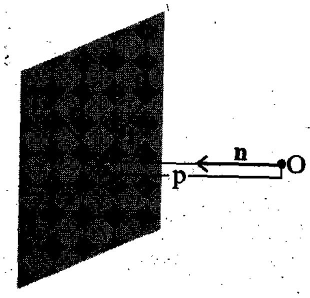

Fig. 17 : The plane Π is the locus of L.

To obtain the equation of the locus, we take the set of all such $\mathbf{r}_1$ (see Fig. 17). So, we change $\mathbf{r}_1$ to the position vector $\mathbf{r}$ of a current point P on the locus. The required equation of the locus, therefore, is

$\mathbf{r}.\mathbf{n} = p$, which is the **normal form of the equation of a plane.**

The Cartesian equivalent of this equation is obtained by putting $\mathbf{r} = (x, y, z)$ and $\mathbf{n} = (l, m, n)$, where *l, m, n* are the direction cosines of **n**. (Note that, since **n** is a unit vector, its coordinates are its direction cosines.)

Then $(x, y, z).(l, m, n) = p$, i.e., $lx + my + nz = p$.

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You need to ask your learners to solve several problems like these. You could do the following ones yourself and with them.

E24) Find the vector equation of the locus of a point which is always equidistant from two fixed points.

E25) Find the equations of the locus of a point which moves such that its perpendicular distances from two intersecting planes, given by the equations $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, are equal. What is the corresponding problem, and the equations formed, in two-dimensional space?

E26) A variable plane is at a fixed distance p from the origin. It meets the coordinate axes in A, B and C, respectively. Through these points, planes are drawn parallel to the coordinate planes. Find the locus of their point of intersection.

E27) Prove that the locus of the line which intersects the lines $y - z = 1, x = 0$; $z - x = 1, y = 0$; $x - y = 1, z = 0$ is $x^2 + y^2 + z^2 - 1 = 2(yz + zx + xy)$.

With this we end our discussion on problems related to studying 3D-geometry. We shall continue our study on the use of 3D in mechanics in the next two units. But first, let us take a brief look at what we have done here.

10.6 SUMMARY

In this unit, we have focussed on the following points.

1. How to help familiarise students with the conventions used for representing three-dimensional objects in two dimensions. With enough activities of the kind suggested, students would find it easier to visualise 3D in 2D.

2. The choice of the origin and the axes is arbitrary. Given a situation, they can be so chosen that the equations involved are reduced to the simplest possible form.

3. The equation representing a rigid body is altered by a translation or rotation of the axes. However, the shape and size of the rigid body is not altered under such transformations.

4. How to help students visualise lines in 3D, and pairs of skew lines in particular.

5. The relationships involving direction cosines, in various cases.

6. How to help students become comfortable with problems requiring them to find loci, in 2D or in 3D, in vectorial or in Cartesian forms.

10.7 COMMENTS ON EXERCISES

E1) (i) The coordinates of the points A, B, C, A′, B′, C′ and O are (x,0,0), (0,y,0), (0,0,c), (0,y,z), (x,0,z), (x,y,z) and (0,0,0), respectively.

(ii) The feet of the perpendiculars to the xOy, yOz and zOx plane have their co-ordinates $(-1,2,0)$, $(0,2,4)$, and $(-1,0,-4)$, respectively.

(iii) 5

(iv) $(-3, -2, 1)$

E2) A good way is to ask them to divide into groups, and make cardboard cut-outs of these planes. Then they could intermesh the three cut-outs to see how the octants form.

The students could also be asked to choose different points in the classroom as origins and identify the co-ordinates of different points in the room w.r.t. the new origin. You could get them to focus on how, as the origin changes, the coordinates of a point w.r.t. the origin also change, and they fall in different octants.

E3) Over here you could do exercises with two objectives:

i) showing that the origin can be any point;
ii) familiarising students with negative coordinates.

For instance, ask a student to pick any point in the room as the origin. With reference to this point, ask another student to give 3 coordinate axes. Then, ask other students to give points corresponding to, say, $(-1, 0, 0)$, $(1, -2, 1)$, and so on.

E4) In 2D, one can think of examples like graphing the relationship between the distance a plane flies, and the time it flies, if its velocity is 300 km/hr. Then, a convenient choice of unit length along the axes would be 1 and 300, respectively.

Think of similar examples in 3D.

E5) To introduce the students to the vectorial approach, you need to start by taking unit vectors **i, j, k** along the axes. Then you need to show that P(x,y,z) is also given as $\mathbf{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where x, y, z are the projections of **OP** along the axes. You could do this by giving the students several exercises of drawing vectors from O, finding their projections along the axes, and comparing these with the Cartesian coordinates of the endpoints. Do they see the relationship? It may help to do such exercises in 2D first, and then extend them to 3D.

E6) The direction of a vector of magnitude 0 is indeterminate because its initial and terminal points coincide. Thus, **0** can have any direction.

E7) 0 is a scalar and **0** is a vector with magnitude 0. What students must realise is why an equation like $\mathbf{a} = 0$ is meaningless, one side being a vector and the other a scalar.

E8) i) For shifting the origin to $(1,-1)$, the transformation equations are

$$x = x' + 1, \quad y = y' - 1.$$

Under this translation, the equation 2x−3y−5 = 0 becomes

$2(x'+1)-3(y'-1)-5=0$, i.e., $2x'-3y'=0$.

Ask the students : Does the translation mean that the line has also shifted? Was the original equation a line too? etc.

ii) The transformation equations are

$x = x' + 3,\ y = y' - 2.$

Under this transformation, the given equation becomes

$(x'+3)^2+(y'-2)^2-6(x'+3)+4(y'-2)+4=0$,

i.e., $x'^2+y'^2=9$.

E9) The given equation can be rewritten as

$(x+1)^2+(y-4)^2-1=0$

If you shift the origin to (−1,4), this becomes $x'^2+y'^2=1$.

Ask your students what the object presented by the original equation was? Has its shape or size altered under the translation?

E10) Think of situations in which the equations become simpler to deal with after a translation of the axes. E.g., see Problem 1 (given after E21 of this unit).

E11) Here, you would need to see if they understand that though the equation representing a rigid body changes under translation, the body itself does not change size or shape. You could draw an object, and then ask them to show what the object would look like after a translation, for example.

E12) The point about rotation of axes is that while the origin remains fixed, the axes in 3 dimensions change. You need to help your students appreciate that the objects in space change their position relative to the axes, but not relative to the origin, nor with reference to each other. As a result this transformation of axes does not alter anything in the body. One activity could be to get your students to place things on their desks, taking the axes along the edges of the desk. Then, without moving the objects, rotate the edges without moving the axes, showing the new axes by strings or lines drawn on the desks. They would see that everything remains the same relative to each other; only the coordinates of the point, obtained from projections on the 3 axes, change.

E13) Take a translation and rotation in 2D to start with, and see why this does not hold in general. Then you can take examples in 3D to see why this is not true.

E14) Is it possible to always have $\angle XOX' = \angle YOY' = \angle ZOZ' = \theta$? Think about the relationship between the *l*s, *m*s and *n*s. Also see E20.

E15) Can the students show you that there are an infinite number of planes passing through a given point parallel to a given line? Can you show them why this is so?

E16) i) An infinite number of lines
ii) Only one line
iii) An infinite number of lines

How would you help them to come out with proofs of these facts?

E17) The direction cosines are proportional to themselves. (What is the constant of proportionality?) Therefore, they are also drs.

E18) If l, m, n are the dcs of the line, then $l = m = n$ and $l^2 + m^2 + n^2 = 1$.

$$\therefore l = \pm\frac{1}{\sqrt{3}}, m = \pm\frac{1}{\sqrt{3}}, n = \pm\frac{1}{\sqrt{3}},$$

where the same sign, positive or negative, is to be taken throughout.

E19) You could take the cube to be a unit cube with three coterminous edges as the axes (see Fig.18). The four diagonals of the cube will have direction ratios 1, 1,1 ; –1, 1, 1 ; 1, –1, 1 ; 1,1, –1.

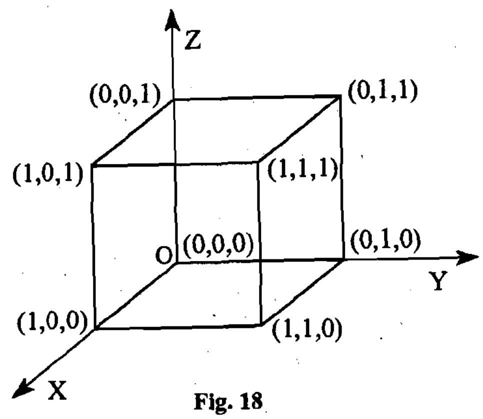

Fig. 18

Let l, m, n be the dcs of the line which makes angles α, β, γ and δ with the four diagonals. Then

$$\cos\alpha = \frac{l+m+n}{\sqrt{3}}, \cos\beta = \frac{-l+m+n}{\sqrt{3}}, \cos\gamma = \frac{l-m+n}{\sqrt{3}}, \cos\delta = \frac{l+m-n}{\sqrt{3}}$$

Squaring and adding these, we get the result.

E20) The cosines of the angles which the x-axis makes with the three lines are l_1, l_2, l_3. Similarly the cosines of the angles which the y-axis and z-axis make with the three lines are $m_1, m_2, m_3; n_1, n_2, n_3$, respectively.

With reference to the given lines the dcs of the axes are, therefore,

$l_1, l_2, l_3; m_1, m_2, m_3; n_1, n_2, n_3$.

Also, these are mutually perpendicular lines. The result follows.

E21) The direction ratios of the given vector are 1, –2, 3. So, its direction cosines are $\frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$.

E22) Let l, m, n be the dcs of the line of shortest distance. Since this line is perpendicular to the given lines, we get

$2l + 3m + 4n = 0, 3l + 4m + 5n = 0$.

Eliminating l, m, n, we get $l = \frac{-1}{\sqrt{6}}, m = \frac{2}{\sqrt{6}}, n = \frac{-1}{\sqrt{6}}$.

The shortest distance is the projection of the join of any two points P and Q (on each of the given lines) onto the line of shortest distance. Taking P(1,2,3) and Q(2,4,5), we get the shortest distance $= (2-1)l + (4-2)m + (5-3)n = \frac{1}{\sqrt{6}}$.

E23) a) Many students may think that this is a disc with centre O and diameter PQ. It is a disc. However, this is the set of all possible positions PQ can take, with O lying on PQ. So, O can coincide with P, or with Q, or be in between. All these cases need to be covered. So it would be a disc with centre O and **radius** PQ.

b) For instance, did some students just draw one segment? And, if so, what did this indicate to you of their understanding of 'locus'?

E24) Let's say the fixed points are A and B. You can take the middle point of the line segment AB as the origin of reference, with the points A and B having position vectors $-\mathbf{a}$ and $\mathbf{a}$. This choice makes the equation simpler. If the position vector of P_1 is $\mathbf{r}_1$, then the equidistance condition of the problem gives us

$$|\mathbf{r}_1 - (-\mathbf{a})| = |\mathbf{r}_1 - \mathbf{a}| \Leftrightarrow \mathbf{r}_1.\mathbf{a} = 0.$$

You could help your students deduce that this means $\mathbf{a}$ is orthogonal to $\mathbf{r}_1$. All points that are equidistant from A and B have a position vector that is orthogonal to $\mathbf{a}$.The equation of the locus, therefore, is obtained by changing $\mathbf{r}_1$ to $\mathbf{r}$, the position vector of a current point on the locus. Hence, the required equation of the locus, is $\mathbf{r}.\mathbf{a} = 0$.

In fact, this is the equation of a plane passing through the origin and normal to the vector $\mathbf{a}$.

E25) If (x_1, y_1, z_1) is a point on the locus, then

$$\frac{a_1x_1 + b_1y_1 + c_1z_1 + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{|a_2x_1 + b_2y_1 + c_2z_1 + d_2|}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

The required equations, therefore, are obtained by generalising (x_1, y_1, z_1) to (x,y,z), i.e.,

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{|a_2x + b_2y + c_2z + d_2|}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$, which is a line.

In 2-space the problem becomes finding the locus of a point which moves such that its perpendicular distance from each of two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are equal. This locus could be found on the same lines as above.

E26) $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$ lie on the plane.

So, the equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

The planes are $x = a$, $y = b$ and $z = c$. Their point of intersection is (a,b,c). We know that as the plane varies, its distance from $(0,0,0)$ remains p. So,

$$\frac{1}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2}} = p$$

Therefore, the locus of (a,b,c) is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}.$$

E27) The planes through the given lines are

$$(y - z - 1) + ax = 0, (z - x - 1) + by = 0, (x - y - 1) + cz = 0. \qquad (1)$$

These planes will intersect in a line if

$$\begin{vmatrix} a & 1 & -1 \\ -1 & b & 1 \\ 1 & -1 & c \end{vmatrix} = 0, \text{ i.e., } abc + a + b + c = 0 \qquad (2)$$

Eliminating a,b,c from (1) and (2) gives us the locus, which is the given one.