UNIT 15 CONGRUENCE AND CONSTRUCTION OF TRIANGLES

Structure

15.1 INTRODUCTION

Geometry, in general, and triangles in particular, have always fascinated human beings. Termed as the polygon of least number of sides or the first closed rectilinear figure in geometry, the triangle was one of the initial figures used for tesselations [blocks used for tile construction].
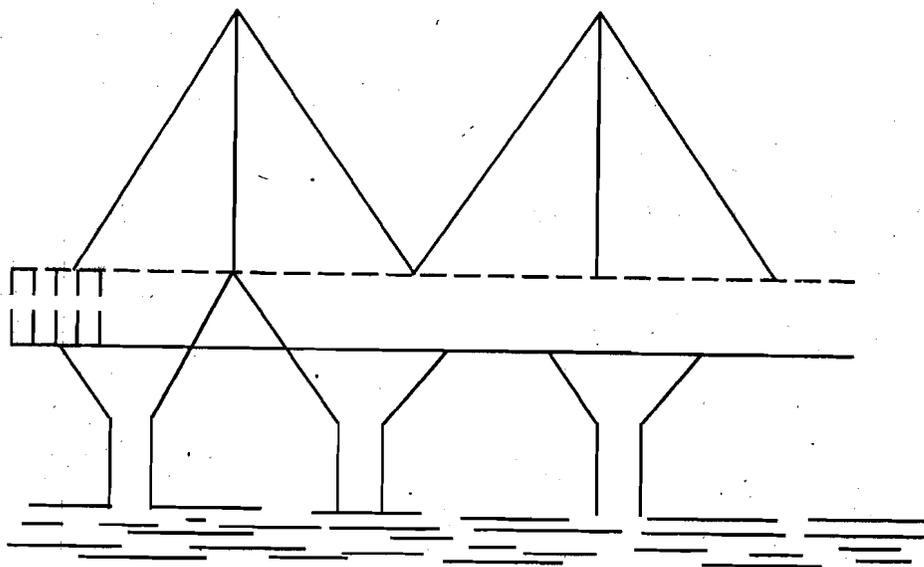

Fig. 15.1

Look at the mosaics of early architecture and those early constructed buildings. Discovery of the various properties of a triangle resulted in its use in the construction of bridges. Heavy poles, and even furniture require triangle edges in their support in order to retain their shape.

In this unit, we will introduce ourselves to the properties so very special to this unique figure.

15.2 OBJECTIVES

After studying this unit, you, the pupil teacher, will be able to effectively:

- explain the concept of equality and similarity together with illustrations;

- explain and apply the concept of congruence to geometric figures in general;

- make the students understand the symbol ≅;

- put across the importance of 1-1 correspondence;

- make the students understand that given SSS, SAS, ASA (AAS), RHS we always get triangles of the same size and shape.

- make the students understand that the four conditions - SSS, SAS, ASA(AAS), RHS whenever satisfied by the given triangles make them congruent; and

- apply this knowledge to various problems.

15.3 COMPARISON OF TWO FIGURES

Main Teaching Points : a) Comparing shapes and sizes.

b) The size determined by area.

Teaching-Learning Process : Hold in your hand two cut-outs of the same shape.

Ask   :   Look at these two cut-outs What can you tell about these cut-outs? (In case there is no response in the beginning, you may bring in leading suggestions like: they look alike, their sizes are not the same, what else ...... there may be responses regarding their volumes, their corners. their sides, ... these are ruled out).

Repeat this line of interaction with 3 or 4 pairs of cut-outs.

Ask   :   When we look at two things or articles, what questions come to our mind regarding their shapes, size or any other characteristics.

Ask   :   Do we try to know which one is bigger? Which one is smaller?

Ask   :   What are we comparing?

When we say that one is bigger than the other, what are we comparing?

Is it their weights?

or is it their lengths?

or is it the space each covers?
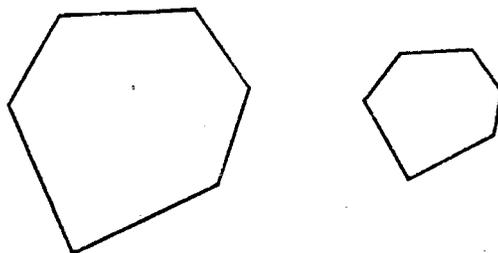
Fig. 15.2(a)

Ask   :   Which one is bigger?

How do you know?

Explain : By covering them with squared transparent paper.

Repeat the same questions with the following figures :-
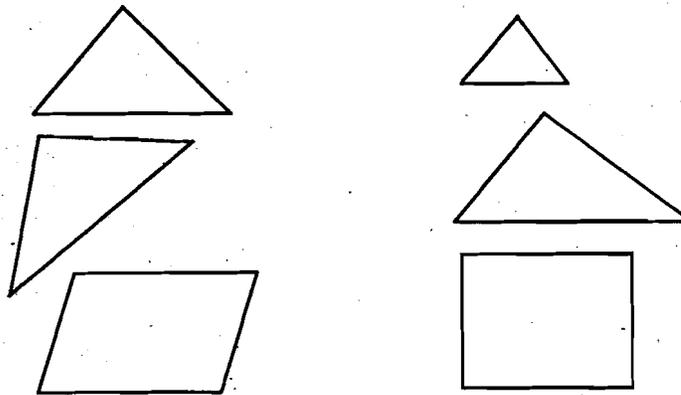
Fig. 15.2(b)

Conclude

Two figures can be different in shape and different in size.

Two figures can be different in shape and same in size.

Two figures can be same in shape and different in size.

Two figures can be same in shape and same in size.

Size of the figures is measured by their area.

Methodology used: Demonstration combined with proper discussion is used to illustrate the point.

Note : We can use transparencies to show movement of one figure into another (super position).

Check Your Progress

Notes : a) Write your answers in the space given below.

b) Compare your answers with the one given at the end of the unit.

1. Explain : Which of the following figures have the same shape and size?
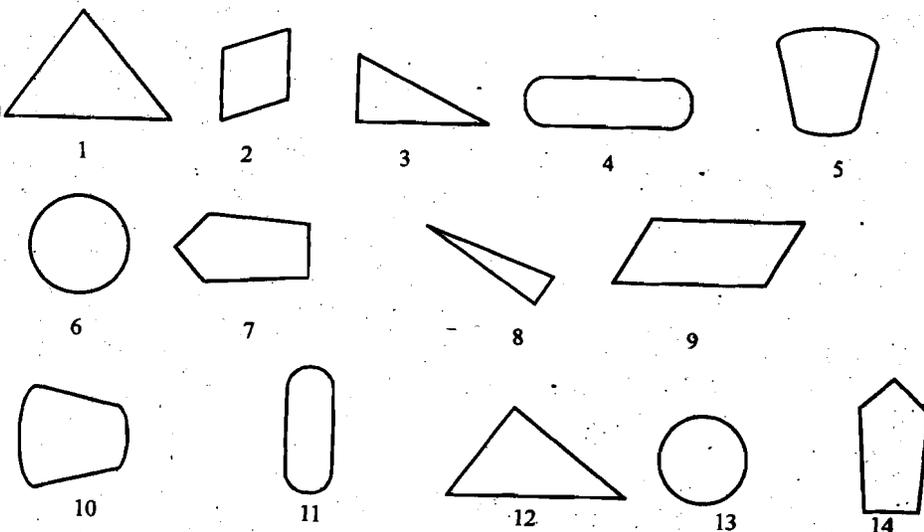
15.4   SIMILARITY

Main Teaching Points: a)   Geometrical figures of the same shape are called similar.

b)   The size of similar figures may be same or different.

Teaching-Learning Process: Here we are concerned with the shape of the figures. Hold in
your hand two circles

Ask   :   What can you say about the shapes of these two figures ?

Repeat with pairs of squares and equilateral triangles. You may
even consider pairs of regular figures.

Ask   :   Do the two have the same size/area/ (the figures chosen must
include those of different sizes and the same sizes).

Activity 1   :   Draw two line segments of 5 cm and 7.5 cm. length. Name them
AB and XY.

On A and X draw angles of 60° with AB and XY respectively.

Similarly on B and Y draw angles of 75°

Let the arms of the two angles (other than the common arm)
intersect in C and Z respectively.

Ask   :   What shapes do you get?

Place the two triangles such that A falls on X and AB on XY. Next
repeat the same thing with B falling on Y and BA on XZ.

What can you say about the positions of AC and XZ?

Do the two triangles have the same shape? Do they have same
size?

Explain   :   The two triangles have the same shape but different sizes.

Ask   :   What will happen if AB and XY both are of the same length?

Will you get two triangles of the same shape?

What about their sizes?

Explain   :   The triangles are of the same shape and the same size.

Activity 2   :   Hold a triangular template in front of a candle. Form an image on
a plain sheet of paper.

Now move the template towards the flame and away from the
flame.

Ask   :   What happens to the shadow?

Do you have the same shape always? What care you have to take
to get a sharp image?

Activity 3   :   Repeat Activity 2 with a template of some different
figures/shapes.

Ask   :   What can you say about your passport size photograph and the
same in the post-card size?

Explain   :   We get figures of the same shape. But their sizes may be same or
different.

When the shapes of two figures are alike we say that figure A **is similar** to figure B.

In particular

Δ ABC is similar to Δ XYZ

We write this as

$$\Delta\,ABC \sim \Delta\,XYZ$$

Conclude : Two figures are similar if they are of the same shape. Their sizes may or may not be the same.

Methodology used : Learning by doing is the best approach. Let the students draw the figures and help them to reach the conclusion.

Check Your Progress

Notes : a) Write your answers in the space given below.

b) Compare your answers with the one given at the end of the unit.

2. Draw a triangle of sides measuring 3 cm, 4 cm, 5 cm and another one of sides 6 cm, 8 cm 10 cm. What can you say about their shapes? Measure their angles and compare. How will you demonstrate that the triangles are similar ?

....

....

....

3. Draw a triangle having two sides of 4 cm and 5 cm and their included angle of 45°. Draw another triangle having sides measuring 6 cm and 7.5 cm and their included angle of 45°.

Compare the two triangles.

What can you say about their angles?

Find the ratio of the third side of one triangle with the third side of the other. Compare it with ratios of other two pairs of corresponding sides. How will you demonstrate that the triangles are similar?

....

....

....

4. Look at the following figures and point out similar figures.

(A) (B) (C) (D) (E)

(F) (G) (H) (I) (J)

(K) (L) (M) (N) (O) (P) (Q) (R)

15.5   CONGRUENCE

15.5   CONGRUENCE
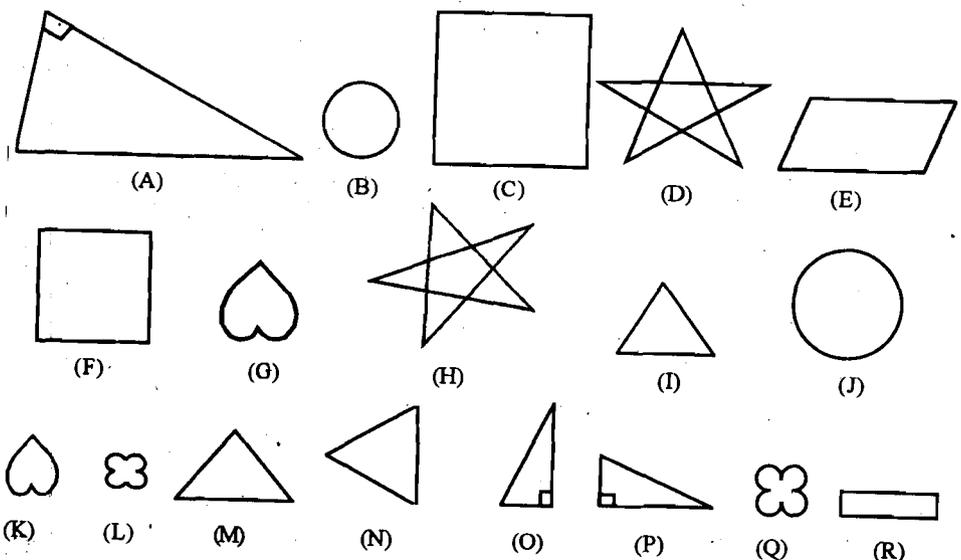
Main Teaching Point : Congruent figures have same shape and same size.

Teaching-Learning Process

15.5.1   Definition

You have seen above that we can have two figures of the same size and also we can have two figures of the same shape.
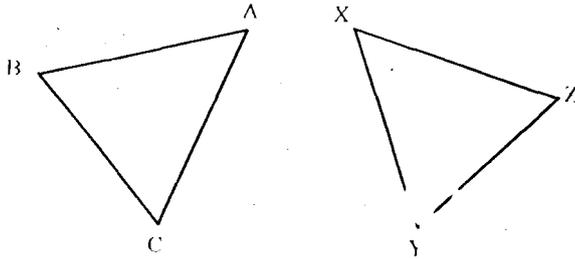
Fig. 15.3

Ask   :   Are the triangles ABC and XYZ of the same shape? Can you confirm it by turning around Δ ABC or Δ XYX and then comparing?

What can you say about their sizes?

Repeat with the following pairs of triangles:
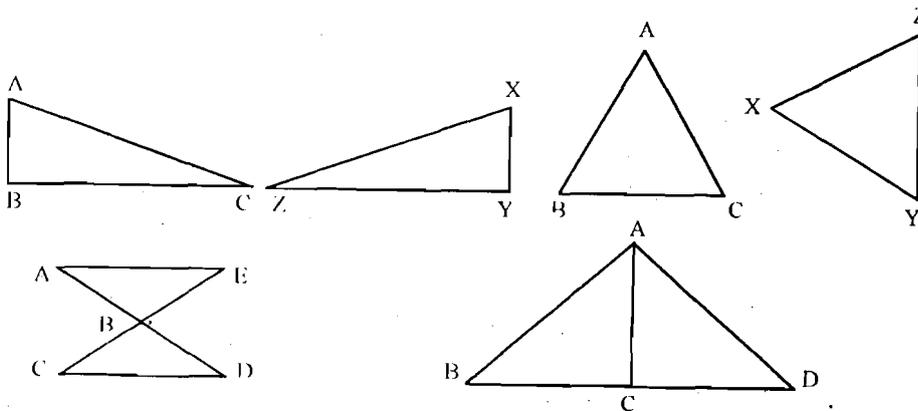
Fig. 15.4

In all the above pairs the size of one triangle is equal to the other and also the shape of one triangle is same as the shape of the other.

In other words the triangles have the same size and are also similar.

The concept of equality of size and similarity of shape is combined in congruence. We say that one figure is congruent to the other.

Symbol ≅ is used to denote congruence of two figures.

Methodology used: Demonstration-cum-discussion leading to the conclsion.

15.5.2 Line Segments and Angles

Main Teaching Points : a) Line-segments of equal length are congruent.

b) Angles of the same measure are congruent.

Teaching-Learning Process

Ask : Given two line-segments what information will you need to decide whether they are congruent?

Draw : Two line-segments of 3 cm each. Do they look alike (similar)? Do they have the same length?

Explain So you have a pair of congruent line-segments. Try with as many pairs.

What do we conclude?

Line-Segments of Equal Length Are Congruent

Ask : In the same way let us consider a pair of angles. What information will you need to say that the pair of angles are congruent?

Draw two angles of 60° each.

Are these angles of the same measure?

Explain : So, you have a pair of congruent angles.

You may try with more pairs.

What do we conclude?

Two angles are congruent if they have the same measure

Methodology used : Demonstration-cum-discussion is used to lead the students to conclusion.

15.5.3 Correspondence and Congruence

Main Teaching Points : a) There are many correspondences possible between two figures.

b) Some of them may be congruences and others may not.

Teaching-Learning Process : In two sets (consider two teams participating in a hockey match) there is element-to-element (a man-to-man) matching or association. This association in mathematics is termed as 1-1 correspondence.
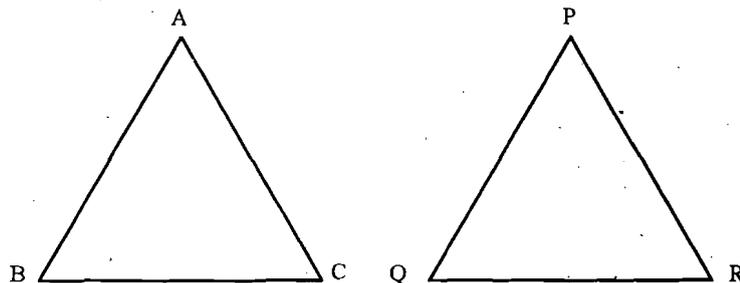

Fig. 15.5

Ask : In how many different ways can the vertices A, B, C of $\triangle$ ABC be set in 1-1 correspondence with vertices P, Q, R of $\triangle$ PQR?

We can set up 1-1 correspondence in six different ways, namely:

(1) A — P (2) A — P (3) A — R (4) A — R (5) A — Q (6) A — Q
    B — Q   B — R   B — Q   B — P   B — P   B — R
    C — R   C — Q   C — P   C — Q   C — R   C — P

Explain : 1-1 correspondence between vertices of the triangles determine the 1-1 correspondence between angles and sides of the two triangles.

e.g in   A — P

        B — Q

        C — R

$\angle A - \angle P$ and AB – PQ

$\angle B - \angle Q$ and AC – PR

$\angle L - \angle R$ and BC – QR

We also call this correspondence as **ABC ↔ PQR**

Activity 1 : Let us superimpose Δ PQR on Δ ABC according to the 1-1 correspondence described above.
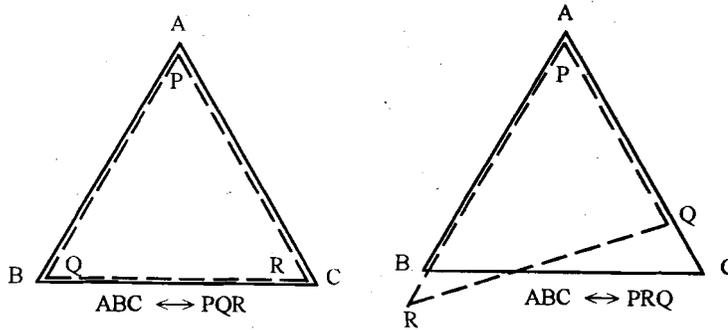
Fig. 15.6

Draw figures for other correspondences also. In how many cases a correspondence results in completely covering one figure over the other. (This happens only in ABC ↔ PQR).

Activity 2 : Take a Δ ABC in which AB = AC. Make a carbon copy of this triangle and name it Δ PQR. Superimpose and find out how many correspondences result in covering one figure over the other completely. In case of ABC ↔ PQR and ABC ↔ PRQ, one Δ covers the other completely.

Activity 3 : Take Δ ABC which is equilateral.

Make Δ PQR which is a carbon-copy of Δ ABC.

Superimpose and investigate in this case also as to how many correspondences result in covering one triangle over the other completely.

Explain : In all correspondences, one triangle covers the other completely.

Methodology used : This is purely an activity-based method. Let the students do themselves and reach at the conclusion.

15.5.4 Congruence of Triangles

Main Teaching Point : Defining congruent triangles.

Teaching-Learning Process : This leads us informally to state that "If there exists at least one correspondence between two triangles such that if we superimpose one triangle over the other, and they cover each other completely, then the two triangles are congruent.

Under activity (1) we found that $\triangle$ ABC $\cong \triangle$ PQR which implies that

$\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$. $AB = PQ$, $BC = QR$, $AC = PR$

Explain : We frequently use the symbol = in place of $\cong$ for line-segments and angles. This usage is widely prevalent and it does not cause any misunderstanding.

We name the triangles according to the correspondence which results in a congruence. This helps in immediately pointing out the pairs of corresponding angles and corresponding sides which are congruent (or equal). We conclude that: "If three angles of a triangle are congruent (equal) to three angles of another triangle each to each and their corresponding sides are congruent (equal), then the two triangles are said to be congruent".

Including the converse of the above statement we define congruence of triangles as below:

Definition : Two triangles are congruent if and only if (iff) 'their angles are equal each to each and their corresponding sides are equal'.

Reinforcement : Cut-out the following pairs of figures and discover the corresponding vertices. Write the corresponding parts that are equal.
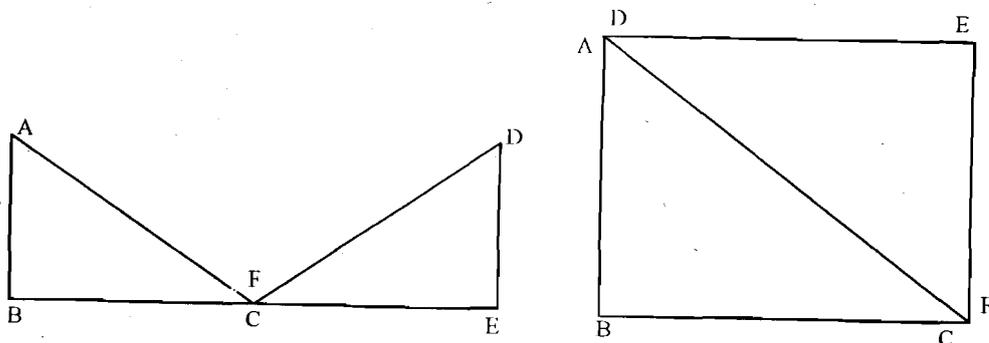

Fig. 15.7

Correctly name the triangles which are congruent.

You may devise more activities of this type.

Methodology used: Deductive reasoning is used. From the activities of the previous sections, he students can deduce the definition.

15.6 CONSTRUCTING TRIANGLES WITH AVAILABLE INFORMATION

If a triangle is given, one can find out measures of angles and sides of the triangle. Conversely, if data about sides and angles of a triangle are given, let us investigate whether we can construct the triangle.

Ask : Do you think you need to know data about all sides and angles of a triangle to be able to construct it.

What data is required so that you can construct a unique triangle?

Explain : We can construct a triangle uniquely if we can fix its three vertices. Generally this is possible if measurements of three parts of which one must be a side are given. (The teacher should remember that ambiguous case is an exception to this rule).

15.6.1 Three Sides (SSS)

Main Teaching Point: We get a unique triangle if we know all its the sides.

Teaching-Learning Process : Let us construct a triangle whose sides measure 3 cm, 4 cm, and 5 cm using ruler and compass only.
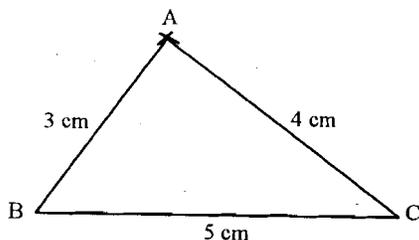
Fig. 15.8

Let every student draw BC = 5 cm

Now Ask : How can you locate A such that A is 3 cm from B and 4 cm from C?

Lead the response till you have the answer : By drawing arcs (parts of a circle) with radius 3 cm and 4 cm and centres B and C respectively.

Ask : What do you observe about the triangle drawn by you and the one drawn by your neighbouring student?

Expected Response : They are all similar and equal in size, i.e., they are congruent.

Ask them to verify by measuring and writing the corresponding angles.

You may repeat by having another set of sides.

Conclusion·

1. Given the measures of the three sides of a triangle, we can construct the triangle uniquely.

2. Triangles drawn with the same measures of sides will be congruent.

3. Given two triangles measures of whose corresponding sides are equal, the triangles are congruent. This is called SSS Congruence.

Methodology used : It is purely an activity-based method. Let the students work it out for themselves and reach the desired conclusion.

15.6.2 Two Sides and the included Angle (SAS)

Main Teaching Point: Two angles and an included side determines a triangle uniquely.

Teaching-Learning Process : Let us construct a triangle ABC which has BC = 4.5 cm, $\angle C = 50^\circ$ and $\angle B = 70^\circ$.
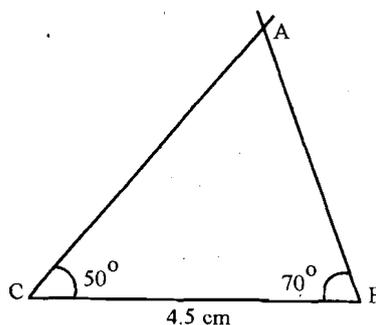
Fig. 15.9

Every student draws a line-segment (BC) equal to 4.5 cm. Then he draws $\angle B = 70^\circ$ and $\angle C = 50^\circ$, the non-common arms of the two angles intersect at A.

Ask   : What can you say about the triangle drawn by you and the ones drawn by your neighbours?

Explain   : They are congruent.

         You may repeat by changing the measures.

Conclusions

1. Given the measures of two angles and the included side, we can construct the triangle uniquely.

2. Triangles drawn with the same measures of two angles and the included side will be congruent.

3. Given two triangles such that the measures of one of their sides and the two angles on these sides are equal, the triangles are congruent.

This is called ASA congruence.

Methodology used: It is purely an activity-based method. Let the students work it out for themselves and reach the desired conclusion.

15.6.3 Two Angles and a Side (ASA, AAS)

Main Teaching Point : Two sides and an included angle determines a unique triangle.

Teaching-Learning Process : Let us construct a triangle XYZ which has XY = 3.5 cm, $\angle Y = 45^\circ$, YZ = 4.5 cm.

[Repeat the steps as in 15.6.1 and 15.6.2.]

Draw XY = 3.5 cm, measure $\angle Y$ as 45°, on this arm from Y take 4.5 cm and mark the point as Z, join XZ.

Conclusion

1. Given the measures of two sides and the included angle of a triangle, we can construct a unique triangle.

2. Triangles drawn with the same measures of two sides and the included angle will be congruent.

3. Given two triangles such that the measures of two of their sides and the included angle are correspondingly equal, the triangles are congruent.

This is called SAS congruence.

Methodology used: It is purely an activity-based method, let the students work it out for themselves and reach the desired conclusion.

15.6.4 The Hypotenuse and One Side of a Right Triangle (RHS)

Main Teaching Point : If in a right triangle, one side and hypotenuse are given then we get a unique triangle.

Teaching-Learning Process : Let us construct a triangle ABC right angled at B which has its hypotenuse AC = 5 cm and side BC = 4 cm.
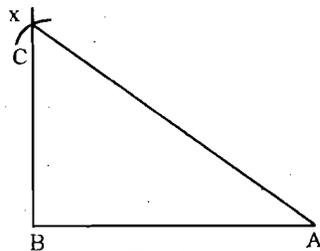

Fig. 15.10

Every student draws a line-segment BC = 4 cm, then at B makes an angle $\angle XBC = 90^\circ$.

Ask   :   How can you locate A on BX such that CA = 5 cm?

Lead the responses till you have the answer : By drawing an arc (part of a circle) with centre at C and radius = 5 cm.

Now repeat the comparison and construction as in the case of previous sections.

Conclusions

1. Given the measures of hypotenuse and a side of a right triangle we can construct the triang.

2. Right triangles drawn with the same measures of hypotenuse and a side are all congruent.

3. Given two right triangles whose hypotenuse and one side each are equal, the triangles are congruent.

This is called RHS congruence.

Methodology used : It is purely an activity-based method. Let the students work out themselves and reach the desired conclusion.

15.7   TRIANGLE — A RIGID FIGURE

Main Teaching Point : A triangle cannot be deformed.

Teaching-Learning Process

Ask   :   What is the geometric figure that you see when you look at a bridge, an electric pole or any other heavy structure?

Ask   :   Why do you think the triangle is used for this purpose?

Let us conduct an experiment.

Experiment   Take four strips of wood long enough to be joined to form a four-sided figure.
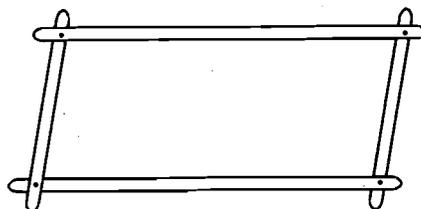
Fig. 15.11(a)

Fix these sticks, as shown in the figures with nails and screws.

Now try to flex the figure to change its shape.

What do you observe ?

Now repeat by forming a five sided figures.
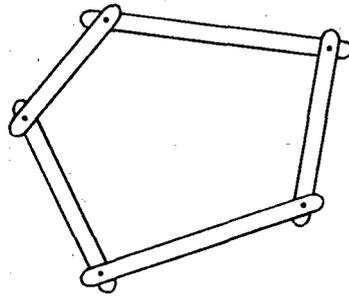
Fig. 15.11(b)

What do you observe?

Let us now form a triangular shape.
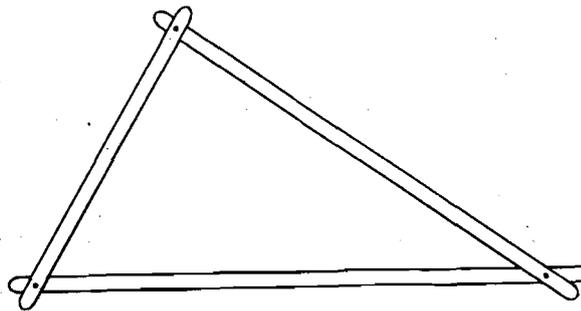
Fig. 15.11(c)

What do you find ?

Now you must have understood why in structures we employ a triangle.

Explain   : The three sticks cannot be moved even slightly. Therefore, the triangle is the most rigid structure which cannot be deformed.

Methodology used : The students can learn this concept best by performing the activity themselves.

15.8 CONGRUENCE OF TRIANGLES

15.8.1 Conditions of Congruence

Main Teaching Point : To list out the four conditions under which two triangles are congruent.

Teaching-Learning Process : You have seen in 15.6 that we have congruent triangles in four cases:

  SSS   : When the three sides of one triangle are equal to the corresponding three sides of another triangle.

  SAS   When two sides and the included angle of one triangle are equal to the corresponding parts of the other triangle.

| ASA | : | When two angles and the included side of one triangle are equal to the corresponding parts of the other traingle. |

OR

AAS	:	When one side and any two angles of one triangle are equal to the corresponding parts of the other triangle.
AAS	:	Easily leads to ASA because when two angles of two triangles are equal, the third angle will be equal to the third angle.
RHS	:	When the hypotenuse and one side of one right triangle are equal to the corresponding parts of the other right triangle.

Methodology used : Deductive logic is used. The students have already reached the conclusions in 15.6.·

15.8.2 Some Applications

Main Teaching Point : Use of different conditions of congruence.

Teaching-Learning Process : In triangle ABC if AB = AC then using SSS congruence prove that $\angle B = \angle C$
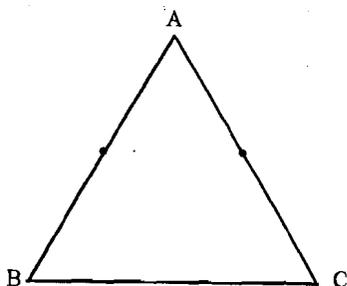
Fig. 15.12

AB = AC

This means can we now write

ABC ↔ ACB?

$\triangle ABC \cong \triangle ACB$ (SSS)

AB = AC

AC = AB

BC = CB

Can you now say

$\angle B = \angle C$? Yes, because you will recall that CORRESPONDENCE AND CORRESPONDING PARTS play a very important role in congruence. ·

Ask : In $\triangle$ ABC and $\triangle$ DEF if AB = DE, BC = EF and AC = DF then

$\angle A = ?$, $\angle B = ?$ and $\angle C = ?$

Can we say that in $\triangle$ ABC and $\triangle$ DEF

If $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$ than is AB = DE, BC = EF, and AC = DF?

Explain : Corresponding angles being equal does not imply that corresponding sides must also be equal. (This means that AAA does not result into a congruence, though it leads to similarity).

In Δ ABC, if AB = AC, then using SAS congruence, prove that ∠B = ∠C.
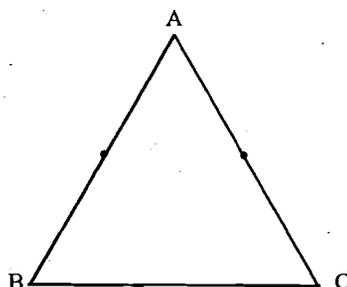
Fig. 15.13

Consider the correspondence ABC ↔ ACB

$\quad$ AB = AC

$\quad$ AC = AB

and ∠A = ∠A

∴ Δ ABC ≅ Δ ACB .... (SAS)

∴ ∠B = ∠C

From the above two results, can you say that 'base angles of an isosceles triangles are equal'?

Can you use this congruence in the following case?
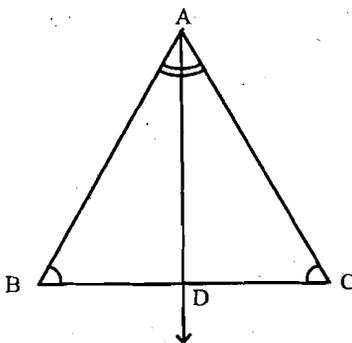
Fig. 15.14

AD bisects ∠A of isosceles triangle ABC. Prove that D is the mid point of BC.

Consider the correspondence

ABD ↔ ACD

Are Δ ABD and Δ ACD congruent ? Why?

1. ....

2. .. ..

3. ....

State the result :

Using RHS congruence

In Δ ABC AB = AC, parpendicular from A to BC meets BC at D. Prove that ∠B = ∠C , BD = DC
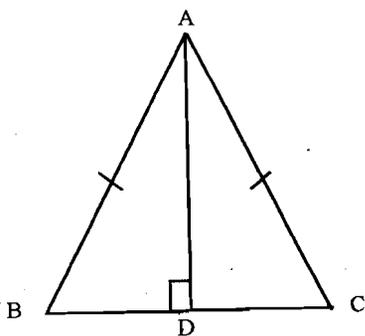
Fig. 15.15

Right $\triangle ABD \cong$ Right $\triangle ACD$

Why ? State the correspondence. So, BD = DC

State the result.

Methodology used : Heuristic approach should be used. Discussion should be initiated and lead the students towards the right answer.

Check Your Progress

Notes : a) Write your answers in the space given below.

b) Compare your answers with the one given at the end of the unit.

5. In the following diagrams are the triangles in each pair congruent? Why? Equal parts are shown by similar markings. Explain.
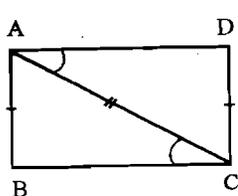
(a)
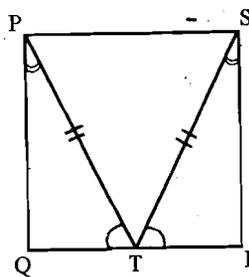
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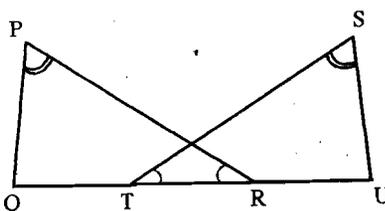
(c)
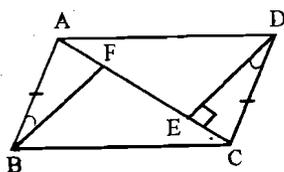
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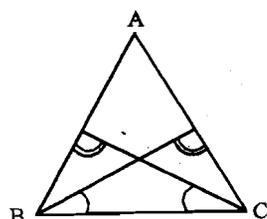
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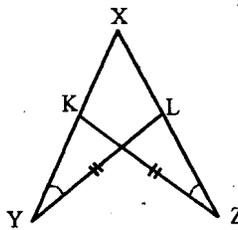
(f)

15.9   LET US SUM UP

1. Figures of the same shape but not necessarily of the same size are similar.

   Similarity of triangles is dependent on the equality of the corresponding angles.

3. Figures of the same shapes may be of different size.

   Triangles of the same shape and same size are congruent.

5. We can construct a unique triangle when we are given:

   ● measures of three sides SSS

   ● measures of two sides and the included angle SAS

- measures of any two angles and one side AAS or ASA

- a right triangle with hypotenuse and one side RHS.

7.   A triangle is a rigid figure.

8.   Two triangles are congruent when

- their corresponding sides are equal SSS

- they have two pairs of corresponding sides and the included angle equal

- two angles and a side of one triangle are equal to the corresponding parts of the other.

- the triangles are right angled and the hypotenuse and one side of one triangle are equal to the corresponding parts of the other.

15.10   UNIT-END ACTIVITIES

1.   Given the information in each part of the question, fill up the blanks .

For △ ABC and △ DEF

	The correspondence is	Two triangle(s) are similar or congruent	Why
a) $\angle A = \angle D$ $\angle B = \angle E$	ACB ↔ ————	————	———
b) $AC = DE$ $AB = DF$ $\angle A = \angle D$	ABC ↔ ————	————	———
c) $\angle A = \angle F$ $\angle B = \angle D$	————— ↔**DEF**	————	———
d) $\angle F = \angle B$ $\angle E = \angle C$ $FE = BC$	————— ↔**DEF**	————	———
e) $\angle D = \angle A = 90°$ $EF = BC$ $DE = AB$	ABC ↔———	————	———

2.   Segments AB and CD bisect each other at O. Prove that

a)   △ OAC ≅ △ OBD

b)   AC = DB

3.   BP bisects $\angle ABC$. PE and PD make equal angles with BP and meet AB and AC at D and E respectively prove that △ BPE ≅ △ BPD.

4.   AE = ED,

EF = EF

Prove that AB = CD
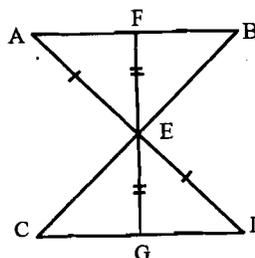
5. DA = DC, BA = BC

Prove that EA = EC
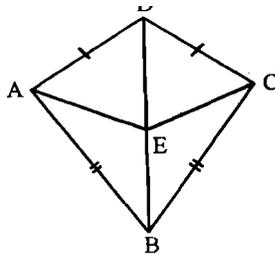
6   DC = AB

∠1 = ∠2

∠3 = ∠4

Prove AE = CF
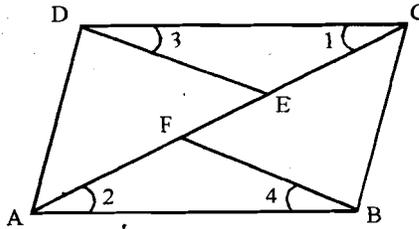
AD = BC

∠1 = 50°

∠2 = 50°

Prove △ ABC ≅ △ BAD
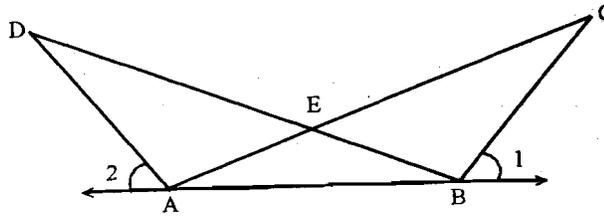
AB = AD

AC = AE

Prove AF bisects ∠ BAD
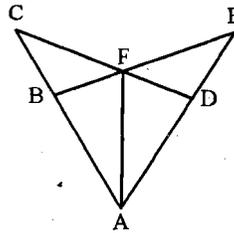
9. AC = AD

BC = BD

AP = AQ

Prove ∠3 = ∠4
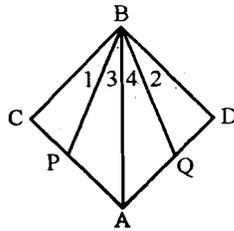
10. In the figure of question 9:

BA bisects ∠ CAD

∠1 = ∠2, ∠3 = ∠4

Prove BC = BD

15.11   ANSWERS TO CHECK YOUR PROGRESS

1. 1 and 12, 2 and 9, 3 and 8, 4 and 11, 5 and 10, 6 and 13, 7 and 14 have same shape and size.

2. The two triangles are similar. Their corresponding angles are equal.

3. The two triangles are similar.

Their corresponding angles are equal. The ratio of the third pair of sides is same as the ratio of the first two pairs of sides.

4. Pairs of similar figures: A and O, B and J, C and F, D and H, G and K, I and M, L and Q.

5. a)   Triangles are not congruent because angles are not included between the sides.

    b) △ PQT = △ SRT - (ASA)   e) △ BCM = △ CBN - (AAS)

    c) △ PQR = △ SUT - (AAS)   f) △ XYL = △ XZK - (AAS)

    d) △ ABF = △ CDE - (AAS)