UNIT 13 PROPOSITIONAL CALCULUS

13.1 INTRODUCTION

According to the theory of evolution, human beings have evolved from the lower species over many millenia. The chief asset that made humans "superior" to their ancestors was the ability to reason. How well this ability has been used for scientific and technological development is common knowledge. But no systematic study of logical reasoning seems to have been done for a long time. The first such study that has been found is by the Greek philosopher Aristotle (384-322 BC). In a modified form, this type of logic seems to have been taught through the Middle Ages.

Then came a major development in the study of logic, its formalisation in terms of mathematics. It was mainly Leibniz (1646-1716) and George Boole (1815-1864) who seriously studied and developed this theory, called **symbolic logic**. It is the basics of this theory that we aim to introduce you to in this unit and the next one.

In the introduction to the block you have read about what symbolic logic is. Using it we can formalise our arguments and logical reasoning in a manner that can easily show if the reasoning is valid, or is a fallacy. How we symbolise the reasoning is what is presented in this unit.

More precisely, in Section 13.2 (i.e., Sec.13.2, in brief) we talk about what kind of sentences are acceptable in mathematical logic. We call such sentences statements or propositions. You will also see that a statement can either be true or false. Accordingly, as you will see, we will give the statement a truth value T or F.

In Sec.13.3 we begin our study of the logical relationship between propositions. This is called propositional calculus. In this we look at some ways of connecting simple propositions to obtain more complex ones. To do so, we use logical connectives like "and" and "or". We also introduce you to other connectives like "not", "implies" and "implies and is implied by". At the same time we construct tables that allow us to find the truth values of the compound statements that we get.

In Sec.1.4 we consider the conditions under which two statements are "the same". In such a situation we can safely replace one by the other.

And finally, in Sec.13.5, we talk about some common terminology and notation which is useful for quantifying the objects we are dealing with in a statement.

It is important for you to study this unit carefully, because the other units in this block are based on it. Please be sure to do the exercises as you come to them. Only then will you be able to achieve the following objectives.

Objectives

After reading this unit, you should be able to develop in your learners the ability to

- distinguish between propositions and non-propositions;
- identify and use logical connectives;
- construct the truth table of any compound proposition;
- identify and use logically equivalent statements;
- identify and use logical quantifiers.

Let us now begin our discussion on mathematical logic.

13.2 PROPOSITIONS

Consider the sentence 'In 2003, the President of India was a woman.' When you read this declarative sentence, you can immediately decide whether it is true or false. And so can anyone else. Also, it wouldn't happen that some people say that the statement is true and some others say that it is false. Everybody would have the same answer. So, this sentence is either **universally true** or **universally false**.

Similarly, 'An elephant weighs more than a human being.' is a declarative sentence which is either true or false, but not both. In mathematical logic we call such sentences **statements** or **propositions**.

On the other hand, consider the declarative sentence 'Women are more intelligent than men.'. Some people may think it is true while others may disagree. So, it is neither universally true nor universally false. Such a sentence is not acceptable as a statement or proposition in mathematical logic.

Note that **a proposition should be either uniformly true or uniformly false.** For example, 'An egg has protein in it.', and 'The Prime Minister of India has to be a man.' are both propositions, the first one true and the second one false.

Would you say that the following are propositions?

'Watch the film.'
'How wonderful!'
'What did you say?'

Actually, none of them are declarative sentences. (The first one is an order, the second an exclamation and the third is a question.) And therefore, none of them are propositions.

Now for some mathematical propositions! You must have studied and created many of them while doing mathematics. Some examples are

Two plus two equals four.
Two plus two equals five.
$x + y > 0$ for $x > 0$ and $y > 0$.
A set with n elements has 2^n subsets.

Of these statements, three are true and one false (which one?).

Now consider the algebraic sentence '$x + y > 0$'. Is this a proposition? Are we in a position to determine whether it is true or false? Not unless we know the values that x and y can take. For example, it is false for $x = 1, y = -2$ and true if $x = 1, y = 0$. Therefore,
'$x + y > 0$' is **not** a proposition, while
'$x + y > 0$ for $x > 0, y > 0$' is a proposition.

Why don't you try this short exercise now?

E1) Which of the following sentences are statements? What are the reasons for your answer?
- i) The sun rises in the West.
- ii) How far is Delhi from here?
- iii) Smoking is injurious to health.
- iv) There is no rain without clouds.
- v) What a beautiful day!
- vi) She is an engineering graduate.
- vii) $2^n + n$ is an even number for infinitely many n.
- viii) $x + y = y + x$ for all $x, y \in \mathbf{R}$.
- ix) Mathematics is fun.
- x) $2^n = n^2$.

Usually, when dealing with propositions, we shall denote them by lower case letters like p, q, etc. So, for example, we may denote

'Ice is always cold.' by p, or
'$\cos^2\theta + \sin^2\theta = 1$ for $\theta \in [0, 2\pi]$' by q.
We shall sometimes show this by saying
p: Ice is always cold., or
q: $\cos^2\theta + \sin^2\theta = 1$ for $\theta \in [0, 2\pi]$.

Now, given a proposition, we know that it is either true or false, but not both. If it is **true**, we will allot it the **truth value T**. If it is **false**, its **truth value will be F**. So, for example, the truth value of

'Ice melts at 30°C.' is F, while that of '$x^2 \geq 0$ for $x \in \mathbf{R}$' is T.

Sometimes, as in the context of logic circuits(see Unit 15), we will use 1 instead of T and 0 instead of F.

Here are some exercises for you now.

E2) Give the truth values of the propositions in E1.

E3) Give two propositions each, the truth values of which are T and F, respectively. Also give two examples of sentences that are not propositions.

Let us now look at ways of connecting simple propositions to obtain compound statements.

13.3 LOGICAL CONNECTIVES

When you're talking to someone, do you use very simple sentences only? Don't you use more complicated ones which are joined by words like 'and', 'or', etc.? In the same way, most statements in mathematical logic are combinations of simpler statements joined by words and phrases like 'and', 'or', 'if ... then', 'if and only if', etc. These words and phrases are called **logical connectives**. There are 6 such connectives, which we shall discuss one by one.

13.3.1 Disjunction

Consider the sentence 'Alice or the mouse went to the market.'. This can be written as 'Alice went to the market or the mouse went to the market.' So, this statement is actually made up of two simple statements connected by 'or'. We have a term for such a compound statement.

Definition: The **disjunction** of two propositions p and q is the compound statement **p or q**, denoted by $\mathbf{p \vee q}$.
For example, 'Zarina has written a book or Singh has written a book.' is the disjunction of p and q, where
p : Zarina has written a book, and
q : Singh has written a book.

Similarly, if p denotes '$2 > 0$' and q denotes '$2 < 5$', then $p \vee q$ denotes the statement '2 is greater than 0 or 2 is less than 5.'.

Let us now look at how the truth value of $p \vee q$ depends upon the truth values of p and q. For doing so, let us look at the example of Zarina and Singh, given above. If even one of them has written a book, then the compound statement $p \vee q$ is true. Also, if both have written books, the compound statement $p \vee q$ is again true. Thus, if the truth value of even one out of p and q is T, then that of '$p \vee q$' is T. Otherwise, the truth value of $p \vee q$ is F. This holds for any pair of propositions p and q. To see the relation between the truth values of p, q and $p \vee q$ easily, we put this in the form of a table (Table 1), which we call a **truth table**.

Table 1 : Truth table for disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

How do we form this table? We consider the truth values that p can take — T or F. Now, when p is true, q can be true or false. Similarly, when p is false, q can be true or false. In this way there are 4 possibilities for the compound proposition $p \vee q$. Given any of these possibilities, we can find the truth value of $p \vee q$. For instance, consider the third possibility, i.e., p is false and q is true. Then, by definition, $p \vee q$ is true. In the same way, you can check that the other rows are consistent.

Let us consider an example.

Example 1: Obtain the truth value of the disjunction of 'The earth is flat.' and '$3 + 5 = 2$'.

Solution: Let p denote 'The earth is flat.' and q denote '$3 + 5 = 2$'. Then we know that the truth values of both p and q are F. Therefore, the truth value of $p \vee q$ is F.

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Try an exercise now.

E4) Write down the disjunction of the following propositions, and give its truth value.
   i) $2 + 3 = 7$,
   ii) Radha is an engineer.

We also use the term **'inclusive or'** for the connective we have just discussed. This is because $p \vee q$ is true even when both p and q are true. But, what happens when we want to ensure that only one of them should be true? Then we have the following connective.

Definition: The **exclusive disjunction** of two propositions p and q is the statement **'Either p is true or q is true, but both are not true.'**. We denote this by $\mathbf{p \oplus q}$.

So, for example, if p is '$2 + 3 = 5$' and q the statement given in E4(ii), then $p \oplus q$ is the statement 'Either $2 + 3 = 5$ or Radha is an engineer.'. This will be true only if Radha is not an engineer.

In general, how is the truth value of $p \oplus q$ related to the truth values of p and q? This is what the following exercise is about.

E5) Write down the truth table for $\oplus$. Remember that $p \oplus q$ is not true if both p and q are true.

Now let us look at the logical analogue of the coordinating conjunction 'and'.

13.3.2 Conjunction

As in ordinary language, we use 'and' to combine simple propositions to make compound ones. For instance, '$1 + 4 \neq 5$ and Prof. Rao teaches Chemistry.' is formed by joining '$1 + 4 \neq 5$' and 'Prof. Rao teaches Chemistry' by 'and'. Let us define the formal terminology for such a compound statement.

Definition: We call the compound statement **'p and q'** the **conjunction** of the statements p and q. We denote this by $\mathbf{p \wedge q}$.

For instance, '$3 + 1 \neq 7 \wedge 2 > 0$' is the conjunction of '$3 + 1 \neq 7$' and '$2 > 0$'. Similarly, '$2 + 1 = 3 \wedge 3 = 5$' is the conjunction of '$2 + 1 = 3$' and '$3 = 5$'.

Now, when would $p \wedge q$ be true? Do you agree that this could happen only when **both** p and q are true, and not otherwise? For instance, '$2 + 1 = 3 \wedge 3 = 5$' is not true because '$3 = 5$' is false.

So, the truth table for conjunction would be as in Table 2 alongside..

Table 2 : Truth table for conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

To see how we can use the truth table above, consider an example.

Example 2: Obtain the truth value of the conjunction of '$2 \div 5 = 1$' and 'Padma is in Bangalore.'.

Solution: Let $p : 2 \div 5 = 1$, and
q: Padma is in Bangalore.

Then the truth value of p is F. Therefore, from Table 2 you will find that the truth value of $p \wedge q$ is F.

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Why don't you try an exercise now?

E6) Give the set of those real numbers x for which the truth value of $p \wedge q$ is T, where
$p : x > -2$, and $q : x + 3 \neq 7$

If you look at Tables 1 and 2, do you see a relationship between the truth values in their last columns? You would be able to formalise this relationship after studying the next connective.

13.3.3 Negation

You must have come across young children who, when asked to do something, go ahead and do exactly the opposite. Or, when asked if they would like to eat, say rice and curry, will say 'No', the 'negation' of yes! Now, if p denotes the statement 'I will eat rice.', how can we denote 'I will not eat rice.'? Let us define the connective that will help us do so.

Definition: The **negation** of a proposition p is **'not p'**, denoted by **$\sim$ p**.

For example, if p is 'Dolly is at the study centre.', then $\sim$ p is 'Dolly is **not** at the study centre.'. Similarly, if p is 'No person can live without oxygen.', $\sim$ p is 'At least one person can live without oxygen.'.

Now, regarding the truth value of $\sim$ p, you would agree that it would be T if that of p is F, and vice versa. Keeping this in mind you can try the following exercises.

E7) Write down $\sim$ p, where p is
i) $0 - 5 \neq 5$
ii) $n > 2$ for every $n \in \mathbf{N}$.
iii) Most Indian children study till Class 5.

E8) Write down the truth table of negation.

Let us now discuss the conditional connectives, representing 'If $\cdots$, then $\cdots$' and 'if and only if'.

13.3.4 Conditional Connectives

Consider the proposition 'If Ayesha gets 75% or more in the examination, then she will get an A grade for the course.'. We can write this statement as 'If p, then q', where
p: Ayesha gets 75% or more in the examination, and
q: Ayesha will get an A grade for the course.
This compound statement is an example of the implication of q by p.

Definition: Given any two propositions p and q, we denote the statement **'If p, then q'** by $p \to q$. We also read this as 'p **implies** q', or 'p is **sufficient** for q', or 'p **only if** q'. We also call p the **hypothesis** and q the **conclusion**. Further, a statement of the form $p \to q$ is called a **conditional statement** or a **conditional proposition**.

So, for example, in the conditional proposition 'If m is in **Z**, then m belongs to **Q**.' the hypothesis is '$m \in \mathbf{Z}$' and the conclusion is '$m \in \mathbf{Q}$'.

Mathematically, we can write this statement as
$m \in \mathbf{Z} \rightarrow m \in \mathbf{Q}$.

Let us analyse the statement $p \rightarrow q$ for its truth value. Do you agree with the truth table we've given below (Table 3)? You may like to check it out while keeping an example from your surroundings in mind.

Table 3 : Truth table for implication

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

You may wonder about the third row in Table 3. But, consider the example '$3 < 0 \rightarrow 5 > 0$'. Here the conclusion is true regardless of what the hypothesis is. And therefore, the conditional statement remains true. In such a situation we say that the **conclusion is vacuously true.**

Why don't you try this exercise now?

E9) Write down the proposition corresponding to $p \rightarrow q$, and determine the values of x for which it is **false**, where

$p : x + y = xy$ where $x, y \in \mathbf{R}$

$q : x \not< 0$ for every $x \in \mathbf{Z}$.

Now, consider the implication 'If Jahanara goes to Baroda, then she doesn't participate in the conference at Delhi.'. What would its converse be? To find it, the following definition may be useful.

Definition: The **converse** of $p \rightarrow q$ is $q \rightarrow p$. In this case we also say 'p is **necessary** for q', or 'p **if** q'.

So, in the example above, the converse of the statement would be 'If Jahanara doesn't participate in the conference at Delhi, then she goes to Baroda.'. This means that Jahanara's non-participation in the conference at Delhi is necessary for her going to Baroda.

Now, what happens when we combine an implication and its converse? To show '$p \rightarrow q$ and $q \rightarrow p$', we introduce a shorter notation.

Definition: Let p and q be two propositions. The compound statement

$$(p \rightarrow q) \wedge (q \rightarrow p)$$

is the **biconditional** of p and q. We denote it by $p \longleftrightarrow q$, and read it as 'p **if and only if** q'. We usually shorten 'if and only if' to **iff**.
We also say that 'p **implies and is implied by** q', or 'p is **necessary and sufficient for** q'.

The two connectives $\rightarrow$ and $\longleftrightarrow$ are called the **conditional connectives.**

For example, 'Sudha will gain weight if and only if she eats regularly.' means that 'Sudha will gain weight if she eats regularly **and** Sudha will eat regularly if she gains weight.'.

One point that may come to your mind here is whether there's any difference in the two statements $p \longleftrightarrow q$ and $q \longleftrightarrow p$. When you study Sec.13.4 you will realise why they are inter-changeable.

Let us now consider the truth table of the biconditional, i.e., of the **two-way**

implication. To obtain its truth values, we need to use Tables 2 and 3, as you will see in Table 4. This is because, to find the value of $(p \to q) \wedge (q \to p)$ we need to know the values of each of the simpler statements involved.

Table 4 : Truth table for two-way implication

p	q	$p \to q$	$q \to p$	$p \longleftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

As you can see from the last column of the table (and from your own experience), $p \longleftrightarrow q$ is true only when both p and q are true or both p and q are false. In other words, $p \longleftrightarrow q$ is true only when p and q have the same truth values. Thus, for example, 'Parimala is in America iff $2 + 3 = 5$' is true only if 'Parimala is in America.' is true.

Here are some related exercises.

E10) For each of the following compound statements, first identify the simple propositions p, q, r, etc., that are combined to make it. Then write it in symbols, using the connectives, and give its truth value.

i) If triangle ABC is equilateral, then it is isosceles.
ii) a and b are integers if and only if ab is a rational number.
iii) If Raza has five glasses of water and Sudha has four cups of tea, then Shyam will not pass the math examination.
iv) Mariam is in Class 1 or in Class 2.

E11) Write down two propositions p and q for which $q \to p$ is true but $p \longleftrightarrow q$ is false.

Now, how would you determine the truth value of a proposition which has more than one connective in it? For instance, does $\sim p \vee q$ mean $(\sim p) \vee q$ or $\sim (p \vee q)$? We discuss some rules for this below.

13.3.5 Precedence Rule

While dealing with operations on numbers, you would have realised the need for applying the BODMAS rule. According to this rule, when calculating the value of an arithmetic expression, we first calculate the value of the Bracketed portion, then apply Of, Division, Multiplication, Addition and Subtraction, in this order. While calculating the truth value of compound propositions involving more than one connective, we have a similar **convention** which tells us which connective to apply first.

Why do we need such a convention? Suppose we didn't have an order of preference, and want to find the truth of, say, $\sim p \vee q$. Some of us may consider the value of $(\sim p) \vee q$, and some may consider $\sim (p \vee q)$. The truth values can be different in these cases. For instance, if p and q are both true, then $(\sim p) \vee q$ is true, but $\sim (p \vee q)$ is false. So, for the purpose of unambiguity, we agree to such an order or rule. Let us see what it is.

The rule of precedence: The **order of preference** in which the connectives are applied in a formula of propositions that has no brackets is

i) $\sim$
ii) $\wedge$
iii) $\vee$ and $\oplus$
iv) $\rightarrow$ and $\longleftrightarrow$

Note that the 'inclusive or' and 'exclusive or' are both third in the order of preference. However, if both these appear in a statement, we first apply the left most one. So, for instance, in $p \vee q \oplus \sim p$, we first apply $\vee$ and then $\oplus$. The same applies to the 'implication' and the 'biconditional', which are both fourth in the order of preference.

To clearly understand how this rule works, let us consider an example.

Example 3: : Write down the truth table of $p \rightarrow q \wedge \sim r \longleftrightarrow r \oplus q$

Solution: We want to find the required truth value when we are given the truth values of p, q and r. According to the rule of precedence given above, we need to first find the truth value of $\sim r$, then that of $(q \wedge \sim r)$, then that of $(r \oplus q)$, and then that of $p \rightarrow (q \wedge \sim r)$, and finally the truth value of $[p \rightarrow (q \wedge \sim r)] \longleftrightarrow r \oplus q$.

So, for instance, suppose p and q are true, and r is false. Then $\sim r$ will have value T, $q \wedge \sim r$ will be T, $r \oplus q$ will be T, $p \rightarrow (q \wedge \sim r)$ will be T, and hence, $p \rightarrow q \wedge \sim r \longleftrightarrow r \oplus q$ will be T.

You can check that the rest of the values are as given in Table 5. Note that we have 8 possibilities $(= 2^3)$ because there are 3 simple propositions involved here.

Table 5 : Truth table for $p \rightarrow q \wedge \sim r \longleftrightarrow r \oplus q$

p	q	r	$\sim r$	$q \wedge \sim r$	$r \oplus q$	$p \rightarrow q \wedge \sim r$	$p \rightarrow q \wedge \sim r \longleftrightarrow r \oplus q$
T	T	T	F	F	F	F	T
T	T	F	T	T	T	T	T
T	F	T	F	F	T	F	F
T	F	F	T	F	F	F	T
F	T	T	F	F	F	T	F
F	T	F	T	T	T	T	T
F	F	T	F	F	T	T	T
F	F	F	T	F	F	T	F

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You may now like to try some exercises on the same lines.

E12) In Example 3, how will the truth values of the compound statement change if you first apply $\longleftrightarrow$ and then $\rightarrow$?

E13) In Example 3, if we replace $\oplus$ by $\wedge$, what is the new truth table?

E14) Form the truth tables of $p \wedge q \vee \sim r$ and $(p \wedge q) \vee (\sim r)$ and see where they differ.

E15) How would you bracket the following formulae to correctly interpret them? [For instance, $p \vee \sim q \wedge r$ would be bracketed as $p \vee ((\sim q) \wedge r)$.]

i) $\sim p \vee q$,
ii) $\sim q \rightarrow \sim p$,
iii) $p \rightarrow q \longleftrightarrow \sim p \vee q$,
iv) $p \oplus q \wedge r \rightarrow \sim p \vee q \longleftrightarrow p \wedge r$.

So far we have considered different ways of making new statements from old ones. But, are all these new ones distinct? Or are some of them the same? And "same" in what way? This is what we shall now consider.

13.4 LOGICAL EQUIVALENCE

'Then you should say what you mean', the March Hare went on.
'I do,' Alice hastily replied, 'at least ... at least I mean what I say — that's the same thing you know.'
'Not the same thing a bit!' said the Hatter. 'Why, you might just as well say that "I see what I eat" is the same thing as "I eat what I see"!'

–from 'Alice in Wonderland'
by Lewis Carroll

In mathematics, as in ordinary language, there can be several ways of saying the same thing. In this section we shall discuss what this means in the context of logical statements.

$\sim q \rightarrow \sim p$ is the **contrapositive** of the proposition $p \rightarrow q$.

Consider the statements 'If Lala is rich, then he must own a car.', and 'If Lala doesn't own a car, then he is not rich.'. Do these statements mean the same thing? If we write the first one as $p \rightarrow q$, then the second one will be $(\sim q) \rightarrow (\sim p)$. How do the truth values of both these statements compare? We find out in the following table.

Table 6

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Consider the last two columns of Table 6. You will find that '$p \rightarrow q$' and '$\sim q \rightarrow \sim p$' have the same truth value for every choice of truth values of p and q. When this happens, we call them equivalent statements.

Definition: We call two propositions r and s **logically equivalent** provided they have the same truth value for every choice of truth values of the simple propositions involved in them. We denote this fact by $\mathbf{r \equiv s}$.

So, from Table 6 we find that $(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$.

You can also check that $(\mathbf{p} \longleftrightarrow \mathbf{q}) \equiv (\mathbf{q} \longleftrightarrow \mathbf{p})$ for any pair of propositions p and q.

As another example, consider the following equivalence that is often used in mathematics. You could also apply it to obtain statements equivalent to 'Neither a borrower, nor a lender be.'!

Example 4: For any two propositions p and q, show that
$\sim (p \vee q) \equiv \sim p \wedge \sim q$.

Solution: Consider the following truth table.

Table 7

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim (p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

You can see that the last two columns of Table 7 are identical. Thus, the truth values of $\sim (p \vee q)$ and $\sim p \wedge \sim q$ agree for every choice of truth values of p and q.

Therefore, $\sim (\mathbf{p} \vee \mathbf{q}) \equiv \sim \mathbf{p} \wedge \sim \mathbf{q}$.

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Fig. 1: Augustus De Morgan (1806-1871) was born in Madurai.

The equivalence you have just seen is one of **De Morgan's laws**. You have already come across these laws in the context of set operations in MTE-04.

The other law due to De Morgan is similar : $\sim (\mathbf{p} \wedge \mathbf{q}) \equiv \sim \mathbf{p} \vee \sim \mathbf{q}$.

In fact, there are several such laws about equivalent propositions. **Some** of them are the following, where, as usual, p, q and r denote propositions.

a) **Double negation law** : $\sim (\sim p) \equiv p$

b) **Idempotent laws**: $p \wedge p \equiv p$,
$p \vee p \equiv p$

c) **Commutativity**: $p \vee q \equiv q \vee p$
$p \wedge q \equiv q \wedge p$

d) **Associativity**: $(p \vee q) \vee r \equiv p \vee (q \vee r)$
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

e) **Distributivity**: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

We ask you to prove these laws now.

E16) Show that the laws given in (a)-(e) above hold true.

E17) Prove that the relation of 'logical equivalence' is an equivalence relation.

E18) Check whether $(\sim p \vee q)$ and $(p \to q)$ are logically equivalent.

The laws given above and the equivalence you have checked in E18 are commonly used, and therefore, **useful to remember**. You will also be applying them in Unit 15 in the context of switching circuits.

Let us now consider some propositional formulae which are always true or always false. Take, for instance, the statement 'If Bano is sleeping and Pappu likes ice-cream, then Bano is sleeping.'. You can draw up the truth table of this compound proposition and see that it is always true. This leads us to the following definition.

Definition: A compound proposition that is true for all possible truth values of the simple propositions involved in it is called a **tautology**. Similarly, a proposition that is false for all possible truth values of the simple propositions that constitute it is called a **contradiction**.

Let us look at some examples of such propositions.

Example 5: Verify that $p \wedge q \wedge \sim p$ is a contradiction and $p \rightarrow q \longleftrightarrow \sim p \vee q$ is a tautology.

Solution: Let us simultaneously draw up the truth tables of these two propositions below.

Table 8

p	q	$\sim p$	$p \wedge q$	$p \wedge q \wedge \sim p$	$p \rightarrow q$	$\sim p \vee q$	$p \rightarrow q \longleftrightarrow \sim p \vee q$
T	T	F	T	F	T	T	T
T	F	F	F	F	F	F	T
F	T	T	F	F	T	T	T
F	F	T	F	F	T	T	T

Looking at the fifth column of the table, you can see that $p \wedge q \wedge \sim p$ is a contradiction. This should not be surprising since $p \wedge q \wedge \sim p \equiv (p \wedge \sim p) \wedge q$ (check this by using the various laws given above).

And what does the last column of the table show? Precisely that $p \rightarrow q \longleftrightarrow \sim p \vee q$ is a tautology.

* * *

Why don't you try an exercise now?

E19) Let $\mathcal{T}$ denote a tautology (i.e., a statement whose truth value is always T) and $\mathcal{F}$ a contradiction. Then, for any statement p, show that

i) $p \vee \mathcal{T} \equiv \mathcal{T}$
ii) $p \wedge \mathcal{T} \equiv p$
iii) $p \vee \mathcal{F} \equiv p$
iv) $p \wedge \mathcal{F} \equiv \mathcal{F}$

Another way of proving that a proposition is a tautology is to use the properties of logical equivalence. Let us look at the following example.

Complementation law: $q \wedge \sim q$ is a contradiction.

Example 6: Show that $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$ is a tautology.

Solution: $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$

$\equiv [(\sim p \vee q) \wedge \sim q] \rightarrow \sim p$, using E18, and symmetricity of $\equiv$.

$\equiv [(\sim p \wedge \sim q) \vee (q \wedge \sim q)] \rightarrow \sim p$, by De Morgan's laws.

$\equiv [(\sim p \wedge \sim q) \vee \mathcal{F}] \rightarrow \sim p$, since $q \wedge \sim q$ is always false.

$\equiv (\sim p \wedge \sim q) \rightarrow \sim p$, using E18.

which is a tautology.
And therefore the proposition we started with is a tautology.

* * *

The laws of logical equivalence can also be used to prove some other logical equivalences, without using truth tables. Let us consider an example.

Example 7: Show that $(p \rightarrow \sim q) \wedge (p \rightarrow \sim r) \equiv \sim [p \wedge (q \vee r)]$.

Solution: We shall start with the statement on the left hand side of the equivalence that we have to prove. Then, we shall apply the laws we have listed above, or the equivalence in E18, to obtain logically equivalent statements. We shall continue this process till we obtain the statement on the right hand side of the equivalence given above. Now

$(p \rightarrow \sim q) \wedge (p \rightarrow \sim r)$

$\equiv (\sim p \vee q) \wedge (\sim p \vee \sim r)$, by E18

$\equiv \ \sim p \ \vee (\sim q \wedge \sim r)$, by distributivity

$\equiv \ \sim p \ \vee [\sim (q \vee r)]$, by De Morgan's laws

$\equiv \ \sim [p \wedge (q \vee r)]$, by De Morgan's laws

So we have proved the equivalence that we wanted to.

***

You may now like to try the following exercises on the same lines.

E20) Use the laws given in this section to show that
$\sim (\sim p \wedge q) \wedge (p \vee q) \equiv p$.

E21) Write down the statement 'If it is raining and if rain implies that no one can go to see a film, then no one can go to see a film.' as a compound proposition. Show that this proposition is a tautology, by using the properties of logical equivalence.

E22) Give an example, with justification, of a compound proposition that is neither a tautology nor a contradiction.

Let us now consider proposition-valued functions.

13.5 LOGICAL QUANTIFIERS

In Sec.13.2, you read that a sentence like 'She has gone to Patna.' is not a proposition, unless who 'she' is is clearly specified.

Similarly, '$x > 5$' is not a proposition unless we know the values of x that we are considering. Such sentences are examples of 'propositional functions'.

Definition: A **propositional function**, or a **predicate**, in a variable x is a sentence p(x) involving x that becomes a proposition when we give x a definite value from the set of values it can take. We usually denote such functions by p(x), q(x), etc. The set of values x can take is called the **universe of discourse**.

So, if p(x) is '$x > 5$', then p(x) is not a proposition. But when we give x particular values, say $x = 6$ or $x = 0$, then we get propositions. Here, p(6) is a true proposition and p(0) is a false proposition.

Similarly, if q(x) is 'x has gone to Patna.', then replacing x by 'Taj Mahal' gives us a false proposition.

Note that a predicate is usually not a proposition. But, of course, every proposition is a propositional function in the same way that every real number is a real-valued function, namely, the constant function.

Now, can all sentences be written in symbolic form by using only the logical connectives? What about sentences like 'x is prime and $x + 1$ is prime for some x.'? How would you symbolise the phrase 'for some x', which we can rephrase as 'there exists an x'? You must have come across this term often while studying mathematics. **We use the symbol '$\exists$' to denote this quantifier, 'there exists'**. The way we use it is, for instance, to rewrite 'There is at least one child in the class.' as
'$(\exists\, x \text{ in } U)p(x)$',
where p(x) is the sentence 'x is in the class.' and U is the set of all children.

$\exists$ is called the **existential quantifier**.

Now suppose we take the negative of the proposition we have just stated. Wouldn't it be 'There is no child in the class.'? We could symbolise this as

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'for all x in U, $q(x)$' where x ranges over all children and $q(x)$ denotes the sentence 'x is not in the class.', i.e., $q(x) \equiv \sim p(x)$.
We have **a mathematical symbol for the quantifier 'for all', which is '$\forall$'**. So the proposition above can be written as
'$(\forall x \in U)q(x)$', or '$q(x), \forall x \in U$'.

$\forall$ is called the **universal quantifier**.

An example of the use of the existential quantifier is the true statement

$(\exists x \in \mathbf{R})(x + 1 > 0)$, which is read as 'There exists an x in **R** for which $x + 1 > 0$.'.

Another example is the false statement
$(\exists x \in \mathbf{N})(x - \frac{1}{2} = 0)$, which is read as 'There exists an x in **N** for which $x - \frac{1}{2} = 0$.'.

An example of the use of the universal quantifier is
$(\forall x \notin \mathbf{N})(x^2 > x)$, which is read as 'for every x not in $\mathbf{N}, x^2 > x$.'.
Of course, this is a false statement, because there is at least one $x \notin \mathbf{N}, x \in \mathbf{R}$, for which it is false.

We often use both quantifiers together, as in the statement called **Bertrand's postulate**:
$(\forall n \in \mathbf{N}\setminus\{1\})(\exists x \in \mathbf{N})$ (x is a prime number and $n < x < 2n$).
In words, this is 'for every integer $n > 1$ there is a prime number lying strictly between n and 2n.'

As you have already read in the example of a child in the class,
$(\forall x \in U)p(x)$ is logically equivalent to $\sim (\exists x \in U)(\sim p(x))$. Therefore,
$\sim (\forall x \in U)p(x) \equiv \sim\sim (\exists x \in U)(\sim p(x)) \equiv (\exists x \in U)(\sim p(x))$.

This is one of the rules for negation that relate $\forall$ and $\exists$. The two rules are
$\sim (\forall \mathbf{x} \in \mathbf{U})\mathbf{p}(\mathbf{x}) \equiv (\exists \mathbf{x} \in \mathbf{U})(\sim \mathbf{p}(\mathbf{x}))$, and

$\sim (\exists \mathbf{x} \in \mathbf{U})\mathbf{p}(\mathbf{x}) \equiv (\forall \mathbf{x} \in \mathbf{U})(\sim \mathbf{p}(\mathbf{x}))$
where U is the set of values that x can take.

Now, consider the proposition
'There is a criminal who has committed every crime.'
We could write this in symbols as
$(\exists c \in A)(\forall x \in B)$(c has committed x)
where, of course, A is the set of criminals and B is the set of crimes (determined by law).
What would its negation be? It would be
$\sim (\exists c \in A)(\forall x \in B)$(c has committed x)
$\equiv (\forall c \in A)[\sim (\forall x \in B)$(c has committed x)
$\equiv (\forall c \in A)(\exists x \in B)$(c has not committed x).
We can interpret this as 'For every criminal, there is a crime that this person has not committed.'.

A predicate can be a function in two or more variables.

These are only some examples in which the quantifiers occur singly, or together. Sometimes you may come across situations (as in E23) where you would use $\exists$ or $\forall$ twice or more in a statement. It is in situations like this or worse [say, $(\forall x_1 \in U_1)(\exists x_2 \in U_2)(\exists x_3 \in U_3)(\forall x_4 \in U_4) \ldots (\exists x_n \in U_n)p]$ where our rule for negation comes in useful. In fact, applying it, in a trice we can say that the negation of this seemingly complicated example is
$(\exists x_1 \in U_1)(\forall x_2 \in U_2)(\forall x_3 \in U_3)(\exists x_4 \in U_4) \ldots (\forall x_n \in U_n)(\sim p)$.

Why don't you try some exercises now?

E23) How would you present the following propositions and their negations using logical quantifiers? Also interpret the negations in words.

i) The politician can fool all the people all the time.
ii) Every real number is the square of some real number.
iii) There is a lawyer who never tells lies.

E24) Write down suitable mathematical statements that can be represented by the following symbolic propositions. Also write down their negations. What is the truth value of your propositions?
i) $(\forall x)(\exists y)p$
ii) $(\exists x)(\exists y)(\forall z)p$.

And finally, let us look at a very useful quantifier, which is very closely linked to $\exists$. You would need it for writing, for example, 'There is one and only one key that fits the desk's lock.' in symbols. The symbol is $\exists!$ x which stands for '**there is one and only one x**' (which is the same as '**there is a unique** x' or '**there is exactly one** x').

So, the statement above would be $(\exists!\, x \in A)$(x fits the desk's lock), where A is the set of keys.

For other examples,, try and recall the statements of uniqueness in the mathematics that you've studied so far. What about 'There is a unique circle that passes through three non-collinear points in a plane.'? How would you represent this in symbols? If x denotes a circle, and y denotes a set of 3 non-collinear points in a plane, then the proposition is
$(\forall\, y \in P)(\exists!\, x \in C)$(x passes through y).

Here C denotes the set of circles, and P the set of sets of 3 non-collinear points.

And now, some short exercises for you!

E25) Which of the following propositions are true (where x, y are in **R**)?
i) $(x \geq 0) \rightarrow (\exists\, y)(y^2 = x)$
ii) $(\forall\, x)(\exists!\, y)(y^2 = x^3)$
iii) $(\exists\, x)(\exists!\, y)(xy = 0)$
iv) $\sim (\exists\, x)(\exists!\, y)(x + y = 0)$.

Before ending the unit, let us take a quick look at what we have covered in it.

13.6 SUMMARY

In this unit we have considered the following points.
1. What a mathematically acceptable statement (or proposition) is.
2. The definition and use of logical connectives:
   Given propositions p and q,
   i) their disjunction is 'p or q', denoted by $p \vee q$;
   ii) their exclusive disjunction is 'either p or q', denoted by $p \oplus q$;
   iii) their conjunction is 'p and q', denoted by $p \wedge q$;
   iv) the negation of p is 'not p', denoted by $\sim p$;
   v) 'if p, then q' is denoted by $p \rightarrow q$;
   vi) 'p if and only if q' is denoted by $p \longleftrightarrow q$;
3. The truth tables corresponding to the 6 logical connectives.

4. Rule of precedence : In any compound statement involving more than one connective, we first apply '$\sim$', then '$\wedge$', then '$\vee$' and '$\oplus$', and last of all '$\longrightarrow$' and '$\longleftrightarrow$'.
5. The meaning and use of logical equivalence, denoted by '$\equiv$'.
6. The following laws about equivalent propositions:
   i) **De Morgan's laws :** $\sim (p \wedge q) \equiv \sim p \vee \sim q$
   $\sim (p \vee q) \equiv \sim p \wedge \sim q$
   ii) **Double negation law :** $\sim (\sim p) \equiv p$
   iii) **Idempotent laws:** $p \wedge p \equiv p$,
   $p \vee p \equiv p$
   iv) **Commutativity:** $p \vee q \equiv q \vee p$
   $p \wedge q \equiv q \wedge p$
   v) **Associativity:** $(p \vee q) \vee r \equiv p \vee (q \vee r)$
   $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
   vi) **Distributivity:** $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
   $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
   vii) $(\sim p \vee q) \equiv p \longrightarrow q$ (ref. E18).
7. Logical quantifiers : 'For every' denoted by '$\forall$', 'there exists' denoted by '$\exists$', and 'there is one and only one' denoted by '$\exists!$'.
8. The rule of negation related to the quantifiers:
   $\sim (\forall x \in U)p(x) \equiv (\exists x \in U)(\sim p(x))$
   $\sim (\exists x \in U)p(x) \equiv (\forall x \in U)(\sim p(x))$

Now we have come to the end of this unit. You should have tried all the exercises as you came to them. You may like to check your solutions with the ones we have given below.

13.7 COMMENTS ON EXERCISES

E1) (i), (iii), (iv), (vii), (viii) are statements because each of them is universally true or universally false.
(ii) is a question.
(v) is an exclamation.
The truth or falsity of (vi) depends upon who 'she' is.
(ix) is a subjective sentence.
(x) will only be a statement if the value(s) n takes is/are given.
Therefore, (ii), (v), (vi), (ix) and (x) are not statements.

E2) The truth value of (i) is F, and of all the others is T.

E4) The disjunction is
'2+3 = 7 or Radha is an engineer.'.
Since '2+3 = 7' is always false, the truth value of this disjunction depends on the truth value of 'Radha is an engineer.'. If this is T, then we use the third row of Table 1 to get the required truth value as T. If Radha is not an engineer, then we get the required truth value as F.

E5)

Table 9: Truth table for 'exclusive or'

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

E6) p will be a true proposition for $x \in]-2, \infty[$.
q will be a true proposition for $x \neq 4$.
Therefore $p \wedge q$ will be true for every x such that $x \in]-2, \infty[$ and $x \neq 4$, i.e., for $x \in]-2, 4[\cup]4, \infty[$.

E7) i) $0 - 5 = 5$
ii) 'n is not greater than 2 for every $n \in \mathbf{N}$.', or 'There is at least one $n \in \mathbf{N}$ for which $n \leq 2$.'
iii) There are some Indian children who do not study till Class 5.

E8) **Table 10: Truth table for negation**

p	$\sim$ p
T	F
F	T

E9) $p \to q$ is the statement 'If $x + y = xy$ for $x, y \in \mathbf{R}$, then $x \not< 0$ for every $x \in \mathbf{Z}$'.
In this case, q is false. Therefore, the conditional statement will be true if p is false also, and it will be false for those values of x and y that make p true.

So, $p \to q$ is false for all those real numbers x of the form $\frac{y}{y-1}$, where $y \in \mathbf{R} \setminus \{1\}$. This is because if $x = \frac{y}{y-1}$ for some $y \in \mathbf{R} \setminus \{1\}$, then $x + y = xy$, i.e., p will be true.

E10) i) $p \to q$, where $p : \triangle ABC$ is equilateral, and $q : \triangle ABC$ is isosceles. If q is true, then $p \to q$ is true. If q is false, then $p \to q$ is true only when p is false. So, if $\triangle ABC$ is an isosceles triangle, the given statement is always true. Also, if $\triangle ABC$ is not isosceles, then it can't be equilateral either. So the given statement is again true.

ii) p : a is an integer.
q : b is an integer.
r : ab is a rational number.
The given statement is $(p \wedge q) \longleftrightarrow r$.
Now, if p is true and q is true, then r will be true.
If $p \wedge q$ is false, it can happen that r is still true.
So, $(p \wedge q) \longleftrightarrow r$ will be true if $p \wedge q$ is true, or when $p \wedge q$ is false and r is false.
In all the other cases $(p \wedge q) \longleftrightarrow r$ will be false.

iii) p : Raza has 5 glasses of water.
q : Sudha has 4 cups of tea.
r : Shyam will pass the math exam.
The given statement is $(p \wedge q) \to \sim r$.
This is true when $\sim r$ is true, or when r is true and $p \wedge q$ is false.
In all the other cases it is false.

iv) p : Mariam is in Class 1.
q : Mariam is in Class 2.
The given statement is $p \oplus q$.
This is true only when p is true or when q is true.

E11) There are infinitely many such examples. You need to give one in which p is true but q is false.

E12) Obtain the truth table. The last column will now have entries TTFTTTTT.

E13) According to the rule of precedence, given the truth values of p, q, r you should first find those of $\sim r$, then of $q \wedge \sim r$, and $r \wedge q$, and $p \rightarrow q \wedge \sim r$, and finally of $(p \rightarrow q \wedge \sim r) \longleftrightarrow r \wedge q$.

Referring to Table 5, the values in the sixth and eighth columns will be replaced by

$r \wedge q$
T
F
F
F
T
F
F
F

and

$p \rightarrow q \wedge \sim r \longleftrightarrow r \wedge q$
F
F
T
T
T
F
F
F

E14) They should both be the same, viz.,

p	q	r	$\sim r$	$p \wedge q$	$(p \wedge q) \vee (\sim r)$
T	T	T	F	T	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	T	F	T
F	T	T	F	F	F
F	T	F	T	F	T
F	F	T	F	F	F
F	F	F	T	F	T

E15) i) $(\sim p) \vee q$

ii) $(\sim q) \rightarrow (\sim p)$

iii) $(p \rightarrow q) \longleftrightarrow [(\sim p) \vee q]$

iv) $[[p \oplus (q \wedge r)] \rightarrow [(\sim p) \vee q]] \longleftrightarrow (p \wedge r)$

E16) a)

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

The first and third columns prove the double negation law.

c)

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

The third and fourth columns prove the commutativity of $\vee$.

The other laws can be similarly proved.

E17) For any three propositions p, q, r:

i) $p \equiv p$ is trivially true.

ii) if $p \equiv q$, then $q \equiv p$ ($\because$ if p has the same truth value as q for all choices of truth values of p and q, then clearly q has the same truth values as p in all the cases.)

iii) if $p \equiv q$ and $q \equiv r$, then $p \equiv r$ (reason as in (ii) above).

Thus, $\equiv$ is reflexive, symmetric and transitive.

$\because$ denotes 'because'.

E18)

p	q	$\sim p$	$\sim p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The last two columns show that $[(\sim p) \vee q] \equiv (p \rightarrow q)$.

E19) i)

p	$\mathcal{T}$	$p \vee \mathcal{T}$
T	T	T
F	T	T

The second and third columns of this table show that $p \vee \mathcal{T} \equiv \mathcal{T}$.

iv)

p	$\mathcal{F}$	$p \wedge \mathcal{F}$
T	F	F
F	F	F

The second and third columns of the adjoining table show that $p \wedge \mathcal{F} \equiv$ F.

You can similarly check (ii) and (iii).

E20) $\sim (\sim p \wedge q) \wedge (p \vee q)$
$\equiv (\sim (\sim p) \vee \sim q) \wedge (p \vee q)$, by De Morgan's laws .
$\equiv (p \vee \sim q) \wedge (p \vee q)$, by the double negation law.
$\equiv p \vee (\sim q \wedge q)$, by distributivity
$\equiv p \vee \mathcal{F}$, where $\mathcal{F}$ denotes a contradiction
$\equiv p$, using E 19 .

E21) p: It is raining.
q: Nobody can go to see a film.
Then the given proposition is
$[p \wedge (p \rightarrow q)] \rightarrow q$
$\equiv p \wedge (\sim p \vee q) \rightarrow q$, since $(p \rightarrow q) \equiv (\sim p \vee q)$
$\equiv (p \wedge \sim p) \vee (p \wedge q) \rightarrow q$, by De Morgan's law
$\equiv \mathcal{F} \vee (p \wedge q) \rightarrow q$, since $p \wedge \sim p$ is a contradiction
$\equiv (\mathcal{F} \vee p) \wedge (\mathcal{F} \vee q) \rightarrow q$, by De Morgan's law
$\equiv p \wedge q \rightarrow q$, since $\mathcal{F} \vee p \equiv p$.
which is a tautology.

E22) There are infinitely many examples. One such is:
'If Venkat is on leave, then Shabnam will work on the computer.'.
This is of the form $p \rightarrow q$. Its truth values will be T or F, depending on those of p and q.

E23) i) $(\forall\, t \in [0,\infty[)(\forall\, x \in H)p(x,t)$ is the given statement where $p(x,t)$ is the predicate 'The politician can fool x at time t seconds.', and H is the set of human beings.
Its negation is $(\exists\, t \in [0,\infty[)(\exists\, x \in H)(\sim p(x,t))$, i.e., there is somebody who is not fooled by the politician at least for one moment.

ii) The given statement is
$(\forall\, x \in \mathbf{R})(\exists\, y \in \mathbf{R})(x = y^2)$.
Its negation is
$(\exists\, x \in \mathbf{R})(\forall\, y \in \mathbf{R})(x \neq y^2)$, i.e.,
there is a real number which is not the square of any real number.

iii) The given statement is
$(\exists\, x \in L)(\forall\, t \in [0,\infty[)p(x,t)$, where L is the set of lawyers and

p(x, t) : x does not lie at time t.
The negation is
$(\forall\, x \in L)(\exists\, t \in [0, \infty[)(\sim p)$
i.e., every lawyer tells a lie at some time.

E24) i) For example,
$(\forall\, x \in \mathbf{N})(\exists\, y \in \mathbf{Z})\left(\frac{x}{y} \in \mathbf{Q}\right)$ is a true statement.
Its negation is
$(\exists\, x \in \mathbf{N})(\forall\, y \in \mathbf{Z})\left(\frac{x}{y} \notin \mathbf{Q}\right)$
You can try (ii) similarly.

E25) (i), (iii) are true.
(ii) is false (e.g., for $x = -1$ there is no y such that $y^2 = x^3$).
(iv) is equivalent to $(\forall\, x \in \mathbf{R})[\sim (\exists!\, y \in \mathbf{R})(x + y = 0)]$, i.e., for every x there is no unique y such that $x + y = 0$. This is clearly false, because for every x **there is** a unique $y(= -x)$ such that $x + y = 0$.